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Rank Approach for Equality Relations of BLUPs in Linear Mixed Model and Its Transformed Model

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Abstract

A linear mixed model (LMM) $\mathscr{M}: \mathbf{y} = \mathbf{X}\beta + \mathbf{Zu} + \varepsilon$ with general assumptions and its transformed model $\mathscr{T}: \mathbf{Ty} = \mathbf{TX}\beta + \mathbf{TZu} + \mathbf{T}\varepsilon$ are considered. This work concerns the comparison problem of predictors under $\mathscr{M}$ and $\mathscr{T}$. Our aim is to establish equality relations between the best linear unbiased predictors (BLUPs) of unknown vectors under two LMMs $\mathscr{M}$ and $\mathscr{T}$ through their covariance matrices by using various rank formulas of block matrices and elementary matrix operations.

1. Introduction

Throughout this note, the symbol $\mathbb{R}^{m\times n}$ denotes the set of all $m\times n$ real matrices. $\mathbf{A}'$, $\mathbf{A}^{+}$, $r(\mathbf{A})$ and $\mathscr{C}(\mathbf{A})$ stand for the transpose, the Moore–Penrose generalized inverse, the rank, and the column space of $\mathbf{A}\in\mathbb{R}^{m\times n}$, respectively. $\mathbf{I}_m$ refers the $m\times m$ identity matrix. Furthermore, $\mathbf{E_A} = \mathbf{A}^{\perp} = \mathbf{I}_m - \mathbf{AA}^{+}$ represents the orthogonal projector for $\mathbf{A}\in\mathbb{R}^{m\times n}$.
A linear mixed model (LMM), formulated by

$$\mathscr{M}: \mathbf{y} = \mathbf{X}\beta + \mathbf{Zu} + \varepsilon, \tag{1.1}$$

where $\mathbf{y}\in\mathbb{R}^{n\times 1}$ is a vector of observable response variables, $\mathbf{X}\in\mathbb{R}^{n\times k}$ and $\mathbf{Z}\in\mathbb{R}^{n\times p}$ are known matrices of arbitrary rank, $\beta\in\mathbb{R}^{k\times 1}$ is a vector of fixed but unknown parameters, $\mathbf{u}\in\mathbb{R}^{p\times 1}$ is a vector of unobservable random effects, and $\varepsilon\in\mathbb{R}^{n\times 1}$ is an unobservable vector of random errors. LMMs include fixed and random effects and supply helpful tools to explain the variability of model parameters affecting response variables. In statistical inferences of analysis requirements, LMMs may need to be transformed. One of the various transformations is the linear transformation of a given model which is obtained by pre-multiplying the model by a given matrix. In such case, for given transformation matrix $\mathbf{T}\in\mathbb{R}^{m\times n}$, transformed model of $\mathscr{M}$ is obtained as follows

$$\mathscr{T}: \mathbf{Ty} = \mathbf{TX}\beta + \mathbf{TZu} + \mathbf{T}\varepsilon. \tag{1.2}$$

We consider the following vector including all unknown vectors under the models $\mathscr{M}$ and $\mathscr{T}$ to establish simultaneous results on predictors:

$$\phi = \mathbf{K}\beta + \mathbf{Gu} + \mathbf{H}\varepsilon = \mathbf{K}\beta + \begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \varepsilon\end{bmatrix} \tag{1.3}$$

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for given $\mathbf{K} \in \mathbb{R}^{s\times k}$, $\mathbf{G} \in \mathbb{R}^{s\times p}$, and $\mathbf{H} \in \mathbb{R}^{s\times n}$. We assume the following general assumptions for considered models:

$$\mathrm{E}\begin{bmatrix}\mathbf{u}\\ \varepsilon\end{bmatrix} = \mathbf{0} \text{ and } \mathrm{D}\begin{bmatrix}\mathbf{u}\\ \varepsilon\end{bmatrix} = \mathrm{cov}\left\{\begin{bmatrix}\mathbf{u}\\ \varepsilon\end{bmatrix}, \begin{bmatrix}\mathbf{u}\\ \varepsilon\end{bmatrix}\right\} = \begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix} := \Sigma,$$

where $\Sigma \in \mathbb{R}^{(n+p)\times(n+p)}$ is a positive semi-definite matrix of arbitrary rank and all the elements of Σ are known.
Let $\mathbf{A} = \begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}$. Then we obtain

$$\mathrm{E}(\mathbf{y}) = \mathbf{X}\beta,\ \mathrm{D}(\mathbf{y}) = \begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}' = \mathbf{A}\Sigma\mathbf{A}' := \mathbf{R},$$

$$\mathrm{E}(\phi) = \mathbf{K}\beta,\ \mathrm{D}(\phi) = \begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}' = \mathbf{B}\Sigma\mathbf{B}' := \mathbf{S},$$

$$\mathrm{cov}(\phi, \mathbf{y}) = \begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}' = \mathbf{B}\Sigma\mathbf{A}' := \mathbf{C}.$$

