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Applications of Taylor Collocation Method and Lambert W Function to the Systems of Delay Differential Equations

Necdet Bildik and Sinan Deniz

Abstract. In this paper, the systems of delay differential equations with initial conditions are solved by using Taylor Collocation Method and Lambert W Function and we tried to show the appropriate method by comparing the solution process of the system of these equations. All numerical computations have been performed on the computer algebraic system Matlab.

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1. INTRODUCTION

Time-delay systems are also called systems with aftereffect or dead-time, hereditary systems, equations with deviating argument or differential-difference equations. They belong to the class of functional differential equations (FDEs) which are infinite dimensional, as opposed to ordinary differential equations (ODEs) (Richard, 2003).

In daily life, as we construct mathematical modeling to solve problems, we usually use the initial value problems such that

$$\begin{cases} y'(t) = f(t, y(t)), & t \geq t_0 \\ y(t_0) = y_0, \end{cases} \tag{1.1}$$

where t_0 is the starting point and y_0 is initial value. For instance, suppose we want to estimate the amount of population growth in a community. Firstly, we assume that there is not any kind of external influence in this group, like an isolated in a closed box. Let $y(t)$ shows the amount of the population at time t and also we assume that speed of growth proportional to the current population at that moment. We denote this rate with "k" constant. In that case if we show the change of population by $y'(t)$, we can rewrite the system (1.1) as follows [1]:

$$\begin{cases} y'(t) = k.\, y(t), & t \geq t_0 \\ y(t_0) = y_0, \end{cases}$$

The delays are always ignored in the systems to be modeled by using ordinary differential equations. But very small amount of delay in the system can cause large changes in the current case of the system. So, while modeling of majority of encountered problems, using of delay differential equations is more real [1],[8]. In previous modeling, in order to determine population growth we accepted that the rate is only proportional to current population. But generally previous state of the system can significantly affect the future status. We use amount of delay to indicate the status of systems in the past and thus when we modeling the systems, we also take into account the dependencies of systems to past. In this case, when we accept that population change in the community commensurate with the previous population a certain period of time (τ) rather than the current population, we obtain the delay differential equation [1] as follows:

$$\begin{cases} y'(t) = k.\, y(t-\tau), & t \geq t_0, \quad \tau > 0, \\ y(t_0) = \varphi(t), & t_0 - \tau \leq t \leq t_0. \end{cases}$$

The principal difficulty in studying delay differential equations lies in its special transcendental character. The delay operator can be expressed in the form of an infinite series. Delay equations always lead to an infinite spectrum of frequencies. The determination of this spectrum requires a corresponding determination of zeros of certain analytic functions. One of the well-known approximation methods is the Pade approximation, which results in a shortened repeating fraction for the approximation of the characteristic equation of the delay (Lam, 1993; Golub and Van Loan, 1989).

2. Lambert W Function

In this section we examine the first order (scalar), linear and homogeneous delay differential equation systems such that

$$y'(t) + A(t)y(t-\tau) + B(t)y(t) = 0, \quad \tau > 0. \tag{2.1}$$

In this system A and B are $n * n$ types matrices of real valued functions depending on t variable and $\tau > 0$ is a real valued constant. Assuming A and B matrices are accepted as a constant in Eq. (2.1), i.e.:

$$y'(t) + Ay(t-\tau) + By(t) = 0, \quad \tau > 0. \tag{2.2}$$

Here,in order to obtain characteristic equation of the system (2.1) we assume that $y = e^{st}$ is the solution of (2.1) equation. So that this solution provides the given equality. In that case, we get

$$se^{st} + Ae^{s(t-\tau)} + Be^{st} = 0.$$

Dividing both sides of equation by e^{st}

$$sI + Ae^{-s\tau} + B = 0.$$

Rearrange the equation, we find

$$sI = -Ae^{-s\tau} - B.$$

Multiplying by $e^{s\tau}$, τ and $e^{B\tau}$ respectively, we obtain

$$\begin{aligned} sIe^{s\tau} &= -A - Be^{s\tau}, \\ (sI)\tau e^{s\tau} &= (-A)\tau - B\tau e^{s\tau}, \\ (s\tau)Ie^{(s\tau)I} &= (-A)\tau - B\tau e^{s\tau I}, \\ (s\tau)Ie^{(s\tau)I} + B\tau e^{s\tau I} &= (-A)\tau, \end{aligned}$$

$$(sI + B)\tau e^{(s\tau)I} = (-A)\tau,$$
$$(sI + B)\tau e^{(sI+B)\tau} = (-A)\tau e^{B\tau}.$$

