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Fractional approach for multi-dimensional wave-like equations with variable coefficient using an efficient method

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Abstract

In this paper, we study multi-dimensional wave-like equations with variable coefficients within the frame of the fractional calculus using fractional natural decomposition method (FNDM). The considered algorithm is an elegant combination of natural transform and decomposition scheme. Five different cases are considered to illustrate and validate the competence of the projected technique in the present framework. The behaviours of the obtained results have been captured for diverse fractional order. To present the reliability and exactness of the FNDM, the numerical study has been presented. The achieved consequences illuminate that, the projected technique is very effective to analyse and easy to employ to investigate the nature of fractional nonlinear coupled system exemplifying the real-world problems.

Keywords: *Fractional wave-like equations, Caputo derivative, fractional natural decomposition method.*

Değişken katsayılı çok boyutlu dalga benzeri denklemler için kesirli yaklaşım üzerine etkili bir metot

Öz

Bu çalışmanın temel amacı, fraksiyonel doğal ayrıştırma yöntemini (FNDM) kullanarak kesirli operatör çerçevesinde değişken katsayılı çok boyutlu dalga benzeri denklemleri incelemektir. Değerlendirilen algoritma, doğal dönüşüm ve ayrıştırma şemasının güzel bir kombinasyonudur. Mevcut çerçevede öngörülen tekniğin yeterliliğini göstermek ve doğrulamak için beş farklı durum ele alınmıştır. Elde edilen sonuçların davranışları, çeşitli kesirli sıralar için değerlendirilmiştir. FNDM'nin güvenilirliğini ve kesinliğini göstermek için sayısal çalışma sunulmuştur. Elde edilen sonuçlar, öngörülen tekniğin

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analiz edilmesinin çok etkili olduğunu ve gerçek dünya problemlerini örnekleyen kesirli doğrusal olmayan bağlı sistemin doğasını araştırmak için kullanılmasının kolay olduğunu göstermektedir.

Anahtar kelimeler: *Kesirli dalga benzeri denklemler, Caputo türevi, kesirli doğal ayrıştırma yöntemi.*

1. Introduction

The pivotal aim of phenomena is describing using the concept of differentiation and integration is to exemplify the corresponding consequences related to the rate of change in an accurate and interesting manner. Particularly, while analysing processes with variation, changes, chaotic, epidemiology and randomness; these operators are effectively illustrated with supporting tools like software and mathematical algorithms to study the corresponding results. The achieving result of these tools related to nonlinear phenomena is the hot topic in the present era. However, when researchers seeking efficient and methodical tools to study long-range time-based problems, hereditary based models and history related systems, and as they proved classical concept is not suitable and later suggested the novel concept to overcome the limitations, called fractional calculus (FC). The most stimulating leaps in scientific and technological significance are found within FC from the last thirty years.

The models with fractional order enlarge our perceptions of differentiability and amalgamate system memory and non-local properties through distinct class derivatives of FC. These constituents aid to illustrate the diverse problems through various spaces and time scales without segregate the systems into reduced components. The derivatives with fractional order capture or can limn significant features of nonlinear models. Further, the fractional operators are local in nature and which gives by employing integer order operators we can portray the vicissitudes in the neighbourhood of a point but applying non-integer operators one can analyse the variations in a spell [1-6]. These possessions make derivatives with fractional order appropriate to model various phenomena such as signal process, optics, financial models, chaos behaviour, image processing, human diseases and many others [7-25]. The numerical and analytical results for these models play a significant role in illustrating the nature of complex problems. Hyperbolic equations are amongst the most exigent to analyse due to sharp features in their solutions. Recently, many authors investigated many mathematical models exemplifying real world problems with the help of fractional calculus, for instance the mathematical model of HIV-1 infection of CD4+T-cells with conformable fractional operator is investigated in [26], the model arising in falling film problems is numerically studied by authors in [27] using two novel techniques, fractional SIR epidemic model of childhood disease examined with the help of Mittag-Leffler memory in [28], some interesting results integro-differential equation of fractional order ith state dependent delay are derived in [29], nondensely characterized integro-differential equations are studied by authors in [30], the numerical stimulation is presented by researchers in [31] for the coupled fractional Whitham-Broer-Kaup equations describing the propagation shallow water waves, by using Hilfer fractional derivative scholars in [32] derived some stimulating results, by using two different algorithms authors in [33] find the solution for Fokas–Lenells equation, authors in [34] presented the numerical stimulation for the nonlinear equations occurs in ion acoustic waves in plasma with Mittag-Leffler law, and many research considered FC has tool with its fundamental notions and results to derive some essential and stimulating results [35-36].