Further, we assume that $\mathscr{M}$ is consistent, i.e., $\mathbf{y} \in \mathscr{C}\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix}$ holds with probability 1 (wp 1), see, e.g., [1]. The consistency of $\mathscr{T}$ is provided with the condition $\mathbf{Ty} \in \mathscr{C}\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix}$ wp 1. It is easy to see that $\mathscr{T}$ is consistent under the consistency of $\mathscr{M}$.
Predictors under original models and their transformed models have different properties. In some cases, due to linear transformation, observable random vectors in transformed models may preserve enough information to predict unknown vectors under original models. For this reason, establishing relationships and comparisons between these models is statistically useful. In prediction problems, covariance matrices of predictors can be used to establish some statistical properties of analysis such as comparison of predictors. Further, some formulas in matrix algebra such as ranks of matrices offer practical ways for simplifying various complicated matrix equations. The matrix rank method based on the fact that $\mathbf{A} = \mathbf{0}$ if and only if $r(\mathbf{A}) = 0$ is one of the useful methods for deriving algebraic and statistical properties of matrix expressions. This study considers the comparison problem of predictors under an LMM and its transformed model under general assumptions. In particular, we establish equality relations between the best linear unbiased predictors (BLUPs) of unknown vectors under $\mathscr{M}$ and $\mathscr{T}$ through their covariance matrices by using various rank formulas for block matrices, the matrix rank method, and elementary matrix operations. We also give some results for certain specific forms of ϕ which correspond to the best linear unbiased estimators (BLUEs) of unknown parameters under $\mathscr{M}$ and $\mathscr{T}$. To derive the results, we use the following situations to establish equalities between two random vectors, see, e.g., [2] and [3]. Let $\mathbf{u}$ be a random vector

(a) If both $\mathrm{E}(\mathbf{F}_1\mathbf{u} - \mathbf{F}_2\mathbf{u}) = \mathbf{0}$ and $\mathrm{D}(\mathbf{F}_1\mathbf{u} - \mathbf{F}_2\mathbf{u}) = \mathbf{0}$ hold, $\mathbf{F}_1\mathbf{u} = \mathbf{F}_2\mathbf{u}$ holds wp 1.
(b) If both $\mathrm{E}(\mathbf{F}_1\mathbf{u}) = \mathrm{E}(\mathbf{F}_2\mathbf{u})$ and $\mathrm{D}(\mathbf{F}_1\mathbf{u}) = \mathrm{D}(\mathbf{F}_2\mathbf{u})$ hold, the expectation and covariance of $\mathbf{F}_1\mathbf{u}$ and $\mathbf{F}_2\mathbf{u}$ are equal, respectively.

Further, we use the following formulas for ranks of block matrices to establish the results in this study. They are given in the following lemma; see [4] and [5].

Lemma 1.1. *Let $\mathbf{M} \in \mathbb{R}^{m\times n}$, $\mathbf{N} \in \mathbb{R}^{m\times k}$, $\mathbf{P} \in \mathbb{R}^{l\times n}$, and $\mathbf{Q} \in \mathbb{R}^{l\times k}$. Then,*

$$r\begin{bmatrix}\mathbf{M}, & \mathbf{N}\end{bmatrix} = r(\mathbf{M}) + r(\mathbf{E_M N}) = r(\mathbf{N}) + r(\mathbf{E_N M}),$$

$$r\begin{bmatrix}\mathbf{M}\\ \mathbf{P}\end{bmatrix} = r(\mathbf{M}) + r(\mathbf{P E_{M'}}) = r(\mathbf{P}) + r(\mathbf{M E_{P'}}),$$

$$r\begin{bmatrix}\mathbf{M} & \mathbf{N}\\ \mathbf{P} & \mathbf{0}\end{bmatrix} = r(\mathbf{N}) + r(\mathbf{P}) + r(\mathbf{E_N M E_{P'}}), \tag{1.4}$$

$$r\begin{bmatrix}\mathbf{MM}' & \mathbf{N}\\ \mathbf{N}' & \mathbf{0}\end{bmatrix} = r\begin{bmatrix}\mathbf{M}, & \mathbf{N}\end{bmatrix} + r(\mathbf{N}),$$

$$r\begin{bmatrix}\mathbf{M} & \mathbf{N}\\ \mathbf{P} & \mathbf{Q}\end{bmatrix} = r(\mathbf{M}) + r(\mathbf{Q} - \mathbf{PM}^{+}\mathbf{N}) \ \textit{if}\ \mathscr{C}(\mathbf{N}) \subseteq \mathscr{C}(\mathbf{M}) \ \textit{and}\ \mathscr{C}(\mathbf{P}') \subseteq \mathscr{C}(\mathbf{M}'), \tag{1.5}$$

Statistical inference of LMMs is an important part in the data analysis, and some previous and recent studies on relations between predictors under these models can be found in, e.g., [6]-[19], among others. Searching relationships between a linear model and its transformed model is one of the essential issues in linear regression analysis. For transformation approaches of linear models, we may refer [2], [20]-[28].

