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Solutions of Nonlinear Partial Differential Equations Using Generalized Hyperbolic Functions

Yusuf Pandir and Halime Ulusoy

Abstract.
In this article, with the help of generalized hyperbolic functions a new version of the classic sech-function method is defined. The developed method is applied to the nonlinear partial differential equations and a general form of solution function called as " 1-soliton " is obtained. New exact solutions of generalized regularized long wave equation (GRLW) and the (2+1) dimensional Boussinesq equation found. Also, physical reviews of the solutions are added.

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1. INTRODUCTION

Many real-world events are represented by a mathematical model of non-linear differential equations. This situation increases the importance of differential equations and their solutions. The various mathematical methods have been developed with the aim to achieve the solutions of nonlinear partial differential equations. When the applications of these methods are made, the new solution functions and significant physical behavior is determined. To construct exact solutions to nonlinear partial differential equations, some important methods have been introduced such as Kudryashov's method, tanh-coth method, the exponential function method, Hirota method, (G'/G)-expansion method, the trial equation method, and so on [1-19]. Soliton solutions, compactons, singular solitons and other solutions have been obtained by using these methods. These types of solutions are very important and appear in various areas of applied mathematics.

In Section 2, we give the definition and properties of generalized hyperbolic functions. In Section 3, as applications, we obtain exact solutions of the generalized regularized long wave equation and the $(2 + 1)$-dimensional Boussinesq equation.

2. The basic concepts and definition of the generalized hyperbolic functions

In this section, we will define new functions which named the generalized hyperbolic functions for constructing new exact solutions of NPDEs, and then study the properties of these functions.

Definition 2.1 Suppose that ξ is an independent variable, p, q and k are all constants. The generalized hyperbolic sine function is

$$\sinh_a(\xi) = \frac{pa^{k\xi} - qa^{-k\xi}}{2}, \tag{2.1}$$

generalized hyperbolic cosine function is

$$\cosh_a(\xi) = \frac{pa^{k\xi} + qa^{-k\xi}}{2}, \tag{2.2}$$

generalized hyperbolic tangent function is

$$\tanh_a(\xi) = \frac{pa^{k\xi} - qa^{-k\xi}}{pa^{k\xi} + qa^{-k\xi}}, \tag{2.3}$$

generalized hyperbolic cotangent function is

$$\coth_a(\xi) = \frac{pa^{k\xi} + qa^{-k\xi}}{pa^{k\xi} - qa^{-k\xi}}, \tag{2.4}$$

generalized hyperbolic secant function is

$$sech_a(\xi) = \frac{2}{pa^{k\xi} + qa^{-k\xi}}, \tag{2.5}$$

generalized hyperbolic cosecant function is

$$cosech_a(\xi) = \frac{2}{pa^{k\xi} - qa^{-k\xi}}, \tag{2.6}$$

the above six kinds of functions are said generalized new hyperbolic functions. Thus we can prove the following theory of generalized hyperbolic functions on the basis of Definition 2.1.

Theorem 2.1. The generalized hyperbolic functions satisfy the following relations:

$$\cosh_a^2(\xi) - \sinh_a^2(\xi) = pq, \tag{2.7}$$

$$1 - \tanh_a^2(\xi) = pq sech_a^2(\xi), \tag{2.8}$$

$$1 - \coth_a^2(\xi) = -pq cosech_a^2(\xi), \tag{2.9}$$

$$sech_a(\xi) = \frac{1}{\cosh_a(\xi)}, \tag{2.10}$$

$$cosech_a(\xi) = \frac{1}{\sinh_a(\xi)}, \tag{2.11}$$

$$\tanh_a(\xi) = \frac{\sinh_a(\xi)}{\cosh_a(\xi)}, \tag{2.12}$$

$$\coth_a(\xi) = \frac{\cosh_a(\xi)}{\sinh_a(\xi)}. \tag{2.13}$$

The following just part of them are proved here for simplification.

