Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (1) (2022), 228 – 234
DOI : 10.15672/hujms.892294

Research Article

Solvable groups having primitive characters of degree two

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Abstract

In this paper, we are interested in the classification of finite solvable groups having primitive characters of degree two. We first determine all finite solvable groups having both a faithful primitive character of degree two and a faithful real-valued irreducible character. Then we classify all finite solvable groups having at most five nonlinear monolithic characters one of which is primitive of degree two.

Mathematics Subject Classification (2020). 20C15

Keywords. solvable groups, primitive characters, monolithic characters, real-valued characters

1. Introduction

Throughout this paper, all groups are finite. Let G be a group and $\chi \in \mathrm{Irr}(G)$, where $\mathrm{Irr}(G)$ is the set of all irreducible characters of G. We say that χ is monolithic when $G/\ker\chi$ has a unique minimal normal subgroup. For example, all irreducible characters of each of the general linear group $\mathrm{GL}(2,3)$, the special linear group $\mathrm{SL}(2,3)$ and the conformal special unitary group $\mathrm{CSU}(2,3)$ are monolithic. The notations $\mathrm{Irr}_m(G)$ and $\mathrm{Irr}_{1,m}(G)$ are the set of all monolithic characters and all nonlinear monolithic characters of G, respectively. It is immediately seen that $\mathrm{Irr}_m(G/N) \subseteq \mathrm{Irr}_m(G)$ for every $N \trianglelefteq G$. More details of the concepts of monolithic characters can be found in Chapter 30 of [3]. It is well-known that the character χ is called primitive if χ cannot be induced from a character of a proper subgroup of G. Every linear character of G is primitive. It is also known that primitive characters are homogeneous, that is, their restrictions to every normal subgroup are multiples of an irreducible character.

There are a lot of research papers which investigate the structure of a finite group G regarding its irreducible characters, some of which are related to the cardinality of the set of all nonlinear irreducible characters of the group. For example, Seitz has shown in the old paper [8] that if G has exactly one nonlinear irreducible character, then G is an extraspecial 2-group or a Frobenius group, called Seitz Frobenius group in this paper, whose kernel is an elementary abelian group of order p^a and complement is a cyclic group of order

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Received: 06.03.2021; Accepted: 23.09.2021

$p^a - 1$ for some prime p and positive integer a. Later, Berkovich (Berger and independently Nielsen-Hansen and Palfy) has obtained a complete classification of finite groups having two nonlinear irreducible characters in Theorem 31.6 of [3]. These groups are called Berkovich groups in this paper. It is possible to see some relations between the number of nonlinear monolithic characters of a group and the group structure. For instance, if $|\mathrm{Irr}_{1,m}(G)| \leq 4$, then G is solvable, unless $G = G' \times A$, where $G' \cong \mathrm{L}_2(5)$ from Proposition 4 of [3]. Berkovich has characterized all finite groups having exactly one nonlinear monolithic character in Proposition 3 of [3]. Recently, all finite solvable real groups with at most two nonlinear monolithic characters have been classified in [4]. Also, Berkovich et al. have presented some important results related to finite groups having distinct monolithic character degrees in [1]. Motivated by these papers, we prove the following theorems:

Theorem A. *Suppose that G is a solvable group having a faithful real-valued irreducible character. If G has a faithful primitive character of degree 2, then $G \cong SL(2,3)$, $G \cong CSU(2,3)$ or $G \cong GL(2,3)$.*

Theorem B. *Let G be a solvable group with $|Irr_{1,m}(G)| = 4$. Then G has a primitive monolithic character of degree 2 if and only if $G \cong Q_8 \rtimes H$, where H is the abelian Hall $2'$-subgroup of G and $H/H \cap Z(G) \cong C_3$.*

Theorem C. *Let G be a solvable group with $|Irr_{1,m}(G)| = 5$. Assume that N is the intersection of the kernels of all nonlinear monolithic characters of G. If G has a primitive monolithic character of degree 2, then one of the following holds:*