2. Notes on BLUPs in LMMs

To obtain some results of the BLUPs under models $\mathscr{M}$ and $\mathscr{T}$, we need some fundamental facts on BLUPs under LMMs. In this section, we review the predictability conditions and then we give the fundamental BLUP equations and related properties under $\mathscr{M}$ and $\mathscr{T}$.

The predictability requirement of vector ϕ in (1.3) under $\mathscr{M}$ is described as holding the inclusion $\mathscr{C}(\mathbf{K}') \subseteq \mathscr{C}(\mathbf{X}')$. This requirement also corresponds to the estimability of vector $\mathbf{K}\beta$ under $\mathscr{M}$; see, e.g., [29]. For transformed model $\mathscr{T}$, the predictability requirement of vector ϕ is $\mathscr{C}(\mathbf{K}') \subseteq \mathscr{C}(\mathbf{X}'\mathbf{T}')$. It's obvious that the predictability of ϕ under $\mathscr{T}$ shows predictability of ϕ under $\mathscr{M}$.

Let ϕ predictable under $\mathscr{M}$. If there exists $\mathbf{Ly}$ such that

$$\mathrm{D}(\mathbf{Ly}-\phi) = \min \text{ subject to } \mathrm{E}(\mathbf{Ly}-\phi) = \mathbf{0}$$

holds in the Löwner partial ordering, the linear statistic $\mathbf{Ly}$ is defined to be the BLUP of ϕ and is denoted by $\mathbf{Ly} = \mathrm{BLUP}_{\mathscr{M}}(\phi) = \mathrm{BLUP}_{\mathscr{M}}(\mathbf{K}\beta + \mathbf{Gu} + \mathbf{H}\varepsilon)$, is originated from [30]. If $\mathbf{G} = \mathbf{0}$ and $\mathbf{H} = \mathbf{0}$, $\mathbf{Ly}$ corresponds the BLUE of $\mathbf{K}\beta$, denoted by $\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\beta)$, under $\mathscr{M}$.

We have the following comprehensive result for the algebraic expressions of the BLUPs of ϕ and also properties of the BLUPs; as a detailed study for linear random effects models see [3].

Lemma 2.1. *Let $\mathscr{T}$ be as given in (1.2) and let ϕ in (1.3) be predictable under $\mathscr{T}$. In this case,*

$$\mathrm{E}(\mathbf{L}_t\mathbf{Ty}-\phi) = \mathbf{0} \text{ and } \mathrm{D}(\mathbf{L}_t\mathbf{Ty}-\phi) = \min \Leftrightarrow \mathbf{L}_t\left[\mathbf{TX}, \quad \mathbf{TRT}'(\mathbf{TX})^{\perp}\right] = \left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]. \tag{2.1}$$

The equation in (2.1) is called the fundamental BLUP *equation and*

$$\mathrm{BLUP}_{\mathscr{T}}(\phi) = \mathbf{L}_t\mathbf{Ty} = \left(\left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{T} + \mathbf{U}_t\mathbf{W}_t^{\perp}\mathbf{T}\right)\mathbf{y}, \tag{2.2}$$

where $\mathbf{U}_t \in \mathbb{R}^{s\times m}$ is arbitrary and $\mathbf{W}_t = \left[\mathbf{TX}, \quad \mathbf{TRT}'(\mathbf{TX})^{\perp}\right]$. In particular,

(a) $\mathbf{L}_t$ is unique $\Leftrightarrow r\left[\mathbf{TX}, \quad \mathbf{TRT}'(\mathbf{TX})^{\perp}\right] = m$.
(b) $\mathrm{BLUP}_{\mathscr{T}}(\phi)$ is unique wp 1 $\Leftrightarrow \mathscr{T}$ is consistent.
(c) The rank of matrix $\mathbf{W}_t$ satisfies $r\left[\mathbf{TX}, \quad \mathbf{TRT}'(\mathbf{TX})^{\perp}\right] = r\left[\mathbf{TX}, \quad \mathbf{TRT}'\right]$.
(d) $\mathrm{BLUP}_{\mathscr{T}}(\phi)$ satisfies

$$\mathrm{D}[\mathrm{BLUP}_{\mathscr{T}}(\phi)] = \left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TRT}'\left(\left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\right)',$$

$$\mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{T}}(\phi)] = \left(\left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TA}-\mathbf{B}\right)\Sigma\left(\left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TA}-\mathbf{B}\right)'. \tag{2.3}$$