The study of wave-like equations is very essential in understanding and capturing the various complex phenomena arisen in connected areas of physics. For instance, the vibrations of acoustic or string waves in a pipe and the velocity of the wave are evaluated by the physical properties of the material through which it propagates. These equations are also employed to study and analyse fluid migration at depth in the crust through fluid-saturated porous media [36], coupling currents in a flat two-layere multi-strand cable having superconductivity [37], elastic waves in soils with non-homogeneous [38] and earthquake stresses [39]. By the aid of single wave-like equation with time-dependent in the flat cables associated with the magnetic field, the gradually decaying long current loops with the additional established effects are illustrated [37]. Here, we hire the time-fractional wave-like equation [40-43]:

$$D_t^{\mu} v(x,y,z,t) = f(x,y,z)v_{xx} + g(x,y,z)v_{yy} + h(x,y,z)v_{zz}, 1 < \mu \leq 2, \quad t > 0, \tag{1}$$

respectively associate with the Neumann boundary conditions and initial conditions

$$\begin{aligned} v_x(0,y,z,t) &= f_1(y,z,t), & v_x(a,y,z,t) &= f_2(y,z,t), \\ v_y(x,0,z,t) &= g_1(x,z,t), & v_y(x,b,z,t) &= g_2(x,z,t), \\ v_z(x,y,0,t) &= h_1(x,y,t), & v_z(x,y,c,t) &= h_2(x,y,t), \end{aligned} \tag{2}$$

And

$$v(x,y,z,0) = \chi(x,y,z), \qquad v_t(x,y,z,0) = \psi(x,y,z). \tag{3}$$

Although the modelling the nonlinear and complex real-world problems is a difficult job, finding the solution for the corresponding systems of equations is very hard in order to analyse the corresponding behaviour. Since we can solve some linear equations without the essence of composite tools but when we want to study highly nonlinear phenomena, we should have an efficient and accurate method to find the solution. There are numerous algorithms in this connection, but each technique has its own equipment and limitations. For instance, some of the essentials of the method to convert nonlinear to linear and partial to ordinary, some schemes requires additional polynomials to evaluate, some methods requires more time for evaluations, few of the has complicated solution procedure and others. In this connection, the Adomian decomposition method (ADM) was proposed by George Adomian in 1984 with the help of Adomian polynomials [44]. Along with the arbitrary external parameter, Adomian polynomials are generalizing to a Maclaurin series and offers rapid convergence in the obtained results. Soon after, it has been widely employed by many physicists and mathematicians to find the solution for numerous classes of models in order to analyse corresponding consequences and capture them with stimulating results. However, scholars always look forward to the development of new tools to examine phenomena with methodical and accurate solution procedure by reducing the computational level.

Some issues and limitations are pointed by authors about ADM related to huge computation and time taken for evaluation and others. Later, Rawashdeh and Maitama nurtured new modified scheme to overcome these limitations, called fractional natural decomposition method (FNDM) by the help of natural transform [45] with ADM to solve fractional differential equation [46, 47]. From lost two years, FNDM is significantly and efficiently employed to the diverse class of real-world models [48-52].

2. Preliminaries

In this segment, we presented some basic and essentials notions of FC [1-6].