(i) *$G/N \cong SL(2,3) \times F$, where F is a Seitz Frobenius group whose kernel is an elementary abelian 2-group.*

(ii) *$G/N \cong M \rtimes L$, where L acts Frobeniusly on M, an elementary abelian p-group for some prime p, and $|M| = |L| + 1$. Also, the Frobenius complement L has a primitive monolithic character of degree 2 and $|Irr_{1,m}(L)| = 4$.*

2. Main results

First we recall that an irreducible character χ of a group G is said to be real-valued when $\chi(g)$ is a real number for each element g of G. If all irreducible characters of G are real-valued, then G is called a real group. Similarly, if all irreducible characters of G are rational-valued, then we say that G is a rational group. Now the proof of Theorem A is the following:

Proof of Theorem A. Let ψ be a faithful real-valued irreducible character of G. By Lemma 2.27 (c) of [6], we know that $\psi_{\mathrm{Z}(G)} = \psi(1)\lambda$ for some linear character λ of $\mathrm{Z}(G)$. Then we get $|\mathrm{Z}(G)| = 2$ since λ is a faithful real-valued irreducible character of the cyclic group $\mathrm{Z}(G)$. The Fitting subgroup $\mathrm{F}(G)$ is nonabelian since $\mathrm{Z}(G)$ is the maximal abelian normal subgroup of G from Corollary 6.13 of [6]. Since $\mathrm{Z}(G) = \mathrm{Z}(\mathrm{F}(G))$, we see that $\mathrm{F}(G)$ is an extraspecial 2-group by Corollary 1.4 of [7]. Let θ be a faithful primitive character of degree 2 of G. Then $\theta_{\mathrm{F}(G)} \in \mathrm{Irr}(\mathrm{F}(G))$, and so $|\mathrm{F}(G)| = 8$. It follows that $\mathrm{F}(G) \cong Q_8$ because $\mathrm{F}(G) \ncong D_8$ by the fact that G has a faithful primitive character and $|\mathrm{Z}(G)| = 2$. By Theorem 14.23 of [6], we know that $|G| = 24, 48$ or 120. Observe that if $|G| = 120$, then $G/\mathrm{F}(G)$ is a cyclic group of order 15. This leads to the contradiction that G has an irreducible character of degree 15 by Lemma 18.1 of [7]. If $|G| = 24$, then $\mathrm{F}(G)$ is the Sylow 2-subgroup of G. Hence we get $G \cong \mathrm{SL}(2,3)$. Now assume that $|G| = 48$. Then $G/\mathrm{Z}(G) \cong \mathrm{S}_4$ since all Sylow subgroups of G are not normal. Let $P \in \mathrm{Syl}_2(G)$. Considering $\mathrm{F}(G) \cong Q_8$, we get that P is isomorphic to either generalised quaternion group Q_{16} or semidihedral group SD_{16}. Thus, we have $G \cong \mathrm{CSU}(2,3)$ and $G \cong \mathrm{GL}(2,3)$, respectively. Indeed $\mathrm{SL}(2,3)$, $\mathrm{CSU}(2,3)$ and $\mathrm{GL}(2,3)$ are the groups satisfying the hypothesis of Theorem A. □

We have the following three immediate consequences of Theorem A:

Corollary 2.1. *Let G be a solvable group. If G has a faithful real-valued primitive character of degree 2, then either $G \cong \mathrm{SL}(2,3)$ or $G \cong \mathrm{CSU}(2,3)$.*

Corollary 2.2. *Let G be a solvable real group. If G has a faithful real-valued primitive character of degree 2, then $G \cong \mathrm{CSU}(2,3)$.*

Corollary 2.3. *If G is a solvable rational group, then G has no primitive character of degree 2.*

The following lemmas provide tools for proving Theorems B and C.