Let ϕ in (1.3) be predictable under $\mathscr{M}$. By setting $\mathbf{T} = \mathbf{I}_n$ in Lemma 2.1, we obtain the following well-known results on BLUP of ϕ under $\mathscr{M}$. We may also refer [31] and for deriving the BLUPs under linear random-effects models see, [17].

$$\mathrm{BLUP}_{\mathscr{M}}(\phi) = \mathbf{Ly} = \left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+} + \mathbf{UW}^{\perp}\right)\mathbf{y}, \tag{2.4}$$

$$\mathrm{D}[\mathrm{BLUP}_{\mathscr{M}}(\phi)] = \left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\mathbf{R}\left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\right)',$$

$$\mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)] = \left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\mathbf{A}-\mathbf{B}\right)\Sigma\left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\mathbf{A}-\mathbf{B}\right)', \tag{2.5}$$

where $\mathbf{U} \in \mathbb{R}^{s\times n}$ is arbitrary and $\mathbf{W} = \left[\mathbf{X}, \quad \mathbf{RX}^{\perp}\right]$. Further, we can write the following results.

(a) $\mathbf{L}$ in (2.4) is unique $\Leftrightarrow r\left[\mathbf{X}, \quad \mathbf{RX}^{\perp}\right] = n$.
(b) $\mathrm{BLUP}_{\mathscr{M}}(\phi)$ is unique wp 1 $\Leftrightarrow \mathscr{M}$ is consistent.
(c) The rank of matrix $\mathbf{W}$ satisfies $r\left[\mathbf{X}, \quad \mathbf{RX}^{\perp}\right] = r\left[\mathbf{X}, \quad \mathbf{R}\right]$.

3. Equality relations of BLUPs in LMMs

In this section, we establish equality relations between BLUPs of ϕ under $\mathscr{M}$ and $\mathscr{T}$ through their covariance matrices by using block matrices' rank formulas and elementary matrix operations. Related conclusions are also given for some special forms of ϕ. Equality relations between covariance matrices of BLUPs of ϕ under the models, which is obtained in the following results, correspond to the equality situations given in Section 1, respectively, by combining the following result:

$$\mathrm{E}[\mathrm{BLUP}_{\mathscr{M}}(\phi)] = \mathrm{E}[\mathrm{BLUP}_{\mathscr{T}}(\phi)] = \mathbf{K}\beta.$$

Theorem 3.1. *Let ϕ in (1.3) be predictable under $\mathscr{T}$ in (1.2) (also predictable under $\mathscr{M}$ in (1.1)). Let* $\mathrm{BLUP}_{\mathscr{T}}(\phi)$ *and* $\mathrm{BLUP}_{\mathscr{M}}(\phi)$ *be as given in (2.2) and (2.4), respectively. Then,*

$$\mathrm{BLUP}_{\mathscr{M}}(\phi) = \mathrm{BLUP}_{\mathscr{T}}(\phi) \text{ wp 1}$$

$$\Leftrightarrow r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \mathbf{TRT'} & \mathbf{0} & \mathbf{TX} & \mathbf{TR} \\ \mathbf{X'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X'T'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C} & -\mathbf{CT'} & \mathbf{K} & -\mathbf{K} & \mathbf{0} \end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} + r\begin{bmatrix}\mathbf{TX}, & \mathbf{TR}\end{bmatrix} + r(\mathbf{X}) + r(\mathbf{TX}).$$

Proof. Note from (2.2) and (2.4) that

$$r(\mathrm{D}[\mathrm{BLUP}_{\mathscr{M}}(\phi) - \mathrm{BLUP}_{\mathscr{T}}(\phi)]) = r\left(\begin{bmatrix}\mathbf{K}, & \mathbf{CX}^{\perp}\end{bmatrix}\mathbf{W}^{+}\mathbf{R} - \begin{bmatrix}\mathbf{K}, & \mathbf{CT'}(\mathbf{TX})^{\perp}\end{bmatrix}\mathbf{W}_t^{+}\mathbf{TR}\right)$$
$$= r\left(\begin{bmatrix}[\mathbf{K}, & \mathbf{CX}^{\perp}], & [\mathbf{K}, & \mathbf{CT'}(\mathbf{TX})^{\perp}]\end{bmatrix}\begin{bmatrix}\mathbf{W} & \mathbf{0} \\ \mathbf{0} & -\mathbf{W}_t\end{bmatrix}^{+}\begin{bmatrix}\mathbf{R} \\ \mathbf{TR}\end{bmatrix}\right), \tag{3.1}$$

