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$n-$dimensional Trigonometric Proportions for Rectangular Simplices

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Abstract.
The main aim of this paper is to obtain $n-$dimensional versions of trigonometric proportions for right triangle in terms of hypotenuse and perpendicular volume faces of a rectangular simplices.

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1. INTRODUCTION

Let T be a right triangle with right vertex A_i generated by the points $A_i,\ A_j,\ A_k$ and let $\phi_i = \frac{\pi}{2},\ \phi_j,\ \phi_k$ be internal angles of T. Then it follows from the elementary geometry that the trigonometric proportions for T is given by

$$cos\phi_\ell = \frac{\|\overrightarrow{A_iA_\ell}\|}{\|\overrightarrow{A_\ell A_m}\|}$$

(1.1)

$$sin\phi_\ell = \frac{\|\overrightarrow{A_iA_m}\|}{\|\overrightarrow{A_\ell A_m}\|}$$

where (ℓ, m) is a permutation $\{j,\ k\}$.

In this work, we give two new trigonometric proportions which are analogues of equation (1.1) and extend results of [4] for rectangular $n-$simplex.

2. Dihedral Angles and Trigonometric Proportions for Rectangular Simplices

Let $A_0,\ A_1,\ A_2, ...,\ A_n$ be independent points in $n-$dimensional Euclidean space R^n. Let $\{\overrightarrow{A_0A_1},\ \overrightarrow{A_0A_2}, ...,\ \overrightarrow{A_0A_n}\}$ be a orthogonal set and A rectangular simplex is the convex hull of A_0. ([1, 6, 5]).

Let S_p be $(n-1)-$face of rectangular simplex not containing A_p vertex and $Vol(S_p)$ denotes its volume for $p \in \{0, 1, ..., n\}$. Then, S_0 is called hypotenuse and

S_m is called the perpendicular face of the rectangular simplex for $m \neq 0$ generated by the points $A_0,\ A_1,\ A_2, ...,\ A_n$, respectively. The edge $|A_0A_m| = a_{0m}$ is called the perpendicular edge of the rectangular simplex for $m \in \{1, ..., n\}$.

The dihedral angle is the angle formed at the common edge between S_m and S_n. The dihedral angle is measured by intersecting it with the $(n-1)$-dimensional hyperplane which is perpendicular to the common edge. θ_{mn} represents the measure of the interior dihedral angle between S_m and S_n. Since the edges a_{0m} and a_{0n} are perpendicular, we easily see that $\theta_{mn} = \frac{\pi}{2}$ for all $m, n \in \{1, ..., n\}$, $m \neq n$.

Lemma 2.1. *Let* Ω *be* $n-$*simplex with vertices* $A_0,\ A_1,\ \ldots, A_n$*, and* $e_0,\ e_1,\ \ldots, e_n$ *be outer unit normal of its* $(n-1)-$ *faces with volume* $Vol(S_i)$*, then*

$$Vol(S_j) = \sum_{i=0, i\neq j}^{n} Vol(S_i) \cos\theta_{ij} \tag{2.1}$$

or by matrix notations,

$$\begin{pmatrix} Vol(S_0) \\ Vol(S_1) \\ . \\ . \\ Vol(S_n) \end{pmatrix} = \begin{pmatrix} 0 & \cos\theta_{ij} \\ \cos\theta_{ij} & 0 \end{pmatrix} \begin{pmatrix} Vol(S_0) \\ Vol(S_1) \\ . \\ . \\ Vol(S_n) \end{pmatrix}$$

where θ_{ij} *is the interior dihedral angle and* $0 \leq i, j \leq n$*.*

Proof. By proof of Theorem 3 in [5] (or [1]), it is obvious for any polyhedron.

$$\sum_{i=0}^{n} Vol(S_i) e_i = 0 \tag{2.2}$$

By taking the inner product of both sides with e_j and simplifying it, we have

$$Vol(S_j) = \sum_{i=0, i\neq j}^{n} Vol(S_i) \cos\theta_{ij},\ 0 \leq j \leq n$$

□

Theorem 2.2. *Let* T *be a rectangular simplex with right vertex* A_0 *generated by points* $A_0,\ A_1, ...,\ A_n$*. Then, the trigonometric proportions of* T *at* S_p *face is given by*

$$\cos\theta_{0p} = \frac{Vol(S_p)}{Vol(S_0)}$$

$$\sin\theta_{0p} = \frac{\sqrt{\sum\limits_{k=1,k\neq p}^{n} Vol(S_k)^2}}{Vol(S_0)}$$

Proof. By using $\theta_{pq} = \frac{\pi}{2},\ p,\ q \in \{1,\ 2, ...,\ n\}$ in Lemma 2.1, we see that

$$Vol(S_p) = Vol(S_0)\cos\theta_{0p} \tag{2.3}$$

By substituting equation (2.3) in Theorem A of [1],

$Vol(S_0)^2 = \sum\limits_{k=1}^{n} Vol(S_k)^2$, we obtain that

$$\sin\theta_{0p} = \frac{\sqrt{\sum\limits_{k=1k\neq p}^{n} Vol(S_k)^2}}{Vol(S_0)}$$

□

Now we can give a new analogue of equation (1.1) for dihedral angles of a rectangular simplex in terms of the hypotenuse and perpendicular face volumes as a following corollary.

Corollary 2.3. *Let T be a rectangular simplex with right vertex A_0 generated by points $A_0,\ A_1, ...,\ A_n$. Then the trigonometric proportions of T at S_p face is given by*

$$\cos\theta_{0p} = \frac{Vol(APF)}{Vol(HF)}$$

$$\sin\theta_{0p} = \frac{\sqrt{B}}{Vol(HF)}$$

where APF is Adjacent Perpendicular Face, HF is Hypotenuse Faces, and B is the sum of square of volume of the Opposite Perpendicular Faces.

Proof. It is clear from Theorem 2.2. □

3. Vertex Angles and Trigonometric Proportions for Rectangular Simplices

E^n is a $n-$dimensional Euclidean space with an orientation. Let $\Omega =< A_0, A_1, ..., A_n >$ be an $n-$simplex with vertices $A_0, A_1, ..., A_n$, and let $\Omega_i =< A_0, A_1, ..., A_{i-1}, A_{i+1}, ..., A_n >$ be its facet which lies in a hyperplane $\Pi_i,\ e_i$ the unit outer normal vector of $\Pi_i,\ i = 0, ..., n$. We set

$$D_i = \det(e_0,\ e_1,\ ....,\ e_{i-1},\ e_{i+1},\ ...,\ e_n)$$

Then, $\theta_i = \arcsin|D_i|$ is called the vertex angle at vertex A_i of Ω [8] (or [7]).

Now we are ready to give the following theorem which is another new version of equation (1.1) for vertex angles of a rectangular simplex in terms of hypotenuse and perpendicular facet volumes.

Theorem 3.1. *Let T be a rectangular tetrahedron with right vertex A_0 generated by points $A_0,\ A_1,\ A_2, ...,\ A_n$. Then the trigonometric proportions of T in terms of hypotenuse and perpendicular face volumes for the vertex angle θ_p are given by*

$$\cos\theta_p = \frac{Vol(S_p)}{Vol(S_0)}$$

$$\sin\theta_p = \frac{\sqrt{\sum_{k=1,k\neq p}^{n} Vol(S_k)^2}}{Vol(S_0)}$$

Proof. By using definition of vertex angle and Theorem 2.2, we obtain the conclusion of the theorem. □

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