Definition 1. The integral of a function$f(t) \in C_\delta (\delta \geq -1)$ with respect to fractional Riemann-Liouville is presented as [1-6]

$$J^\alpha f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\vartheta)^{\mu-1} f(\vartheta) d\vartheta. \tag{4}$$

Definition 2. The Caputo fractional derivative of$f \in C_{-1}^n$ is presented as [1-6]

$$D_t^\alpha f(t) = \begin{cases} \dfrac{d^n f(t)}{dt^n}, & \alpha = n \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\vartheta)^{n-\alpha-1} f^{(n)}(\vartheta) d\vartheta, & n-1 < \alpha < n, n \in \mathbb{N}. \end{cases} \tag{5}$$

Definition 3. For the one-parameter, the Mittag-Leffler type function is presented [54] as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \ \alpha > 0, z \in \mathbb{C}. \tag{6}$$

Definition 4. For the function $f(t)$, the natural transform (NT) is denoted by $\mathbb{N}[f(t)]$ for $t \in \mathbb{R}$ and presented with the NT variables s and ω by [55]

$$\mathbb{N}[f(t)] = R(s,\omega) = \int_{-\infty}^{\infty} e^{-st} f(\omega t) dt; \quad s,\omega \in (-\infty,\infty).$$

Now, using Heaviside function $H(t)$ we define the NT as

$$\mathbb{N}[f(t)H(t)] = \mathbb{N}^+[f(t)] = R^+(s,\omega) = \int_0^\infty e^{-st} f(\omega t) dt; \ s,\omega \in (0,\infty) \text{and} t \in \mathbb{R}. \tag{7}$$

Further, for $\omega = 1$, Equation (7) is reduces to the Laplace transform, for $s = 1$ and the Equation (7) represents the Sumudu transform.

Theorem 1 [55]**:** The NT $R_\alpha(s,\omega)$ of the fractional derivative of $f(t)$ Riemann-Liouville sense is symbolized by $D^\alpha f(t)$ and defined as

$$\mathbb{N}^+[D^\alpha f(t)] = R_\alpha(s,\omega) = \frac{s^\alpha}{\omega^\alpha} R(s,\omega) - \sum_{k=0}^{n-1} \frac{s^k}{\omega^{\alpha-k}} [D^{\alpha-k-1} f(t)]_{t=0}, \tag{8}$$

where$R(s,\omega)$is NT of $f(t)$, α is the order and n be any positive integer. Further$n - 1 \leq \alpha < n$.

Theorem 2 [55]**:** The natural transform $R_\alpha(s,\omega)$ of the fractional derivative in Caputo sense of $f(t)$ is symbolize by${}^c D^\alpha f(t)$ and defined as

$$\mathbb{N}^+[{}^cD^\alpha f(t)] = R_\alpha^c(s,\omega) = \frac{s^\alpha}{\omega^\alpha} R(s,\omega) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{\omega^{\alpha-k}} [D^k f(t)]_{t=0}. \tag{9}$$

3. Fundamental solution procedure of the proposed algorithm

In this section, we hired coupled equations to present the basic procedure of the projected scheme with initial conditions

$$D_t^\alpha v(x,t) + Rv(x,t) + Fv(x,t) = h(x,t), \tag{10}$$

and

$$v(x,0) = g(x), \tag{11}$$

where $D^\alpha v(x,t)$ signifies the fractional Caputo derivative of $v(x,t)$, $h(x,t)$ is the source term, R and F respectively the linear and nonlinear differential operator. On plugging *NT* and by the assist of Theorem 2, then Equation (10) gives

$$\begin{aligned} V(x,s,\omega) = \frac{v^\alpha}{s^\alpha} \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{\omega^{\alpha-k}} [D^k v(x,t)]_{t=0} + \frac{\omega^\alpha}{s^\alpha} \mathbb{N}^+[h(x,t)] \\ - \frac{\omega^\alpha}{s^\alpha} \mathbb{N}^+[R\, v(x,,t) + Fv(x,t)]. \end{aligned} \tag{12}$$

On employing inverse NT on Equation (12) to get

$$v(x,t) = H(x,t) - \mathbb{N}^{-1}\left[\frac{\omega^\alpha}{s^\alpha} \mathbb{N}^+[Rv(x,t) + F\, v(x,t)]\right]. \tag{13}$$