Theorem 2.2. The derivative formulae of generalized hyperbolic functions as following

$$\frac{d(\sinh_a(\xi))}{d\xi} = k \ln a \ \cosh_a(\xi), \tag{2.14}$$

$$\frac{d(\cosh_a(\xi))}{d\xi} = k \ln a \ \sinh_a(\xi), \tag{2.15}$$

$$\frac{d(\tanh_a(\xi))}{d\xi} = kpq \ln a \ sech_a^2(\xi), \tag{2.16}$$

$$\frac{d(\coth_a(\xi))}{d\xi} = -kpq \ln a \ cosech_a^2(\xi), \tag{2.17}$$

$$\frac{d(sech_a(\xi))}{d\xi} = -k \ln a \ sech_a(\xi) \tanh_a(\xi), \tag{2.18}$$

$$\frac{d(cosech_a(\xi))}{d\xi} = -k \ln a \ cosech_a(\xi) \coth_a(\xi). \tag{2.19}$$

Proof of (2.16): According to (2.14) and (2.15), we can get

$$\frac{d(\tanh_a(\xi))}{d\xi} = \left(\frac{\sinh_a(\xi)}{\cosh_a(\xi)}\right)' = \frac{(\sinh_a(\xi))' \cosh_a(\xi) - (\cosh_a(\xi))' \sinh_a(\xi)}{\cosh_a^2(\xi)}$$

$$= \frac{k \ln a \ \cosh_a^2(\xi) - k \ln a \ \sinh_a^2(\xi)}{\cosh_a^2(\xi)} = kpq \ln a sech_a^2(\xi). \tag{2.20}$$

Similarly, we can prove other differential coefficient formulae in Theorem 2.2.

Remark 2.1. We see that when $p = 1$, $q = 1$, $k = 1$ and $a = e$ in (2.1)-(2.6), new generalized hyperbolic function $\sinh_a(\xi)$, $\cosh_a(\xi)$, $\tanh_a(\xi)$, $\coth_a(\xi)$, $sech_a(\xi)$ and $cosech_a(\xi)$, degenerate as hyperbolic function $\sinh(\xi)$, $\cosh(\xi)$, $\tanh(\xi)$, $\coth(\xi)$, $sech(\xi)$ and $cosech(\xi)$, respectively. In addition, when $p = 0$ or $q = 0$ in (2.1)-(2.6), $\sinh_a(\xi)$, $\cosh_a(\xi)$, $\tanh_a(\xi)$, $\coth_a(\xi)$, $sech_a(\xi)$ and $cosech_a(\xi)$, degenerate as exponential function $\frac{1}{2}pa^{k(\xi)}$, $\pm\frac{1}{2}qa^{-k(\xi)}$, $2pa^{-k(\xi)}$, $\pm 2qa^{k(\xi)}$ and ± 1, respectively.

3. Applications

Example 1. *Application to the generalized regularized long wave equation*

The generalized regularized long wave equation that will be studied in this paper are given by [20,21]

$$u_t + u_x + \delta u^m u_x - \gamma u_{xxt} = 0, \tag{3.1}$$

where m is a positive integer and δ and γ are positive constants that describe the behavior of the undular bore. This equation is very important in physics since it describes phenomena with weak nonlinearity and dispersion waves including nonlinear transverse waves in shallow water, ion acoustic and magnetohydrodynamic waves in plasma, and phonon packets in nonlinear crystals. Their solutions are kinds of solitary waves named solitons whose shapes are not affected by collision. The hypothesis for solving this equation is

$$u(x,t) = \frac{A}{\cosh_a^s(\eta)}, \tag{3.2}$$

where

$$\eta = Bx - vt. \tag{3.3}$$

Here, in (3.2) and (3.3), A represents the amplitude of the soliton while B is the inverse width of the soliton and v is the velocity of the soliton. The exponent s is unknown at this point and will be evaluated during the course of the derivation of the solutions to (3.1). From (3.2), it is possible to obtain

$$u_t = \frac{Aksv \ln a \tanh_a(\eta)}{\cosh_a^s(\eta)}, \tag{3.4}$$

$$u_x = \frac{-AksB \ln a \tanh_a(\eta)}{\cosh_a^s(\eta)}, \tag{3.5}$$

$$u^q u_x = \frac{-A^{m+1} ksB \ln a \tanh_a(\eta)}{\cosh_a^{s(m+1)}(\eta)}, \tag{3.6}$$