Lemma 2.4. *Let G be a solvable group. Every monolithic character of G is primitive if and only if G is abelian. Also, the character having a maximal kernel among the kernels of nonlinear irreducible characters of G must be imprimitive and monolithic.*

Proof. It is clear that all irreducible characters of G are linear, and so primitive when G is abelian. Now suppose that G is nonabelian and that every monolithic character of G is primitive. Then G has at least one nonlinear irreducible character. Now let $\overline{G} := G/\mathrm{ker}\chi$, where χ is a nonlinear irreducible character of G such that $\mathrm{ker}\chi$ is a maximal kernel among the kernels of all nonlinear irreducible characters of G. Since $\overline{G}/\overline{N}$ is an abelian group for every $1 \neq \overline{N} \trianglelefteq \overline{G}$, the commutator subgroup $\overline{G}'$ is the unique minimal normal subgroup of $\overline{G}$. Thus, χ is monolithic. By Lemma 12.3 of [6], we get that $\overline{G}$ is an M-group, which contradicts with primitivity of χ. This contradiction shows that G is an abelian group if every monolithic character of G is primitive. Also, we can conclude that the character χ having a maximal kernel among the kernels of nonlinear irreducible characters of G must be imprimitive and monolithic. □

Lemma 2.5. *Let G be a solvable group. If $\theta \in \mathrm{Irr}_m(G)$ is primitive of degree 2, then $|\{\psi \in \mathrm{Irr}(G) \mid \mathrm{ker}\psi = \mathrm{ker}\theta\}| \geq 3$.*

Proof. Set $\overline{G} = G/\mathrm{ker}\theta$. If θ is real-valued, we obtain by Corollary 2.1 that $|\{\psi \in \mathrm{Irr}(G) \mid \mathrm{ker}\psi = \mathrm{ker}\theta\}| \geq 3$, as desired. Now we can assume that θ is not real-valued. Thus, the conjugate character $\overline{\theta}$ is a distinct monolithic character of degree 2 because $\mathrm{ker}\theta = \mathrm{ker}\overline{\theta}$. On the other hand, $\overline{G}$ has a unique minimal normal subgroup $\overline{M}$, which is an elementary abelian p-group for some prime p. Thus, $\mathrm{F}(\overline{G})$ is a nonabelian p-group since θ is a faithful primitive character of $\overline{G}$. So we get that $\theta_{\mathrm{F}(\overline{G})} \in \mathrm{Irr}(\mathrm{F}(\overline{G}))$, which leads to the fact that $\mathrm{F}(\overline{G})$ is a 2-group and $|\overline{M}| = 2$. By observing that an irreducible character χ of $\overline{G}$ is faithful if and only if $\overline{M} \nleq \mathrm{ker}\chi$, we may write the following fundamental equalities that

$$|\overline{G}| = |\overline{G} : \overline{G}'| + \sum_{\substack{\chi \in \mathrm{Irr}_1(\overline{G}) \\ \overline{M} \leq \mathrm{ker}\chi}} \chi(1)^2 + \sum_{\substack{\chi \in \mathrm{Irr}_1(\overline{G}) \\ \overline{M} \nleq \mathrm{ker}\chi}} \chi(1)^2, \text{ and} \tag{2.1}$$

$$|\overline{G} : \overline{M}| = |\overline{G} : \overline{G}'| + \sum_{\substack{\chi \in \mathrm{Irr}_1(\overline{G}) \\ \overline{M} \leq \mathrm{ker}\chi}} \chi(1)^2, \tag{2.2}$$

where $\mathrm{Irr}_1(\overline{G})$ is the set of all nonlinear irreducible characters of $\overline{G}$. Hence from the Equalities 2.1 and 2.2, we obtain that

$$\frac{|\overline{G}|}{2} = \sum_{\substack{\chi \in \mathrm{Irr}_1(\overline{G}) \\ \overline{M} \nleq \mathrm{ker}\chi}} \chi(1)^2.$$

If θ and $\overline{\theta}$ are the only faithful irreducible characters of $\overline{G}$, we get the contradiction that

$|\overline{G}| = 16$. Then $\overline{G}$ has at least one more faithful irreducible character, which means that $|\{\psi \in \mathrm{Irr}(G) \mid \ker\psi = \ker\theta\}| \geq 3$. □