By the definition of Lambert W Function, we get the characteristic equation

$$W((sI + B)\tau)e^{W((sI+B)\tau)} = (sI + B)\tau.$$

Rearranging the equation

$$(sI + B)\tau = W(-A\tau e^{B\tau}),$$

$$sI = \frac{1}{\tau}W(-A\tau e^{B\tau}) - B. \tag{2.3}$$

Particularly for $B = 0$, we can get

$$sI = \frac{1}{\tau}W(-A\tau). \tag{2.4}$$

In this case the general solution of the system (2.1) is determined as

$$y(t) = \sum_{-\infty}^{\infty} c_k e^{\left[\frac{1}{\tau}W_k(-A\tau e^{B\tau})-B\right]t}.$$

In this equation, the coefficient matrix c_k is $n*1$ types and it is calculated by means of initial function [2],[4],[5],[9],[10].

Example 1:

$$\begin{cases} y_1' = -y_2(t-1), \\ y_2' = 2y_1(t-2) + y_3(t-2), \\ y_3' = 3y_2(t-1). \end{cases}$$

Let's solve the system of delay differential equation by using Lambert function.
Solution: We write

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix}$$

where $y'(t) = \frac{dy}{dt} = \sum_{j=1}^{3} A_i y(t-\tau_j) = A_1 y(t-\tau_1) + A_2 y(t-\tau_2) + A_3 y(t-\tau_3)$.
Particular solutions of this system of equation are type of $y(t) = ce^{st}$ and it should be $det(sI - \sum_{j=1}^{3} A_i y(t-\tau_j)) = 0$ to get non-zero solutions. Thus particular solution is calculated as:

$$y(t) = ce^{st},$$
$$y_1(t-2) = ce^{s(t-2)},$$
$$y_2(t-1) = ce^{s(t-1)},$$
$$y_3(t-2) = ce^{s(t-2)}.$$

When the matrices of these equations are set up, we obtain

$$\mathbf{det}\begin{pmatrix} sI - A_1 y_1(t-\tau_1) \\ sI - A_2 y_2(t-\tau_2) \\ sI - A_3 y_3(t-\tau_3) \end{pmatrix} = 0.$$

So

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \left(A_1 c e^{s(t-2)} + A_2 c e^{s(t-1)} + A_3 c e^{s(t-2)}\right)$$

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \left(A_1 c e^{st} e^{-2s} + A_2 c e^{st} e^{-s} + A_3 c e^{st} e^{-2s}\right)$$

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}_{3*1} \begin{pmatrix} e^{-2s} & e^{-s} & e^{-2s} \end{pmatrix}_{1*3}$$

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} A_1 e^{-2s} & A_1 e^{-s} & A_1 e^{-2s} \\ A_2 e^{-2s} & A_2 e^{-s} & A_2 e^{-2s} \\ A_3 e^{-2s} & A_3 e^{-s} & A_3 e^{-2s} \end{pmatrix}$$

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0.\, e^{-2s} & -1.\, e^{-s} & 0.\, e^{-2s} \\ 2.\, e^{-2s} & 0.\, e^{-s} & 1.\, e^{-2s} \\ 0.\, e^{-2s} & 3.\, e^{-s} & 0.\, e^{-2s} \end{pmatrix}$$

are found. Thus

$$\mathbf{det} \begin{pmatrix} s & e^{-s} & 0 \\ -2e^{-2s} & s & -e^{-2s} \\ 0 & -3e^{-s} & s \end{pmatrix} = 0.$$