(3.7)

$$u_{xxt} = \frac{As^3k^3B^2v(\ln a)^3 \tanh_a(\eta)}{\cosh_a^s(\eta)} - \frac{\left(As(s+2)k^3B^2(\ln a)^2pqv + As^2(s+2)k^3B^2(\ln a)^3pqv\right)\tanh_a(\eta)}{\cosh_a^{s+2}(\eta)}.$$

These results will now be substituted in Eq. (3.1) to obtain the 1-soliton solution of the generalized regularized long wave equation. Eq. (3.1) by virtue of (3.4)-(3.7) reduces to

$$\frac{Aksv \ln a \tanh_a(\eta)}{\cosh_a^s(\eta)} - \frac{AksB \ln a \tanh_a(\eta)}{\cosh_a^s(\eta)} - \frac{\delta A^{m+1} ksB \ln a \tanh_a(\eta)}{\cosh_a^{s(m+1)}(\eta)}$$

(3.8)

$$+\frac{\left(\gamma As(s+2)k^3B^2(\ln a)^2pqv + \gamma As^2(s+2)k^3B^2(\ln a)^3pqv\right)\tanh_a(\eta)}{\cosh_a^{s+2}(\eta)} - \frac{\gamma As^3k^3B^2v(\ln a)^3 \tanh_a(\eta)}{\cosh_a^s(\eta)} = 0.$$

From (3.8), equating the exponents $s+2$ and $s(q+1)$ gives

$$s + 2 = s(m+1), \tag{3.9}$$

that leads to

$$s = \frac{2}{m}. \tag{3.10}$$

Now from (3.8), the two linearly independent functions are $1/\cosh^{s+j}$ for $j = 0, 2$. Thus setting their coefficients to zero gives

$$v = \pm \frac{A^{\frac{m}{2}} m \sqrt{\delta\,(2\delta A^m \ln a + pq(m+1)(2\ln a + m))}}{kpq(m+1)(2\ln a + m)\sqrt{2\gamma \ln a}}, \tag{3.11}$$

and

$$B = \pm \frac{A^{\frac{m}{2}} m \sqrt{\delta}}{k\sqrt{2\gamma \ln a\,(2\delta A^m \ln a + pq(m+1)(2\ln a + m))}}. \tag{3.12}$$