Corollary 2.6. *Let G be a solvable group. If G has a primitive monolithic character of degree 2, then $|\mathrm{Irr}_{1,m}(G)| \geq 4$.*

Proof. Let θ be a primitive monolithic character of degree 2. By Lemma 2.5, we see that $|\{\psi \in \mathrm{Irr}(G) \mid \ker\psi = \ker\theta\}| \geq 3$, which implies that $|\mathrm{Irr}_{1,m}(G)| \geq 3$. We also know by Lemma 2.4 that G has at least one nonlinear imprimitive and monolithic character whose kernel is maximal among the kernels of nonlinear irreducible characters of G. Thus, we get the desired result $|\mathrm{Irr}_{1,m}(G)| \geq 4$. □

Lemma 2.7. *Let G be a solvable group and $\theta \in \mathrm{Irr}(G)$ primitive monolithic character of degree 2. If $|\mathrm{Irr}_{1,m}(G)| = 4$, then $G/\ker\theta \cong \mathrm{SL}(2,3)$.*

Proof. Let $\overline{G} = G/\ker\theta$. Since θ is a primitive monolithic character of $\overline{G}$, we have $|\mathrm{Irr}_{1,m}(\overline{G})| = 4$ by Corollary 2.6. Because of the fact that an irreducible character having a maximal kernel among the nonlinear irreducible characters of G is imprimitive and monolithic, we conclude by Lemma 2.5 that $\overline{G}$ has three faithful irreducible characters, at least one of which must be real-valued. It follows from Theorem A that $\overline{G} = G/\ker\theta \cong \mathrm{SL}(2,3)$. □

Proof of Theorem B. Let $\theta \in \mathrm{Irr}_m(G)$ be primitive of degree 2. By Lemma 2.7, we know that $G/\ker\theta \cong \mathrm{SL}(2,3)$. Now, suppose that $|\ker\theta| > 1$. Since $\ker\theta$ is the intersection of kernels of all nonlinear monolithic characters of G, we get from Lemma 2 in [2] that $\ker\theta \leq \mathrm{Z}(G)$. Also, note that $\ker\theta \cap G' = 1$ since the intersection of kernels of all monolithic characters of a group is trivial. This leads the fact that $G' \cong (G/\ker\theta)' \cong Q_8$. Now suppose that $2 \mid |\ker\theta|$. Then there exists a chief factor of G such that $|\ker\theta : L| = 2$. The Sylow 2-subgroup of G/L must be equal to the Fitting subgroup $\mathrm{F}(G/L)$. Since G/L has no faithful monolithic character, we obtain that the center $\mathrm{Z}(G/L)$ is an elementary abelian group of order 4. Therefore, G/L has three different minimal normal subgroups of order 2, one of which is $\ker\theta/L$. Say K/L and M/L for other minimal normal subgroups of G/L. Since $\ker\theta \cap G'L = L$ and $(G/\ker\theta)' \cong Q_8$, we obtain that $G'L/L \cong Q_8$, and hence G/L has six linear characters. On the other hand, the factor groups both G/K and G/M are isomorphic to $\mathrm{C}_2 \times \mathrm{A}_4$ because G/L has exactly four nonlinear monolithic characters. It follows that G/L has two nonlinear non-monolithic irreducible characters of degree 3, which leads to a contradiction that

$$48 = |G/L| = 6 + 3 \cdot 2^2 + 3 \cdot 3^2 = 45.$$

This contradiction shows that $2 \nmid |\ker\theta|$. Thus, a Hall $2'$-subgroup of G is abelian and $G' = P \cong Q_8$, where $P \in \mathrm{Syl}_2(G)$. Therefore, we have that $G = P \rtimes H$ and $H/H \cap \mathrm{Z}(G) \cong \mathrm{C}_3$, where H is the Hall $2'$-subgroup of G.