So

$$s\left(s^2 - 3e^{-3s} + 2e^{-3s}\right),$$

$$s_1 = 0,$$

and

$$s^2 = e^{-3s},$$

is calculated. Hence we also get

$$s_2 = \frac{2}{3} W\left(\frac{3}{2}\right),$$

$$s_3 = \frac{2}{3} W\left(-\frac{3}{2}\right),$$

Hence, the general solution to s_2 is

$$\begin{aligned} y(t) &= \ldots + c_{-1} e^{\frac{2}{3} W_{-1}\left(\frac{3}{2}\right) t} + c_0 e^{\frac{2}{3} W_0\left(\frac{3}{2}\right) t} + c_1 e^{\frac{2}{3} W_1\left(\frac{3}{2}\right) t} + \ldots \\ &= \ldots + c_{-1} e^{\frac{2}{3}(-1.12168-4.46634i)t} + c_0 e^{\frac{2}{3}(0.725861)t} + c_1 e^{\frac{2}{3}(-1.12168+4.46634i)t} + \ldots \end{aligned}$$

and the general solution to s_3 is also

$$\begin{aligned} y(t) &= \ldots + c_{-1} e^{\frac{2}{3} W_{-1}\left(-\frac{3}{2}\right) t} + c_0 e^{\frac{2}{3} W_0\left(-\frac{3}{2}\right) t} + c_1 e^{\frac{2}{3} W_1\left(-\frac{3}{2}\right) t} + \ldots \\ &= \ldots + c_{-1} e^{\frac{2}{3}(-1.65090-7.64120i)t} + c_0 e^{\frac{2}{3}(-0.03278+1.54964i)t} + c_1 e^{\frac{2}{3}(-1.65090+7.64120i)t} + \ldots \end{aligned}$$

3. Taylor Collocation Method

Taylor Collocation Method is one of the effective method to find the approximate solutions of systems of linear high-order delay differential equations in the form

$$\sum_{r=0}^{m}\sum_{i=1}^{k} P_{ji}^{r}(t)y_i^{(r)}(\lambda t+\mu) = f_i(t), \quad j = 1,2,\ldots,k \tag{3.1}$$

under the mixed conditions defined as

$$\sum_{j=0}^{m-1} a_{rj}^{n}y_n^{(j)}(a)+b_{rj}^{n}y_n^{(j)}(b)+c_{rj}^{n}y_n^{(j)}(c) = \lambda_{nr}, \quad a \le c \le b, \quad r = 0,1,2,\ldots,m-1, \quad n = 1,2,\ldots,k$$

where $y_i(t)$ is an unknown function, the known functions $P_{ji}^{n}(t)$ and $f_j(t)$ are defined on interval $a \le t \le b$, and also a_{rj}, b_{rj}, c_{rj} and λ_{nr} are appropriate constants.
Our main purpose is to find the approximate solutions of the system (3.1) expressed in the truncated Taylor series form as:

$$y_i(t) = \sum_{n=0}^{N} y_{in}(t-c)^n, \quad y_{in} = \frac{y_i^{(n)}(c)}{n!}, \quad i = 1,2,\ldots,k, \quad a \le t \le b, \tag{3.2}$$

where $y_{in}, (n = 0,1,...,N \textit{ and } i = 1,2,\ldots,k)$ are unknown coefficients, N is any positive integer such that $N \ge m$ [3].

Fundamental Relations

The functions defined by Eq. (3.2) can be written in the matrix forms as:

$$y_i(t) = T(t)A_i, \quad i = 1,2,...,k, \tag{3.3}$$

$$y_i'(t) = T(t)BA_i, \tag{3.4}$$

$$\vdots$$

$$y_i^{(r)}(t) = T(t)B^{(r)}A_i, \tag{3.5}$$

where

$$T(t) = \begin{bmatrix}1 & (t-c) & (t-c)^2\ldots(t-c)^N\end{bmatrix},$$

$$A_i = [y_{i0} \quad y_{i1} \quad y_{i2} \quad \ldots y_{iN}]^T,$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

By substituting $t \to \lambda t + \mu$ in the equation (3.3), we obtain the matrix form

$$y_i(\lambda t+\mu) = T(\lambda t+\mu)A_i. \tag{3.6}$$

The relation between the matrix $T(\lambda t+\mu)$ and $T(t)$ is

$$T(\lambda t+\mu) = T(t)B(\lambda,\mu), \tag{3.7}$$

where for $\lambda \neq 0$ and $\mu \neq 0$

$$\mathbf{B}(\lambda,\mu) = \begin{bmatrix} \binom{0}{0}\lambda^0\mu^0 & \binom{1}{0}\lambda^0\mu^1 & \binom{2}{0}\lambda^0\mu^2 & \dots & \binom{N}{0}\lambda^0\mu^N \\ 0 & \binom{1}{0}\lambda^1\mu^0 & \binom{1}{1}\lambda^1\mu^1 & \dots & \binom{N}{1}\lambda^1\mu^{N-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \binom{N}{N-1}\lambda^{N-1}\mu^1 \\ 0 & 0 & 0 & \dots & \binom{N}{N}\lambda^N\mu^0 \end{bmatrix}.$$