where $\mathbf{W}_t = \begin{bmatrix}\mathbf{TX}, & \mathbf{TRT'}(\mathbf{TX})^{\perp}\end{bmatrix}$ and $\mathbf{W} = \begin{bmatrix}\mathbf{X}, & \mathbf{RX}^{\perp}\end{bmatrix}$. We can apply (1.5) to (3.1) since $\mathscr{C}(\mathbf{TR}) = \mathscr{C}(\mathbf{TRT'}) \subseteq \mathscr{C}(\mathbf{W}_t)$, $\mathscr{C}(\mathbf{R}) \subseteq \mathscr{C}(\mathbf{W})$, $\mathscr{C}\left(\begin{bmatrix}\mathbf{K}, & \mathbf{CT'}(\mathbf{TX})^{\perp}\end{bmatrix}'\right) \subseteq \mathscr{C}(\mathbf{W}_t')$, and $\mathscr{C}\left(\begin{bmatrix}\mathbf{K}, & \mathbf{CX}^{\perp}\end{bmatrix}'\right) \subseteq \mathscr{C}(\mathbf{W}')$ hold. Then, by simplifying Lemma 1.1, and congruence operations, (3.1) is equivalently written as

$$r\begin{bmatrix} \mathbf{X} & \mathbf{RX}^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \mathbf{0} & -\mathbf{TX} & -\mathbf{TRT'}(\mathbf{TX})^{\perp} & \mathbf{TR} \\ \mathbf{K} & \mathbf{CX}^{\perp} & \mathbf{K} & \mathbf{CT'}(\mathbf{TX})^{\perp} & \mathbf{0} \end{bmatrix} - r\begin{bmatrix}\mathbf{X}, & \mathbf{RX}^{\perp}\end{bmatrix} - r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT'}(\mathbf{TX})^{\perp}\end{bmatrix}$$

$$= r\begin{bmatrix} \mathbf{X} & \mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \mathbf{0} & -\mathbf{TX} & -\mathbf{TRT'} & \mathbf{TR} \\ \mathbf{K} & \mathbf{C} & \mathbf{K} & \mathbf{CT'} & \mathbf{0} \\ \mathbf{0} & \mathbf{X'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X'T'} & \mathbf{0} \end{bmatrix} - r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} - r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT'}\end{bmatrix} - r(\mathbf{X}) - r(\mathbf{TX})$$

$$= r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \mathbf{TRT'} & \mathbf{0} & \mathbf{TX} & \mathbf{TR} \\ \mathbf{X'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X'T'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C} & -\mathbf{CT'} & \mathbf{K} & -\mathbf{K} & \mathbf{0} \end{bmatrix} - r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} - r\begin{bmatrix}\mathbf{TX}, & \mathbf{TR}\end{bmatrix} - r(\mathbf{X}) - r(\mathbf{TX}). \tag{3.2}$$

The required result is seen from (3.2) by using the matrix rank method. □

Corollary 3.2. *Let models $\mathscr{M}$ and $\mathscr{T}$ be as given in (1.1) and (1.2), respectively.*

(a) Assume that $\mathbf{K}\beta$ is estimable under $\mathscr{T}$ (also estimable under $\mathscr{M}$). Then

$$\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\beta) = \mathrm{BLUE}_{\mathscr{T}}(\mathbf{K}\beta) \text{ wp 1}$$

$$\Leftrightarrow r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \mathbf{TRT'} & \mathbf{0} & \mathbf{TX} & \mathbf{TR} \\ \mathbf{X'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X'T'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K} & -\mathbf{K} & \mathbf{0} \end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} + r\begin{bmatrix}\mathbf{TX}, & \mathbf{TR}\end{bmatrix} + r(\mathbf{X}) + r(\mathbf{TX}).$$

(b) $\mathbf{X}\beta$ is estimable under $\mathscr{T}$ $\Leftrightarrow r(\mathbf{TX}) = r(\mathbf{X})$ (also note that $\mathbf{X}\beta$ is always estimable under $\mathscr{M}$). Then

$$\mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}\beta) = \mathrm{BLUE}_{\mathscr{T}}(\mathbf{X}\beta) \text{ wp 1} \Leftrightarrow r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{R} \\ \mathbf{0} & \mathbf{TRT'} & \mathbf{TX} & \mathbf{TR} \\ \mathbf{X'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X'T'} & \mathbf{0} & \mathbf{0} \end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} + r\begin{bmatrix}\mathbf{TX}, & \mathbf{TR}\end{bmatrix} + r(\mathbf{X}).$$