Conversely, let $G = P \rtimes H$ be a group as in the theorem, where $P \in \mathrm{Syl}_2(G)$. Since $G' = P \nleq \ker\psi$ for every nonlinear monolithic character ψ of G, the unique minimal normal subgroup of $G/\ker\psi$ is an elementary abelian 2-group. It follows that $H \cap \mathrm{Z}(G) \leq \ker\psi$, and hence $\psi \in \mathrm{Irr}(G/H \cap \mathrm{Z}(G))$. Since $G/H \cap \mathrm{Z}(G) \cong \mathrm{SL}(2,3)$, then we conclude that $|\mathrm{Irr}_{1,m}(G)| = 4$ and G has a primitive monolithic character of degree 2. □

We can give some examples of this type groups in Theorem B as $\mathrm{C}_{15} \times \mathrm{SL}(2,3)$, $\mathrm{C}_3^2 \times \mathrm{SL}(2,3)$ or $\mathrm{C}_9 \times \mathrm{SL}(2,3)$, where C_n is cyclic group of order n.

Before the proof of Theorem C, we note that if G is one of the (a), (b), (c) and (d) in Theorem 31.6 of [3], then it is immediately seen that every nonlinear irreducible character of G is monolithic. But if G is a group as (e) in Theorem 31.6 of [3], then G has a nonlinear faithful character which is non-monolithic because $\mathrm{Z}(G) \cap G' = 1$. We also note that a

Berkovich group G has only one nonlinear faithful irreducible character if and only if G is either as the case (c) or the case (e) in Theorem 31.6 of [3].

Proof of Theorem C. Let $\theta \in \mathrm{Irr}_m(G)$ be primitive of degree 2 and $\widehat{G} = G/\ker\theta$. We first assume that $|\mathrm{Irr}_{1,m}(\widehat{G})| = 5$. By Lemma 2.5, we know that $\widehat{G}$ has at least three faithful monolithic characters. If $\widehat{G}$ has exactly three faithful irreducible characters, then one of them must be real-valued, which is a contradiction by Theorem A. Therefore, $\widehat{G}$ has four faithful monolithic characters, which are not real-valued. Now let $\widehat{M}$ be the unique minimal normal subgroup of $\widehat{G}$. Since $\theta \in \mathrm{Irr}_m(\widehat{G})$ is faithful and primitive of degree 2, we know that the Fitting subgroup $\mathrm{F}(\widehat{G})$ is a nonabelian 2-group and $|\widehat{M}| = 2$. Also, 2 divides the degree of every faithful irreducible character of $\widehat{G}$ by Clifford's Theorem. Suppose that $\theta, \overline{\theta}, \psi$ and $\overline{\psi}$ are all faithful irreducible characters of $\widehat{G}$. By using Equalities 2.1 and 2.2, we obtain that

$$\frac{|\widehat{G}|}{2} = \theta(1)^2 + \overline{\theta}(1)^2 + \psi(1)^2 + \overline{\psi}(1)^2 = 4 + 4 + 4t^2 + 4t^2 = 8(1+t^2),$$