And for $\lambda \neq 0$ and $\mu = 0$ we get

$$\mathbf{B} = \begin{bmatrix} (\lambda)^0 & 0 & 0 & \dots & 0 \\ 0 & (\lambda)^1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & (\lambda)^N \end{bmatrix}.$$

By differentiating both side of the equation (3.7) with respect to t and using the relation (3.5) we get

$$y_i^{(r)}(\lambda t+\mu) = T(t)B(\lambda,\mu)B^r A_i, \quad i = 1,2,\dots,k. \tag{3.8}$$

Thus the matrices $y^r(t), r = 0,1,2,\dots,m$ can be expressed by

$$\mathbf{y^{(r)}}(\lambda\mathbf{t}+\mu) = \begin{bmatrix} y_1^{(r)}(\lambda t+\mu) \\ y_2^{(r)}(\lambda t+\mu) \\ \vdots \\ y_k^{(r)}(\lambda t+\mu) \end{bmatrix} = \begin{bmatrix} T(t)B(\lambda,\mu)B^{(r)}A_1 \\ T(t)B(\lambda,\mu)B^{(r)}A_2 \\ \vdots \\ T(t)B(\lambda,\mu)B^{(r)}A_k \end{bmatrix} =$$

$$\begin{bmatrix} T(t) & 0 & \dots & 0 \\ 0 & T(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T(t) \end{bmatrix} \begin{bmatrix} B(\lambda,\mu) & 0 & \dots & 0 \\ 0 & B(\lambda,\mu) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B(\lambda,\mu) \end{bmatrix} \begin{bmatrix} B^{(r)} & 0 & \dots & 0 \\ 0 & B^{(r)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B^{(r)} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix}$$

or briefly

$$y^{(r)}(\lambda t+\mu) = T^*(t)\widetilde{B}(\lambda,\mu)\widetilde{B}^r A, \tag{3.9}$$

where

$$\mathbf{T^*(t)} = \begin{bmatrix} T(t) & 0 & \dots & 0 \\ 0 & T(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T(t) \end{bmatrix}, \quad \widetilde{\mathbf{B}}(\lambda,\mu) = \begin{bmatrix} B(\lambda,\mu) & 0 & \dots & 0 \\ 0 & B(\lambda,\mu) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B(\lambda,\mu) \end{bmatrix},$$

$$\widetilde{\mathbf{B}}^{(\mathbf{r})} = \begin{bmatrix} B^{(r)} & 0 & \dots & 0 \\ 0 & B^{(r)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B^{(r)} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} [3].$$

Methods of Solution
First, we can write the system (3.1) in the matrix form

$$\sum_{r=0}^{m} P_r(t)y^{(r)}(\lambda t+\mu)=f(t), \tag{3.10}$$

where

$$\mathbf{P_{(r)}(t)}=\begin{bmatrix} P^r_{11}(t) & P^r_{12}(t) & \dots & P^r_{1k}(t)\\ P^r_{21}(t) & P^r_{22}(t) & \dots & P^r_{2k}(t)\\ \vdots & \vdots & \ddots & \vdots\\ P^r_{k1}(t) & P^r_{k2}(t) & \dots & P^r_{kk}(t)\end{bmatrix},\quad \mathbf{y^{(r)}}(\lambda,\mu)=\begin{bmatrix} y_1^{(r)}(\lambda t+\mu)\\ y_2^{(r)}(\lambda t+\mu)\\ \vdots\\ y_k^{(r)}(\lambda t+\mu)\end{bmatrix},\quad \mathbf{f(t)}=\begin{bmatrix} f_1(t)\\ f_2(t)\\ \vdots\\ f_k(t)\end{bmatrix}.$$