Theorem 3.3. *Let ϕ in (1.3) be predictable under $\mathscr{T}$ in (1.2) (also predictable under $\mathscr{M}$ in (1.1)). Let* $\mathrm{BLUP}_{\mathscr{T}}(\phi)$ *and* $\mathrm{BLUP}_{\mathscr{M}}(\phi)$ *be as given in (2.2) and (2.4), respectively. Then*

$$\mathrm{D}[\phi - \mathrm{BLUP}_{\mathscr{T}}(\phi)] = \mathrm{D}[\phi - \mathrm{BLUP}_{\mathscr{M}}(\phi)]$$

$$\Leftrightarrow r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{C'} \\ \mathbf{0} & \mathbf{TRT'} & \mathbf{0} & \mathbf{TX} & \mathbf{TC'} \\ \mathbf{X'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K'} \\ \mathbf{0} & \mathbf{X'T'} & \mathbf{0} & \mathbf{0} & \mathbf{K'} \\ -\mathbf{C} & \mathbf{CT'} & -\mathbf{K} & \mathbf{K} & \mathbf{0} \end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} + r(\mathbf{TX}) + r\begin{bmatrix}\mathbf{TX}, & \mathbf{TR}\end{bmatrix} + r(\mathbf{X}).$$

Proof. By using (2.3) and (1.5), we obtain

$$
\begin{aligned}
& r(\mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]-\mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{T}}(\phi)]) \\
& = r\left(\mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]-\left(\begin{bmatrix}\mathbf{K}, & \mathbf{CT}'(\mathbf{TX})^{\perp}\end{bmatrix}\mathbf{W}_t^{+}\mathbf{TA}-\mathbf{B}\right)\Sigma\left(\begin{bmatrix}\mathbf{K}, & \mathbf{CT}'(\mathbf{TX})^{\perp}\end{bmatrix}\mathbf{W}_t^{+}\mathbf{TA}-\mathbf{B}\right)'\right) \\
& = r\begin{bmatrix}\Sigma & \Sigma\left(\begin{bmatrix}\mathbf{K}, & \mathbf{CT}'(\mathbf{TX})^{\perp}\end{bmatrix}\mathbf{W}_t^{+}\mathbf{TA}\right)'-\Sigma\mathbf{B}' \\ \begin{bmatrix}\mathbf{K}, & \mathbf{CT}'(\mathbf{TX})^{\perp}\end{bmatrix}\mathbf{W}_t^{+}\mathbf{TA}\Sigma-\mathbf{B}\Sigma & \mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]\end{bmatrix}-r(\Sigma) \\
& = r\left(\begin{bmatrix}\Sigma & -\Sigma\mathbf{B}' \\ -\mathbf{B}\Sigma & \mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]\end{bmatrix}+\begin{bmatrix}\Sigma\mathbf{A}'\mathbf{T}' & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix}\mathbf{K}, & \mathbf{CT}'(\mathbf{TX})^{\perp}\end{bmatrix}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{W}_t \\ \mathbf{W}_t' & \mathbf{0}\end{bmatrix}^{+}\right. \\
& \qquad \left.\times\begin{bmatrix}\mathbf{TA}\Sigma & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix}\mathbf{K}, & \mathbf{CT}'(\mathbf{TX})^{\perp}\end{bmatrix}'\end{bmatrix}\right)-r(\Sigma),
\end{aligned}
\tag{3.3}
$$

where $\mathbf{W}_t=\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'(\mathbf{TX})^{\perp}\end{bmatrix}$. We can apply (1.5) to (3.3) since

$$
\mathscr{C}(\mathbf{TA}\Sigma)=\mathscr{C}(\mathbf{TRT}')\subseteq\mathscr{C}(\mathbf{W}_t) \quad \textit{and} \quad \mathscr{C}\left(\begin{bmatrix}\mathbf{K}, & \mathbf{CT}'(\mathbf{TX})^{\perp}\end{bmatrix}'\right)\subseteq\mathscr{C}(\mathbf{W}_t').
$$