From non-homogeneous terms and given initial condition, $H(x,t)$ are exists. The infinite series solution is presented as

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t), \qquad F\, v(x,t) = \sum_{n=0}^{\infty} A_n, \tag{14}$$

where the A_n is indicating the nonlinear terms of $Fv(x,t)$. By using the Equations (13) and (14), we have

$$\sum_{n=0}^{\infty} v_n(x,t) = H(x,t) - \mathbb{N}^{-1}\left[\frac{\omega^\alpha}{s^\alpha} \mathbb{N}^+\left[R \sum_{n=0}^{\infty} v_n(x,t)\right] + \sum_{n=0}^{\infty} A_n\right]. \tag{15}$$

By associating two sides of Equation (15), we get

$$v_0(x,t) = H(x,t),$$
$$v_1(x,t) = -\mathbb{N}^{-1}\left[\frac{\omega^\alpha}{s^\alpha} \mathbb{N}^+[Rv_0(x,t)] + A_0\right],$$
$$v_2(x,t) = -\mathbb{N}^{-1}\left[\frac{\omega^\alpha}{s^\alpha} \mathbb{N}^+[Rv_1(x,t)] + A_1\right],$$

$$\vdots$$

In the same manner, we can achieve the recursive

$$v_{n+1}(x,t) = -\mathbb{N}^{-1}\left[\frac{\omega^\alpha}{s^\alpha}\mathbb{N}^+[Rv_n(x,t)] + A_n\right]. \tag{16}$$

Finally, we define the approximate solutions as

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t).$$

4. Solution for wave-like equations having fractional order

Here, we consider arbitrary order wave-like equations having variable coefficients in ordered to illuminate the exactness and competence of the considered method.

Example 4.1. Consider the 1D fractional wave-like equation [17, 35]:

$$D^\mu v(x,t) = \frac{1}{2}x^2 v_{xx}, \quad 1 < \mu \le 2, \tag{17}$$

associate to initial conditions

$$v(x,0) = x \text{and } v_t(x,0) = x^2 \; . \tag{18}$$

By employing NT on Equation (17), we have

$$\mathbb{N}^+\left[D_t^\mu v(x,t)\right] = \mathbb{N}^+\left[\frac{1}{2}x^2\frac{\partial^2 v}{\partial x^2}\right]. \tag{19}$$

The non-linear operator is defined as

$$\frac{s^\mu}{w^\mu}\mathbb{N}^+[v(x,t)] - \sum_{k=0}^{n-1}\frac{w^{k-\mu}}{s^{k+1-\mu}}[D^k v]_{t=0} = \mathbb{N}^+\left[\frac{1}{2}x^2\frac{\partial^2 v}{\partial x^2}\right]. \tag{20}$$

By Equations (18) and (20), we get

$$\mathbb{N}^+[v(x,t)] = x + x^2 t + \frac{w^\mu}{s^\mu}\mathbb{N}^+\left[\frac{1}{2}x^2\frac{\partial^2 v}{\partial x^2}\right]. \tag{21}$$

On plugging inverse NT to Equation (21), we obtain

$$v(x,t) = x + x^2 t + \mathbb{N}^{-1}\left[\frac{w^\mu}{s^\mu}\mathbb{N}^+\left[\frac{1}{2}x^2\frac{\partial^2 v}{\partial x^2}\right]\right]. \tag{22}$$

Let $v(x,t) = \sum_{n=0}^{\infty} v_n(x,t)$ be the infinite series solution of $v(x,t)$. Now, we rewrite Equation (22) as

$$\sum_{n=0}^{\infty} v_n(x,t) = x + x^2 t + \mathbb{N}^{-1}\left[\frac{w^\mu}{s^\mu}\mathbb{N}^+\left[\frac{1}{2}x^2\sum_{n=0}^{\infty}\frac{\partial^2 v_n}{\partial x^2}\right]\right]. \tag{23}$$