To obtain the Taylor polynomial solutions of the system (3.10) in the form (3.2) , we compute the Taylor coefficients by means of the collocation points defined by

$$t_l=a+\frac{b-a}{N}l,\qquad l=0,1,\dots,N \tag{3.11}$$

where $a\le t\le b$ and $a=t_0<t_1<\dots<t_n=b$.
Substituting the collocation points (3.11) into the matrix Eq. (3.10), then we obtain the matrix form as:

$$\sum_{r=0}^{m} P_rY^{(r)}=F, \tag{3.12}$$

where

$$\mathbf{P_r}=\begin{bmatrix} P_r(t_0) & 0 & \dots & 0\\ 0 & P_r(t_1) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & P_r(t_N)\end{bmatrix},\quad \mathbf{Y^{(r)}}=\begin{bmatrix} y_1^{(r)}(\lambda t_0+\mu)\\ y_2^{(r)}(\lambda t_1+\mu)\\ \vdots\\ y_k^{(r)}(\lambda t_N+\mu)\end{bmatrix},\quad \mathbf{F}=\begin{bmatrix} f(t_0)\\ f(t_1)\\ \vdots\\ f(t_N)\end{bmatrix}.$$

Using the relation (3.9) and the collocation points (3.11), we have

$$y^r(\lambda t_1+\mu)=T^*(t_1)\widetilde{B}(\lambda,\mu)\widetilde{B}^rA,\qquad l=0,1,\dots,N.$$

This system can be written as

$$Y^{(r)}=T\widetilde{B}(\lambda,\mu)\widetilde{B}^rA, \tag{3.13}$$

where $T=[T^*(t_0)\quad T^*(t_1)\quad \dots T^*(t_N)]^T$ and

$$\mathbf{T^*(t_r)}=\begin{bmatrix} T(t_r) & 0 & \dots & 0\\ 0 & T(t_r) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & T(t_r)\end{bmatrix}.$$

Substituting the expression (3.13) into (3.12), we have the fundamental matrix equation

$$\left(\sum_{r=0}^{m} P_rT\widetilde{B}(\lambda,\mu)\widetilde{B}^r\right)A=F. \tag{3.14}$$

Briefly, the fundamental matrix Eq. (3.14) corresponding to Eq.(3.1) can be expressed in the form $WA = F$ which corresponds to a linear algebraic equations of the system with k times $(N + 1)$ unknowns Taylor coefficients such that

$$W = [w_{pq}] = \sum_{r=0}^{m} P_r T \widetilde{B}(\lambda, \mu) \widetilde{B}^r, \quad p, q = 1, 2, \ldots, k(N + 1) \quad [3].$$

Example 1:

$$\begin{cases} y_1' = -y_2(t-1) & y_1(0) = 0 \\ y_2' = 2y_1(t-2) + y_3(t-2) & y_2(0) = -2 \\ y_3' = 2y_2(t-1) & y_3(0) = -2. \end{cases}$$

Solve the delay differential equation system by using Taylor Collocation Method for $-2 \leq t \leq 4$.

Solution: Once considering the set of the equation as follows:

$$\begin{cases} y_1' + y_2(t-1) = 0, \\ y_2' - 2y_1(t-2) - y_3(t-2) = 0, \\ y_3' - 2y_2(t-1) = 0, \end{cases}$$

then we may write the equation in matrix form as:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} y_1(t-1) \\ y_2(t-1) \\ y_3(t-1) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t-2) \\ y_2(t-2) \\ y_3(t-2) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus P1 and P2 are obtained such as

$$\mathbf{P_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $P_2 = I_{12*12}$.

If we take $N = 3$ for $-2 \leq t \leq 4$, we get the collocation points $t_0 = -2, t_1 = 0, t_2 = 2, t_3 = 4$.

Similarly, we write

$$\left(P_0 T \widetilde{B}(1, -1) + P_1 T \widetilde{B}(1, -2) + P_3 T \widetilde{B}(1, 0) \widetilde{B}\right) A = F,$$

where

$$\mathbf{T}=\begin{pmatrix}1&-2&4&-8&0&0&0&0&0&0&0&0\\0&0&0&0&1&-2&4&-8&0&0&0&0\\0&0&0&0&0&0&0&0&1&-2&4&8\\1&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0&0&0\\1&2&4&8&0&0&0&0&0&0&0&0\\0&0&0&0&1&2&4&8&0&0&0&0\\0&0&0&0&0&0&0&0&1&2&4&8\\1&4&16&64&0&0&0&0&0&0&0&0\\0&0&0&0&1&4&16&64&0&0&0&0\\0&0&0&0&0&0&0&0&1&4&16&64\end{pmatrix}$$