Then (3.3) is equivalently written as

$$
\begin{aligned}
& r\begin{bmatrix}\mathbf{0} & -\mathbf{TX} & -\mathbf{TRT}'(\mathbf{TX})^{\perp} & \mathbf{TA}\Sigma & \mathbf{0} \\ -\mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ -(\mathbf{TX})^{\perp}\mathbf{TRT}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{TX})^{\perp}\mathbf{TC}' \\ \Sigma\mathbf{A}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \Sigma & -\Sigma\mathbf{B}' \\ \mathbf{0} & \mathbf{K} & \mathbf{CT}'(\mathbf{TX})^{\perp} & -\mathbf{B}\Sigma & \mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]\end{bmatrix}-r(\Sigma)-2r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'(\mathbf{TX})^{\perp}\end{bmatrix} \\
& = r\begin{bmatrix}-\mathbf{TRT}' & -\mathbf{TX} & -\mathbf{TRT}'(\mathbf{TX})^{\perp} & \mathbf{TC}' \\ -\mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ -(\mathbf{TX})^{\perp}\mathbf{TRT}' & \mathbf{0} & \mathbf{0} & (\mathbf{TX})^{\perp}\mathbf{TC}' \\ \mathbf{CT}' & \mathbf{K} & \mathbf{CT}'(\mathbf{TX})^{\perp} & \mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]-\mathbf{S}\end{bmatrix}-2r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix} \\
& = r\begin{bmatrix}-\mathbf{TRT}' & -\mathbf{TX} & \mathbf{TC}' \\ -\mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{K}' \\ \mathbf{CT}' & \mathbf{K} & \mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]-\mathbf{S}\end{bmatrix}-2r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix}+r[(\mathbf{TX})^{\perp}\mathbf{TRT}'(\mathbf{TX})^{\perp}] \\
& = r\left(\begin{bmatrix}\mathbf{TRT}' & \mathbf{TC}' & \mathbf{TX} \\ \mathbf{CT}' & \mathbf{S} & \mathbf{K} \\ \mathbf{X}'\mathbf{T}' & \mathbf{K}' & \mathbf{0}\end{bmatrix}-\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\right)+r[(\mathbf{TX})^{\perp}\mathbf{TRT}'(\mathbf{TX})^{\perp}]-2r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix}.
\end{aligned}
\tag{3.4}
$$

We can apply (1.5) to (3.4) after setting the expression of $\mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]$ given in (2.5). In this case, in a similar way to obtaining (3.3), (3.4) is equivalently written as

$$
\begin{aligned}
& r\left(\begin{bmatrix}\Sigma & \mathbf{0} & -\Sigma\mathbf{B}' & \mathbf{0} \\ \mathbf{0} & \mathbf{TRT}' & \mathbf{TC}' & \mathbf{TX} \\ -\mathbf{B}\Sigma & \mathbf{CT}' & \mathbf{S} & \mathbf{K} \\ \mathbf{0} & \mathbf{X}'\mathbf{T}' & \mathbf{K}' & \mathbf{0}\end{bmatrix}+\begin{bmatrix}\Sigma\mathbf{A}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix}\mathbf{K}, & \mathbf{CX}^{\perp}\end{bmatrix} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{W} \\ \mathbf{W}' & \mathbf{0}\end{bmatrix}^{+}\begin{bmatrix}\mathbf{A}\Sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{bmatrix}\mathbf{K}, & \mathbf{CX}^{\perp}\end{bmatrix}' & \mathbf{0}\end{bmatrix}\right) \\
& \quad -2r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix}+r[(\mathbf{TX})^{\perp}\mathbf{TRT}'(\mathbf{TX})^{\perp}]-r(\Sigma),
\end{aligned}
\tag{3.5}
$$

where $\mathbf{W}=\begin{bmatrix}\mathbf{X}, & \mathbf{RX}^{\perp}\end{bmatrix}$. We can reapply (1.5) to (3.5) since $\mathscr{C}(\mathbf{A}\Sigma)=\mathscr{C}(\mathbf{R})\subseteq\mathscr{C}(\mathbf{W})$ and $\mathscr{C}\left(\begin{bmatrix}\mathbf{K}, & \mathbf{CX}^{\perp}\end{bmatrix}'\right)\subseteq\mathscr{C}(\mathbf{W}')$. Then from Lemma 1.1, and some congruence operations, (3.5) is equivalently written as