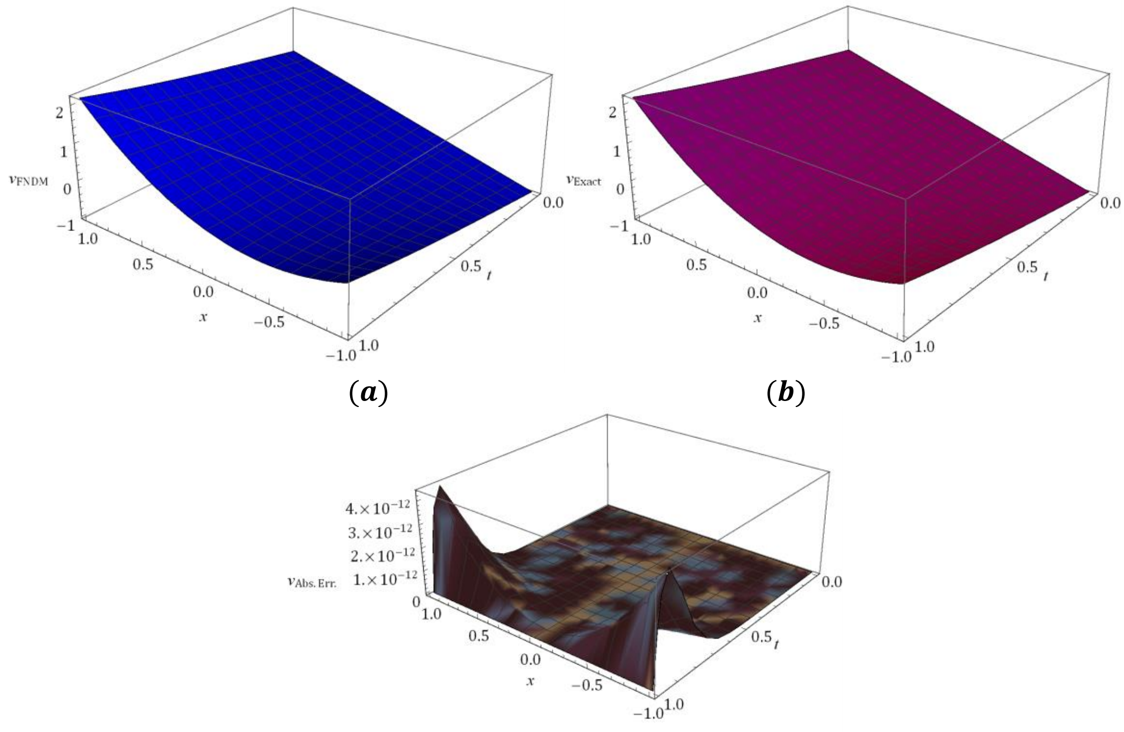
On relating Equation (23) with two sides, we get

$$v_0(x,t) = x + x^2 t, \quad v_1(x,t) = \frac{x^2 t^{\mu+1}}{\Gamma[\mu+2]}, \quad v_2(x,t) = \frac{x^2 t^{2\mu+1}}{\Gamma[2\mu+2]},$$
$$v_3(x,t) = \frac{x^2 t^{3\mu+1}}{\Gamma[3\mu+2]}, \quad v_4(x,t) = \frac{x^2 t^{4\mu+1}}{\Gamma[4\mu+2]}, \quad v_5(x,t) = \frac{x^2 t^{5\mu+1}}{\Gamma[5\mu+2]},$$
$$v_6(x,t) = \frac{x^2 t^{6\mu+1}}{\Gamma[6\mu+2]}, \quad v_7(x,t) = \frac{x^2 t^{7\mu+1}}{\Gamma[7\mu+2]}, \quad v_8(x,t) = \frac{x^2 t^{8\mu+1}}{\Gamma[8\mu+2]},$$
$$v_9(x,t) = \frac{x^2 t^{9\mu+1}}{\Gamma[9\mu+2]}, \quad \cdots$$

Similarly, the rest of the terms of $v_n(n \geq 10)$ can be achieved. Then, we establish the series solutions as