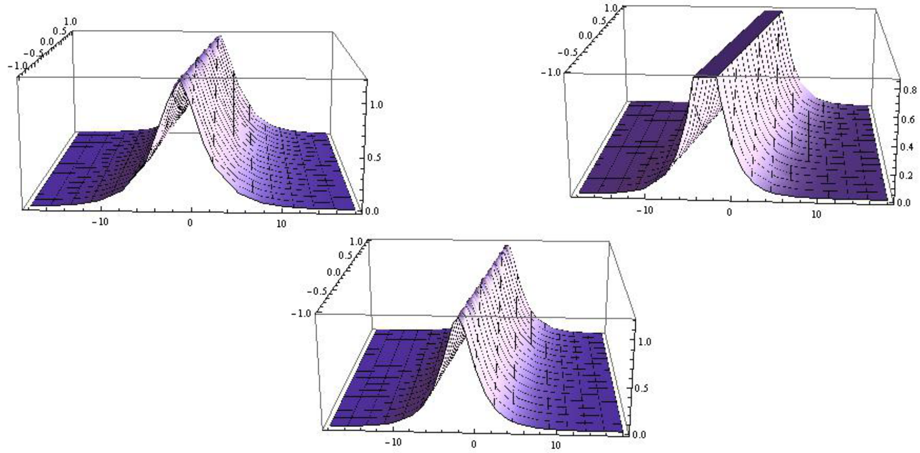

FIGURE 1. Solution of $u(x, t)$ is shown at $A = 2, \quad \delta = 5, \quad \gamma = 2, \quad k = 2, \quad p = 2, \quad q = 4, \quad m = 4.$

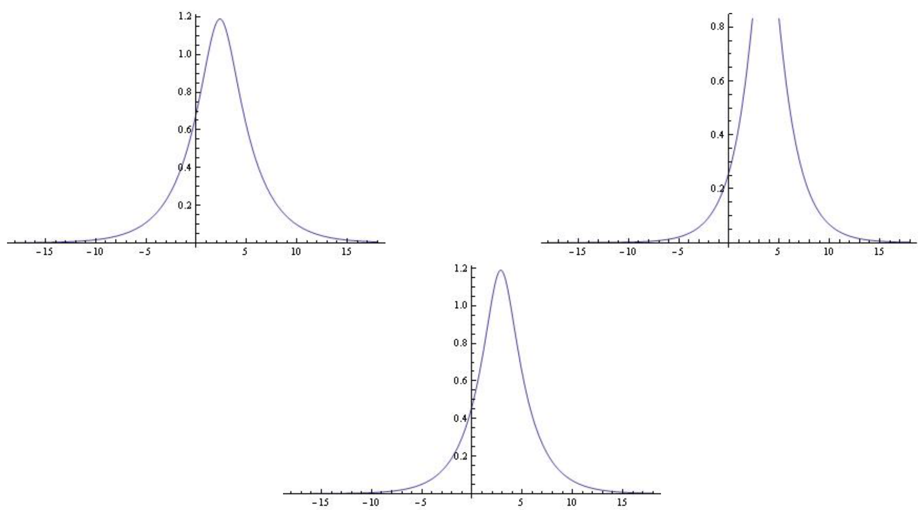

FIGURE 2. The graphs represent the exact approximate solution of Eq. (3.13) for $t = 1$.

Thus the 1-soliton solution of the generalized regularized long wave equation with generalized evolution is given by

$$u(x,t) = \frac{A}{\cosh_a^{\frac{2}{m}}[Bx - vt]}. \tag{3.13}$$

Thus Figure 1. shows the 1-soliton solution of the generalized regularized long wave equation with generalized evolution is given by the free parameters and a takes respectively, Golden Mean, e and 10.

Example 2. *Application to the (2 + 1) dimensional Boussinesq equation*

We consider the (2 + 1) dimensional Boussinesq equation, in the normalized form [22]

$$u_{tt} - u_{xx} - u_{yy} - (u^2)_{xx} - u_{xxxx} = 0. \tag{3.14}$$

The hypothesis for solving these equations is

$$u(x,t) = \frac{A}{\cosh_a^s(\eta)}, \tag{3.15}$$

where

$$\eta = B_1 x + B_2 y - vt. \tag{3.16}$$

Here in (3.15) A is the amplitude of the 1-soliton while v is the velocity of the soliton and B_1 and B_2 are the inverse widths of the solitons. The exponent s is unknown at this point and the value will fall out in the process of deriving the solution of this equation. From (3.15), it is possible to obtain

$$u_{tt} = \frac{k^2 s^2 v^2 A(\ln a)^2}{\cosh_a^s(\eta)} - \frac{k^2 s v^2 A pq(s+1)(\ln a)^2}{\cosh_a^{s+2}(\eta)}, \tag{3.17}$$