where $\psi(1) = 2t$ for some positive integer t. Thus, we find that $3 \nmid |\widehat{G}|$, which is a contradiction by Theorem 14.23 in [6]. Therefore, we get $|\mathrm{Irr}_{1,m}(\widehat{G})| = 4$. It follows from Lemma 2.7 that $\widehat{G} \cong \mathrm{SL}(2,3)$. Now let $\chi \in \mathrm{Irr}(\widehat{G})$, $\chi(1) = 3$ and let $\varphi \in \mathrm{Irr}_{1,m}(G)$ such that $\varphi \notin \mathrm{Irr}(\widehat{G})$. We first consider the case in which $\ker\varphi$ is a maximal kernel among the kernels of all nonlinear irreducible characters of G. Then $|G : \ker\chi\ker\varphi| = 1$ or 3 since $\ker\varphi \neq \ker\chi$ and $G/\ker\chi \cong \mathrm{A}_4$. Let $T := \ker\chi \cap \ker\varphi$. Now assume that $|G : \ker\chi\ker\varphi| = 3$. Then $\ker\chi\ker\varphi/T \in \mathrm{Syl}_2(G/T)$ is an elementary abelian normal subgroup of order 16. Since $\mathrm{Z}(G/T) = 1$, we obtain that G/T is a Frobenius group with $\ker\chi\ker\varphi/T$ the Frobenius kernel. Then G/T has five nonlinear monolithic characters of degree 3, which contradicts with $|\mathrm{Irr}_{1,m}(G)| = 5$. Therefore, we may assume that $|G : \ker\chi\ker\varphi| = 1$. It follows that $G/T \cong \mathrm{A}_4 \times S$, where S, called a Seitz group, has only one nonlinear irreducible character. Thus, we have that G/T has only two nonlinear monolithic characters. Since $|\ker\chi : \ker\theta| = 2$, we see that $T\ker\theta = \ker\chi$. It follows that $G/\ker\varphi \cong \ker\chi/T \cong \ker\theta/N$, where $N = \ker\theta \cap \ker\varphi$. Then $\ker\theta/N$ is a Seitz group because $G/\ker\varphi$ is a Seitz group. Thus, there exists a normal subgroup K of G such that $N < K < \ker\theta$. Note that G/K has four nonlinear monolithic characters, which leads to the fact that the order of $\ker\theta/K$ is odd from Theorem B. Therefore, the Seitz group $G/\ker\varphi$ cannot be an extraspecial 2-group. Then again by Theorem B, the Frobenius kernel of the Seitz group $G/\ker\varphi$ is an elementary abelian 2-group. By considering $\ker\varphi/N \cong \mathrm{SL}(2,3)$, we obtain that $G/N \cong \ker\varphi/N \times \ker\theta/N \cong \mathrm{SL}(2,3) \times F$, where F is a Seitz Frobenius group whose kernel is an elementary abelian 2-group, as desired result in case (i) of the theorem.

It remains to consider the case that $\ker\varphi$ is not a maximal kernel among the kernels of all nonlinear irreducible characters of G. Then we have that $\ker\varphi < \ker\chi$. Let $\overline{G} = G/\ker\varphi$ and $\ker\theta \cap \ker\varphi = L$. We first show that $\ker\varphi < \ker\theta$. We have already that $\ker\varphi \neq \ker\theta$. Now suppose that $\ker\varphi \not< \ker\theta$. We know that $\overline{G}$ cannot be a Berkovich group by the last paragraph given before the proof of Theorem C. Thus, $\ker\chi/\ker\varphi$ is not a chief factor of $\overline{G}$. Let $\overline{M} := M/\ker\varphi$ be a chief factor of $\overline{G}$ such that $M < \ker\chi$, that is, $\overline{M}$ is the unique minimal normal subgroup of $\overline{G}$. Thus, there exists $R \trianglelefteq G$ such that $L < R < \ker\theta$ and $\ker\chi/M \cong \ker\theta/R$. Since G/R has four nonlinear monolithic characters, we see by Theorem B that $2 \nmid |\ker\theta/R|$, and hence $2 \nmid |\ker\chi/M|$. Since $\overline{G}$ is not a nilpotent group it is well-known that $\overline{G}/\Phi(\overline{G})$ is nonabelian, and so $\Phi(\overline{G}) \leq \ker\chi/\ker\varphi$. Assume that the Frattini subgroup $\Phi(\overline{G}) > 1$. Since $\overline{M} \leq \Phi(\overline{G})$ and $\ker\chi/M$ is an abelian group, then $\ker\chi/\ker\varphi$ is a nilpotent group. Therefore, we obtain that $\ker\chi/\ker\varphi = \mathrm{F}(\overline{G})$ is a