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots$$
$$= x + x^2 t + \frac{x^2 t^{\mu+1}}{\Gamma[\mu+2]} + \frac{x^2 t^{2\mu+1}}{\Gamma[2\mu+2]} + \frac{x^2 t^{3\mu+1}}{\Gamma[3\mu+2]} + \frac{x^2 t^{4\mu+1}}{\Gamma[4\mu+2]} + \frac{x^2 t^{5\mu+1}}{\Gamma[5\mu+2]}$$
$$+ \frac{x^2 t^{6\mu+1}}{\Gamma[6\mu+2]} + \frac{x^2 t^{7\mu+1}}{\Gamma[7\mu+2]} + \cdots$$

The exact solution for Equation (17) at $\mu = 2$ is $v(x,t) = x + x^2\sinh(t)$.



(***a***) (***b***) (***c***)

Figure 1. Behaviour of (***a***) obtained results (***b***) analytical solution (***c***)$v_{Abs.Err.} = |v_{Exact} - v_{FNDM}|$ for Ex. 4.1 at $\mu = 2$.

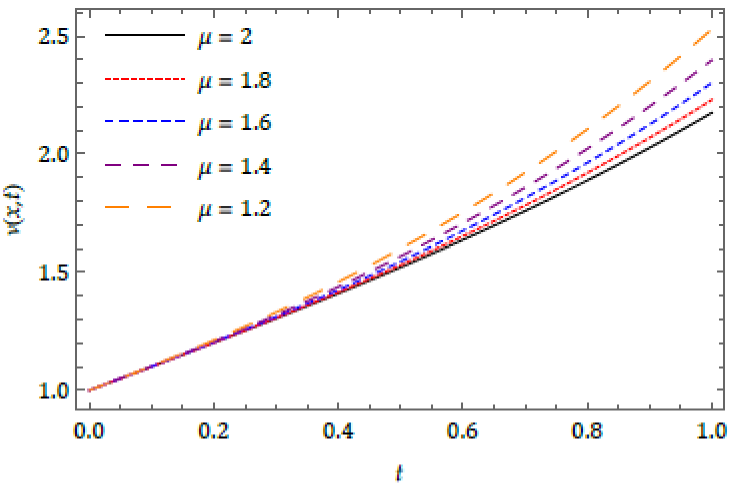


Figure 2. Behaviour of obtained solution for Ex. 4.1 with distinct μ at $x = 1$.

Table 1. Numerical comparison of different order solution for Ex. 4.1 with diverse t and x at $\mu = 2$.

x	t	$\lvert v_{Exact} - v_{FNDM}^{(3)} \rvert$	$\lvert v_{Exact} - v_{FNDM}^{(5)} \rvert$	$\lvert v_{Exact} - v_{FNDM}^{(7)} \rvert$	$\lvert v_{Exact} - v_{FNDM}^{(9)} \rvert$
0.25	0.25	6.57391×10^{-13}	0	0	0
	0.50	3.37159×10^{-10}	1.22818×10^{-15}	6.10623×10^{-16}	6.17562×10^{-16}
	0.75	1.29984×10^{-8}	2.39093×10^{-13}	5.96190×10^{-14}	5.96190×10^{-14}
	1	1.73809×10^{-7}	1.00849×10^{-11}	1.66533×10^{-16}	1.38778×10^{-17}
0.50	0.25	2.62956×10^{-12}	0	0	0
	0.50	1.34864×10^{-9}	4.91274×10^{-15}	2.44249×10^{-15}	2.47025×10^{-15}
	0.75	5.19938×10^{-8}	9.56374×10^{-13}	2.38476×10^{-13}	2.38476×10^{-13}
	1	6.95236×10^{-7}	4.03395×10^{-11}	6.66134×10^{-16}	5.55112×10^{-17}
0.75	0.25	5.91652×10^{-12}	0	0	0
	0.50	3.03443×10^{-9}	1.10467×10^{-14}	5.60663×10^{-15}	5.55112×10^{-15}
	0.75	1.16986×10^{-7}	2.15183×10^{-12}	5.36515×10^{-13}	5.36460×10^{-13}
	1	1.56428×10^{-6}	9.07638×10^{-11}	1.55431×10^{-15}	0
1	0.25	1.05183×10^{-11}	0	0	0
	0.50	5.39454×10^{-9}	1.96509×10^{-14}	9.76996×10^{-15}	9.88098×10^{-15}
	0.75	2.07975×10^{-7}	3.82550×10^{-12}	9.53904×10^{-13}	9.53904×10^{-13}
	1	2.78095×10^{-6}	1.61358×10^{-10}	2.66454×10^{-15}	2.22045×10^{-16}