$$u_{xx} = \frac{k^2 s^2 B_1^2 A(\ln a)^2}{\cosh_a^s(\eta)} - \frac{k^2 s B_1^2 A pq(s+1)(\ln a)^2}{\cosh_a^{s+2}(\eta)}, \tag{3.18}$$

$$u_{yy} = \frac{k^2 s^2 B_2^2 A(\ln a)^2}{\cosh_a^s(\eta)} - \frac{k^2 s B_2^2 A pq(s+1)(\ln a)^2}{\cosh_a^{s+2}(\eta)},, \tag{3.19}$$

$$(u^2)_{xx} = \frac{4k^2 s^2 B_1^2 A^2(\ln a)^2}{\cosh_a^{2s}(\eta)} - \frac{2k^2 s B_1^2 A^2 pq(2s+1)(\ln a)^2}{\cosh_a^{2s+2}(\eta)}, \tag{3.20}$$

$$u_{xxxx} = \frac{k^4 s^4 A B_1^4(\ln a)^4}{\cosh_a^s(\eta)} - \frac{k^4 s A B_1^4(\ln a)^4 pq(s+1)(s+2)^2 + k^4 s^3 A B_1^4(\ln a)^4 pq(s+1)}{\cosh_a^{s+2}(\eta)}. \tag{3.21}$$

Substituting these into (3.14) yields

$$\frac{k^2 s^2 B_1^2 A(\ln a)^2}{\cosh_a^s(\eta)} - \frac{k^2 s B_1^2 A pq(s+1)(\ln a)^2}{\cosh_a^{s+2}(\eta)} - \frac{k^2 s^2 B_1^2 A(\ln a)^2}{\cosh_a^s(\eta)} + \frac{k^2 s B_1^2 A pq(s+1)(\ln a)^2}{\cosh_a^{s+2}(\eta)}$$
$$-\frac{k^2 s^2 B_2^2 A(\ln a)^2}{\cosh_a^s(\eta)} + \frac{k^2 s B_2^2 A pq(s+1)(\ln a)^2}{\cosh_a^{s+2}(\eta)} - \frac{4k^2 s^2 B_1^2 A^2(\ln a)^2}{\cosh_a^{2s}(\eta)} + \frac{2k^2 s B_1^2 A^2 pq(2s+1)(\ln a)^2}{\cosh_a^{2s+2}(\eta)}$$
$$-\frac{k^4 s^4 A B_1^4(\ln a)^4}{\cosh_a^s(\eta)} + \frac{k^4 s A B_1^4(\ln a)^4 pq(s+1)(s+2)^2 + k^4 s^3 A B_1^4(\ln a)^4 pq(s+1)}{\cosh_a^{s+2}(\eta)} = 0. \tag{3.22}$$

Now from (3.22), equating the exponents $2s + 2$ and $s + 4$ and also $2s$ and $s + 2$ gives respectively