$$\widetilde{\mathbf{B}}(\mathbf{1},-\mathbf{1})=\begin{pmatrix}1&-1&1&-1&0&0&0&0&0&0&0&0\\0&1&2&3&0&0&0&0&0&0&0&0\\0&0&1&-3&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0&0\\0&0&0&0&1&-1&1&-1&0&0&0&0\\0&0&0&0&0&1&-2&3&0&0&0&0\\1&2&4&8&0&0&1&-3&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&1&-1&1&-1\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&1&-3\\0&0&0&0&0&0&0&0&0&0&0&1\end{pmatrix}$$

$$\widetilde{\mathbf{B}}(\mathbf{1},-\mathbf{2})=\begin{pmatrix}1&-2&4&-8&0&0&0&0&0&0&0&0\\0&1&-4&12&0&0&0&0&0&0&0&0\\0&0&1&-6&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0&0\\0&0&0&0&1&-2&4&-8&0&0&0&0\\0&0&0&0&0&1&-4&12&0&0&0&0\\0&0&0&0&0&0&1&-6&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&1&-2&4&-8\\0&0&0&0&0&0&0&0&0&1&-4&12\\0&0&0&0&0&0&0&0&0&0&1&-6\\0&0&0&0&0&0&0&0&0&0&0&1\end{pmatrix}$$

$\widetilde{B}(1,0) = I_{12*12}$
and

$$\widetilde{\mathbf{B}} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&2&0&0&0&0&0&0&0&0&0\\ 0&0&0&3&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&3&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&2&0\\ 0&0&0&0&0&0&0&0&0&0&0&3\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

So, we get the equation

$$WA = F.$$

Here,

$$\mathbf{W} = \begin{pmatrix} 0&1&-4&12&1&-3&9&-27&0&0&0&0\\ -2&8&-32&128&0&1&-4&12&-1&4&-16&64\\ 0&0&0&0&-2&6&-18&54&0&1&-4&12\\ 0&1&0&0&1&-1&1&-1&0&0&0&0\\ -2&4&-8&16&0&1&0&0&-1&2&-4&8\\ 0&0&0&0&-2&2&-2&2&0&1&0&0\\ 0&1&4&12&1&1&1&1&0&0&0&0\\ -2&0&0&0&0&1&4&12&-1&0&0&0\\ 0&0&0&0&-2&-2&-2&-2&0&1&4&12\\ 0&1&8&48&1&3&9&27&0&0&0&0\\ -2&-4&-8&-16&0&1&8&48&-1&-2&-4&-8\\ 0&0&0&0&-2&-6&-18&-54&0&1&8&48 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix},$$

Now, let's write the $\overline{W}$ and $\overline{F}$ by using the initial conditions:

$$\overline{\mathbf{W}} = \begin{pmatrix} 0&1&-4&12&1&-3&9&-27&0&0&0&0\\ -2&8&-32&128&0&1&-4&12&-1&4&-16&64\\ 0&0&0&0&-2&6&-18&54&0&1&-4&12\\ 0&1&0&0&1&-1&1&-1&0&0&0&0\\ -2&4&-8&16&0&1&0&0&-1&2&-4&8\\ 0&0&0&0&-2&2&-2&2&0&1&0&0\\ 0&1&4&12&1&1&1&1&0&0&0&0\\ -2&0&0&0&0&1&4&12&-1&0&0&0\\ 0&0&0&0&-2&-2&-2&-2&0&1&4&12\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0 \end{pmatrix}, \overline{\mathbf{F}} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\-2\\-2 \end{pmatrix},$$

Now by equation $\overline{W}A = \overline{F}$, we find

$$\mathbf{A} = \mathbf{inv}(\overline{\mathbf{W}}) * \overline{\mathbf{F}} = \begin{pmatrix} 0 \\ 0.0000 \\ 1.0000 \\ 0.0000 \\ -2.0000 \\ -2.0000 \\ -0.0000 \\ -0.0000 \\ -2.0000 \\ -0.0000 \\ -2.0000 \\ -0.0000 \end{pmatrix},$$

using by Matlab computer programming. Consequently, we write

$$A_1 = [0 \quad 0 \quad 1 \quad 0]^T, \quad A_2 = [-2 \quad -2 \quad 0 \quad 0]^T, \quad A_3 = [-2 \quad 0 \quad -2 \quad 0]^T.$$