p-group, where p is an odd prime. By the fact that $\mathrm{F}(\overline{G}/\Phi(\overline{G})) = \mathrm{F}(\overline{G})/\Phi(\overline{G})$, we get a contradiction that $2 \nmid |\mathrm{F}(\overline{G}/\Phi(\overline{G}))|$. This contradiction shows that $\Phi(\overline{G}) = 1$, and so $\mathrm{F}(\overline{G})$ is an elementary abelian p-group. Then there exists a maximal subgroup $\overline{H}$ of $\overline{G}$ such that $\overline{M} \not\leq \overline{H}$, which gives us $\overline{G} = \overline{M} \cdot \overline{H}$. Also, $\overline{M}$ is the unique minimal normal subgroup of $\overline{G}$ and $\overline{M} \cap \overline{H} \trianglelefteq \overline{G}$, we obtain that $\overline{G} = \overline{M} \rtimes \overline{H}$. Similarly, $\mathrm{F}(\overline{G}) \cap \overline{H} \trianglelefteq \overline{G}$ since $\overline{G} = \mathrm{F}(\overline{G}) \cdot \overline{H}$ and $\mathrm{F}(\overline{G})$ is an abelian group. It follows that $\mathrm{F}(\overline{G}) = \overline{M}$. This leads to the fact that $2 \nmid |\overline{M}|$ because $\mathrm{F}(G/M) \cong V/M \times \ker\chi/M$, where $V/M \cong \mathrm{C}_2^2$. Then the Sylow 2-subgroup of $\overline{G}$ is isomorphic to C_2^2. But we know that φ is the unique irreducible character whose kernel does not contain the unique minimal normal subgroup $\overline{M}$. This contradicts with Corollary 4.5(f) of [1]. This contradiction shows that $\ker\varphi < \ker\theta$.

We know that $\ker\chi/\ker\varphi =: \overline{\ker\chi}$ is the unique maximal kernel among the kernels of all nonlinear irreducible characters of $\overline{G} = G/\ker\varphi$. Then $\Phi(\overline{G}) \leq \overline{\ker\chi}$ since $\overline{G}$ is not nilpotent. Suppose that $\Phi(\overline{G}) > 1$. Then $\overline{\ker\chi}$ is a nilpotent group because $\overline{\ker\chi}/\Phi(\overline{G})$ is nilpotent. By considering the fact that φ is a monolithic character of $\overline{G}$ and $|\ker\chi/\ker\theta| = 2$, then we deduce that $\overline{\ker\chi}$ must be a 2-group. From Theorem B, we see that $\overline{U} := \ker\theta/\ker\varphi$ is the unique minimal normal subgroup of $\overline{G}$. Since φ is the unique faithful character of $\overline{G}$, by Clifford's Theorem and Frobenius Reciprocity we obtain that $\varphi(1) = e(|\overline{U}| - 1)$, where e is a positive integer. Since $\overline{U}$ is a 2-group we get a contradiction that $24 = |\overline{G} : \overline{U}| = e\varphi(1) = e^2(|\overline{U}| - 1)$. Therefore, $\Phi(\overline{G})$ must be trivial, and hence $\mathrm{F}(\overline{G})$ is the unique minimal normal subgroup of $\overline{G}$. It follows that $(|\overline{\ker\chi}/\mathrm{F}(\overline{G})|, |\mathrm{F}(\overline{G})|) = 1$ since $\overline{\ker\chi}/\mathrm{F}(\overline{G})$ is an abelian group by Theorem B. Now assume that $3 \,|\, |\mathrm{F}(\overline{G})|$. Then $3 \nmid |\overline{\ker\chi}/\mathrm{F}(\overline{G})|$ since $(|\overline{\ker\chi}/\mathrm{F}(\overline{G})|, |\mathrm{F}(\overline{G})|) = 1$. Thus, we obtain that a Sylow 3-subgroup of $\overline{G}/\mathrm{F}(\overline{G})$ is a cyclic group of order 3, which is a contradiction by Corollary 4.5(c) of [1]. This contradiction says that $(|\overline{G}/\mathrm{F}(\overline{G})|, |\mathrm{F}(\overline{G})|) = 1$. Remember that $\mathrm{F}(\overline{G})$ is the unique minimal normal subgroup of $\overline{G}$. Thus, $\mathrm{F}(\overline{G})$ is an elemantary abelian p-group for some prime p. By Corollary 4.5(f) of [1], we obtain that $\overline{H}$ acts Frobeniusly on $\mathrm{F}(\overline{G})$, where $\overline{H}$ is a Hall p'-subgroup of $\overline{G}$. That is, $\overline{G} = \mathrm{F}(\overline{G}) \rtimes \overline{H}$ is a Frobenius group and $|\mathrm{Irr}_{1,m}(\overline{H})| = 4$. Also, $|\mathrm{F}(\overline{G})| - 1 = |\overline{H}|$ since φ is the unique faithful irreducible character of $\overline{G}$.