$$
\begin{aligned}
& r\begin{bmatrix}\mathbf{0} & -\mathbf{X} & -\mathbf{RX}^{\perp} & \mathbf{A}\Sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}' & \mathbf{0} \\ -\mathbf{X}^{\perp}\mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}^{\perp}\mathbf{C}' & \mathbf{0} \\ \Sigma\mathbf{A}' & \mathbf{0} & \mathbf{0} & \Sigma & \mathbf{0} & -\Sigma\mathbf{B}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{TRT}' & \mathbf{TC}' & \mathbf{TX} \\ \mathbf{0} & \mathbf{K} & \mathbf{CX}^{\perp} & -\mathbf{B}\Sigma & \mathbf{CT}' & \mathbf{S} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}'\mathbf{T}' & \mathbf{K}' & \mathbf{0}\end{bmatrix}+r[(\mathbf{TX})^{\perp}\mathbf{TRT}'(\mathbf{TX})^{\perp}]-2r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix} \\
& \quad -r(\Sigma)-2r\begin{bmatrix}\mathbf{X}, & \mathbf{RX}^{\perp}\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= r\begin{bmatrix} -\mathbf{R} & -\mathbf{X} & -\mathbf{R}\mathbf{X}^{\perp} & \mathbf{0} & \mathbf{C}' & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}' & \mathbf{0} \\ -\mathbf{X}^{\perp}\mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}^{\perp}\mathbf{C}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T}\mathbf{R}\mathbf{T}' & \mathbf{T}\mathbf{C}' & \mathbf{T}\mathbf{X} \\ \mathbf{C} & \mathbf{K} & \mathbf{C}\mathbf{X}^{\perp} & \mathbf{C}\mathbf{T}' & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}'\mathbf{T}' & \mathbf{K}' & \mathbf{0} \end{bmatrix} + r[(\mathbf{T}\mathbf{X})^{\perp}\mathbf{T}\mathbf{R}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}] - 2r\begin{bmatrix}\mathbf{T}\mathbf{X}, & \mathbf{T}\mathbf{R}\mathbf{T}'\end{bmatrix} \\
&\quad - 2r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} \\
&= r\begin{bmatrix} -\mathbf{R} & -\mathbf{X} & \mathbf{0} & \mathbf{C}' & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{K}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T}\mathbf{R}\mathbf{T}' & \mathbf{T}\mathbf{C}' & \mathbf{T}\mathbf{X} \\ \mathbf{C} & \mathbf{K} & \mathbf{C}\mathbf{T}' & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\mathbf{T}' & \mathbf{K}' & \mathbf{0} \end{bmatrix} + r[(\mathbf{T}\mathbf{X})^{\perp}\mathbf{T}\mathbf{R}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}] + r(\mathbf{X}^{\perp}\mathbf{R}\mathbf{X}^{\perp}) - 2r\begin{bmatrix}\mathbf{T}\mathbf{X}, & \mathbf{T}\mathbf{R}\mathbf{T}'\end{bmatrix} \\
&\quad - 2r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} \\
&= r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{C}' \\ \mathbf{0} & \mathbf{T}\mathbf{R}\mathbf{T}' & \mathbf{0} & \mathbf{T}\mathbf{X} & \mathbf{T}\mathbf{C}' \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ \mathbf{0} & \mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ -\mathbf{C} & \mathbf{C}\mathbf{T}' & -\mathbf{K} & \mathbf{K} & \mathbf{0} \end{bmatrix} + r\begin{bmatrix} \mathbf{T}\mathbf{R}\mathbf{T}' & \mathbf{T}\mathbf{X} \\ \mathbf{X}'\mathbf{T}' & \mathbf{0} \end{bmatrix} - 2r(\mathbf{T}\mathbf{X}) + r\begin{bmatrix} \mathbf{R} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{bmatrix} - 2r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} \\
&\quad - 2r\begin{bmatrix}\mathbf{T}\mathbf{X}, & \mathbf{T}\mathbf{R}\mathbf{T}'\end{bmatrix} - 2r(\mathbf{X}). \qquad (3.6)
\end{aligned}
$$

The required result is seen from (3.6) by using (1.4) and the matrix rank method. □

Corollary 3.4. *Let models $\mathscr{M}$ and $\mathscr{T}$ be as given in (1.1) and (1.2), respectively.*

(a) Assume that $\mathbf{K}\beta$ is estimable under $\mathscr{T}$ (also estimable under $\mathscr{M}$). Then

$$
\begin{aligned}
&\mathrm{D}[\mathrm{BLUE}_{\mathscr{T}}(\mathbf{K}\beta)] = \mathrm{D}[\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\beta)] \\
&\Leftrightarrow r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}\mathbf{R}\mathbf{T}' & \mathbf{0} & \mathbf{T}\mathbf{X} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ \mathbf{0} & \mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ \mathbf{0} & \mathbf{0} & -\mathbf{K} & \mathbf{K} & \mathbf{0} \end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} + r(\mathbf{T}\mathbf{X}) + r\begin{bmatrix}\mathbf{T}\mathbf{X}, & \mathbf{T}\mathbf{R}\end{bmatrix} + r(\mathbf{X}).
\end{aligned}
$$

(b) $\mathbf{X}\beta$ is estimable under $\mathscr{T}$ $\Leftrightarrow r(\mathbf{T}\mathbf{X}) = r(\mathbf{X})$ (also note that $\mathbf{X}\beta$ is always estimable under $\mathscr{M}$). Then

$$
\mathrm{D}[\mathrm{BLUE}_{\mathscr{T}}(\mathbf{X}\beta)] = \mathrm{D}[\mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}\beta)] \Leftrightarrow r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} \\ \mathbf{0} & \mathbf{T}\mathbf{R}\mathbf{T}' & \mathbf{T}\mathbf{X} \\ \mathbf{X}' & -\mathbf{X}'\mathbf{T}' & \mathbf{0} \end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} + r\begin{bmatrix}\mathbf{T}\mathbf{X}, & \mathbf{T}\mathbf{R}\end{bmatrix}.
$$

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