Hence, the system of the solutions is obtained as:

$$y_1(t) = t^2,$$

$$y_2(t) = -2t - 2,$$

$$y_3(t) = -2t^2 - 2.$$

Example 2:

$$\begin{cases} y_1' = -y_2(t-1) & y_1(0) = 0 \\ y_2' = 2y_1(t-2) + y_3(t-2) & y_2(0) = -2 \\ y_3' = 3y_2(t-1) & y_3(0) = -2. \end{cases}$$

Solve the system of delay differential equation by using Taylor Collocation Method for $-2 \le t \le 4$.
Solution: If we do the same process as previous example, we write

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} y_1(t-1) \\ y_2(t-1) \\ y_3(t-1) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t-2) \\ y_2(t-2) \\ y_3(t-2) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From here, we find

$$\mathbf{W} = \begin{pmatrix} 0 & 1 & -4 & 12 & 1 & -3 & 9 & -27 & 0 & 0 & 0 & 0 \\ -2 & 8 & -32 & 128 & 0 & 1 & -4 & 12 & -1 & 4 & -16 & 64 \\ 0 & 0 & 0 & 0 & -3 & 9 & -27 & 81 & 0 & 1 & -4 & 12 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -2 & 4 & -8 & 16 & 0 & 1 & 0 & 0 & -1 & 2 & -4 & 8 \\ 0 & 0 & 0 & 0 & -3 & 3 & -3 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 12 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 4 & 12 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -3 & -3 & -3 & 0 & 1 & 4 & 12 \\ 0 & 1 & 8 & 48 & 1 & 3 & 9 & 27 & 0 & 0 & 0 & 0 \\ -2 & -4 & -8 & -16 & 0 & 1 & 8 & 48 & -1 & -2 & -4 & -8 \\ 0 & 0 & 0 & 0 & -3 & -9 & -27 & -81 & 0 & 1 & 8 & 48 \end{pmatrix}.$$

By using initial conditions, we get

$$\overline{\mathbf{W}} = \begin{pmatrix} 0 & 1 & -4 & 12 & 1 & -3 & 9 & -27 & 0 & 0 & 0 & 0 \\ -2 & 8 & -32 & 128 & 0 & 1 & -4 & 12 & -1 & 4 & -16 & 64 \\ 0 & 0 & 0 & 0 & -2 & 6 & -18 & 54 & 0 & 1 & -4 & 12 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -2 & 4 & -8 & 16 & 0 & 1 & 0 & 0 & -1 & 2 & -4 & 8 \\ 0 & 0 & 0 & 0 & -2 & 2 & -2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 12 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 4 & 12 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -2 & -2 & -2 & 0 & 1 & 4 & 12 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

From here, we obtain the coefficients matrix as:

$$\mathbf{A} = \mathbf{inv}(\overline{\mathbf{W}}) * \overline{\mathbf{F}} = \begin{pmatrix} 0 \\ 0.8910 \\ 0.4265 \\ 0.0758 \\ -2.0000 \\ -1.3175 \\ -0.1991 \\ 0.0095 \\ -2.0000 \\ -2.6730 \\ -1.2796 \\ -0.2275 \end{pmatrix}.$$

Therefore the solutions for $-2 \leq t \leq 4$ and $N = 3$ are:

$$y_1(t) = 0.0758t^3 + 0.4265t^2 + 0.8910,$$

$$y_2(t) = 0.0095t^3 - 0.1991t^2 - 1.3175t - 2,$$

$$y_3(t) = -0.2275t^3 - 1.2796t^2 - 2.6730t - 2.$$

4. Conclusions

Lambert W function is very effective method to get the general solutions of system the of delay differential equations easily. It is obtained that Lambert W function is faster than Taylor Collocation Method but it is not enough to find the exact solutions. Taylor Collocation Method is more appropriate method than Lambert W function when we have the initial conditions. But in this method, as the number of collocation points are increasing, because of the long time of calculation process due to increasing dimension of the matrices, it takes more time to get the solutions. But getting more approximate result than Lambert W function shows that the Taylor Collocation Method better than the presented method.

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Sinan Deniz (sinan.deniz@cbu.edu.tr)–Department of Mathematics, Faculty of Arts and Sciences , Celal Bayar University, Muradiye Campus, 45047 Manisa, Turkey

Necdet Bildik (necdet.bildik@cbu.edu.tr)– Department of Mathematics, Faculty of Arts and Sciences , Celal Bayar University, Muradiye Campus, 45047 Manisa, Turkey