Observe that a group G as in the case (a) or (b) of Theorem C has five nonlinear monolithic irreducible characters, some of which are primitive of degree 2. □

We recall that if G/N is a supersolvable group for $N \trianglelefteq G$, then G is a relative M-group with respect to N by Theorem 6.22 of [6]. This is why if θ is a primitive character of G, then $\theta_N \in \mathrm{Irr}(N)$. We also conclude that there are no primitive characters of even degree of a finite group with a normal 2-complement.

Corollary 2.8. *Let G be a solvable group having five nonlinear irreducible characters. If G has a primitive character of degree* 2*, then $G \cong N \rtimes \mathrm{SL}(2,3)$, where* $\mathrm{SL}(2,3)$ *acts Frobeniusly on $N = \mathrm{E}(5^2)$.*

Proof. Suppose that $\theta \in \mathrm{Irr}(G)$ is primitive of degree 2. Then G has no normal 2-complement, and so it has at least one nonlinear irreducible character χ of odd degree by Corollary (Thompson) 12.2 of [6]. Thus, we obtain by Corollary (Gallagher) 6.17 of [6] that $|G : G'| \leq 4$ because G has exactly five nonlinear irreducible characters. If $|G : G'| = 4$, then $\chi_{G'} \in \mathrm{Irr}(G')$ by Corollary 11.29 of [6], which gives a contradiction $|\mathrm{Irr}_1(G)| \geq 8$ by Corollary 6.17 of [6]. Therefore, $|G : G'| = 2$ or 3, so G has at least two primitive characters of degree 2. By Lemma 2.4, we know that G has at least one imprimitive character. If G has only one imprimitive character, we get a contradiction by Theorem A of [5] that $G \cong \mathrm{SL}(2,3)$. Assume that G has exactly two imprimitive characters. By considering the fact that only the group as in the case (i) of Theorem B in [5] has five nonlinear irreducible characters, we obtain the desired result that $G \cong N \rtimes \mathrm{SL}(2,3)$, where $\mathrm{SL}(2,3)$ acts Frobeniusly on $N = \mathrm{E}(5^2)$. Now, we can assume that G has three imprimitive characters and two nonlinear primitive characters θ and $\overline{\theta}$, because θ is not real-valued by

Corollary 2.1. Also, note that $|G : G'| = 2$, and so G has at least two imprimitive characters of odd degree. Now suppose that $\ker\theta > 1$. Since the intersection of kernels of all nonlinear irreducible characters of G is trivial, we conclude that $|\mathrm{Irr}_1(G/\ker\theta)| \leq 4$. Then $G/\ker\theta$ has one or two imprimitive characters. This gives a contradiction by Theorems A and B of [5]. Thus, we obtain that $\ker\theta = 1$. Since G has a faithful primitive character, we know that $\mathrm{Z}(G)$ is the unique maximal abelian normal subgroup of G by Corollary 6.13 of [6], and hence $\mathrm{F}(G)$ is nonabelian. By Corollary 16.3(d) of [7], we have that $|\mathrm{F}(G)/\mathrm{Z}(G)| = 4$. Thus we obtain that $G/\mathrm{F}(G) \cong \mathrm{S}_3$ by Theorem 14.23 of [6]. Since $\theta_{\mathrm{F}(G)} \in \mathrm{Irr}(\mathrm{F}(G))$, we have by Corollary 6.17 of [6] that G has four irreducible characters of even degree, which is a contradiction. □

Acknowledgment. The work of the authors was supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK). The project number is 119F295.

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