$$2s + 2 = s + 4, \quad 2s = s + 2, \tag{3.23}$$

that leads to

$$s = 2. \tag{3.24}$$

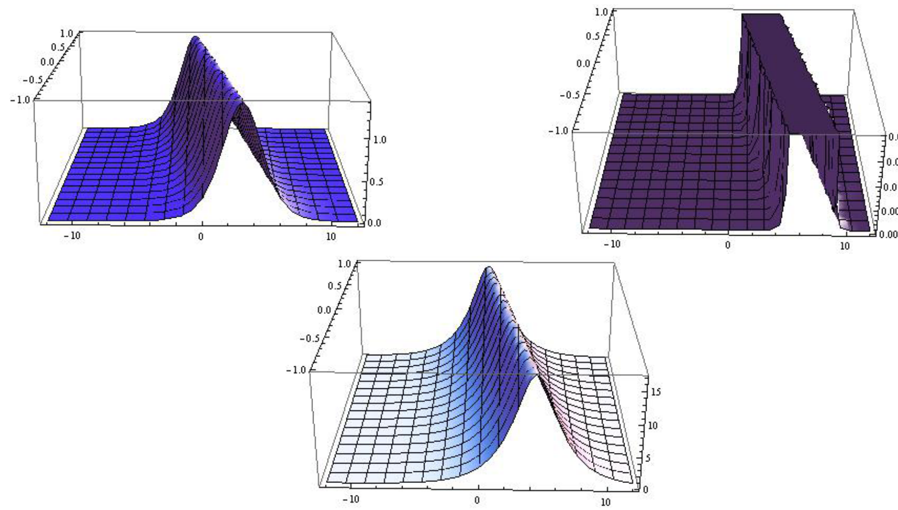

FIGURE 3. Solution of $u(x,t)$ is shown at $B_1 = 2$, $B_2 = 4$, $k = \frac{1}{2}$, $p = 2$, $q = \frac{1}{2}$.

So from (3.22) the four linearly independent functions are $1/\cosh^{s+j}$ and $1/\cosh^{2s+i}$ for $i = 0, 2$ and $j = 0, 2, 4$. Therefore, setting their respective coefficients to zero we obtain A, B_1, B_2 and v as follows:

$$A = 6k^2(\ln a)^2 pqB_1^2, \tag{3.25}$$

and

$$v = \pm\sqrt{B_1^2 + 4k^2(\ln a)^2 B_1^4 + B_2^2}. \tag{3.26}$$

Thus the 1-soliton solution of the generalized regularized long wave equation with generalized evolution is given by

$$u(x,t) = \frac{A}{\cosh_a^2[B_1 x + B_2 y - vt]}. \tag{3.27}$$

Thus Figure 3. shows the 1-soliton solution of the (2 + 1) dimensional Boussinesq equation is given by the free parameters and a takes respectively, Golden Mean, e and 10.

4. Conclusions and Remarks

In this article, a new version of the classical sech-function method is defined with the help of the generalized hyperbolic functions. The developed method is applied to the GRLW equation and the (2+1)-dimensional Boussinesq equation and more general form of solution, also known 1-soliton solution, is obtained. We devise new generalized hyperbolic functions to construct new exact solutions of nonlinear partial differential equations. The number and shape of the solitons in these solutions are related to the values of three parameters k, p, q and size of the regions of the independent variables. Our method can be also applied to other nonlinear partial differential equations.

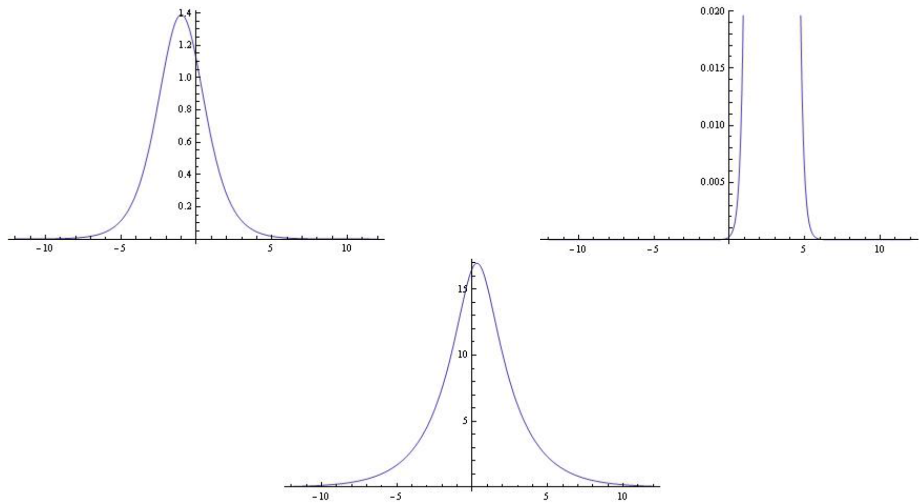

Figure 4. The graphs represent the exact approximate solution of Eq. (3.27) for $t = 1$.

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