Example 4.2. Consider the 2D wave-like equation having fractional order:

$$D^{\mu}v(x,y,t)=\frac{1}{12}\left(x^2v_{xx}+y^2v_{yy}\right),\quad 1<\mu\leq 2, \tag{24}$$

associated with

$$v(x,y,0)=x^4 \text{and } v_t(x,y,0)=y^4\ . \tag{25}$$

By employing NT on Equation (24), we have

$$\mathbb{N}^{+}\left[D_t^{\mu}v(x,y,t)\right]=\mathbb{N}^{+}\left[\frac{1}{12}\left(x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}\right)\right]. \tag{26}$$

The non-linear operator is defined as

$$\frac{s^{\mu}}{w^{\mu}}\mathbb{N}^{+}[v(x,y,t)]-\sum_{k=0}^{n-1}\frac{w^{k-\mu}}{s^{k+1-\mu}}[D^k v]_{t=0}=\mathbb{N}^{+}\left[\frac{1}{12}\left(x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}\right)\right]. \tag{27}$$

By Equations (25) and (27), we get

$$\mathbb{N}^{+}[v(x,y,t)]=x^4+ty^4+\frac{w^{\mu}}{s^{\mu}}\mathbb{N}^{+}\left[\frac{1}{12}\left(x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}\right)\right]. \tag{28}$$

On applying inverse NT to Equation (28), it gives

$$v(x,y,t)=x^4+ty^4+\mathbb{N}^{-1}\left[\frac{w^{\mu}}{s^{\mu}}\mathbb{N}^{+}\left[\frac{1}{12}\left(x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}\right)\right]\right]. \tag{29}$$

Assume that, the infinite series solution for $v(x,y,t)$is$v(x,y,t)=\sum_{n=0}^{\infty}v_n(x,y,t)$.
Now, we rewrite Equation (29) as

$$\sum_{n=0}^{\infty}v_n(x,y,t)=x^4+ty^4+\mathbb{N}^{-1}\left[\frac{w^{\mu}}{s^{\mu}}\mathbb{N}^{+}\left[\frac{1}{12}\left(x^2\sum_{n=0}^{\infty}\frac{\partial^2 v_n}{\partial x^2}+ y^2\sum_{n=0}^{\infty}\frac{\partial^2 v_n}{\partial y^2}\right)\right]\right]. \tag{30}$$

On relating Equation (30) with two sides, we get

$$v_0(x,y,t)=x^4+ty^4,\ \ v_1(x,y,t)=\left(\frac{x^4}{\Gamma[\mu+1]}+\frac{ty^4}{\Gamma[\mu+2]}\right)t^{\mu},\ \ v_2(x,y,t)=\left(\frac{x^4}{\Gamma[2\mu+1]}+\frac{ty^4}{\Gamma[2\mu+2]}\right)t^{2\mu},$$

$$v_3(x,y,t)=\left(\frac{x^4}{\Gamma[3\mu+1]}+\frac{ty^4}{\Gamma[3\mu+2]}\right)t^{3\mu},\quad v_4(x,y,t)=\left(\frac{x^4}{\Gamma[4\mu+1]}+\frac{ty^4}{\Gamma[4\mu+2]}\right)t^{4\mu},$$

$$v_5(x,y,t) = \left(\frac{x^4}{\Gamma[5\mu+1]} + \frac{ty^4}{\Gamma[5\mu+2]}\right)t^{5\mu}, \quad v_6(x,y,t) = \left(\frac{x^4}{\Gamma[6\mu+1]} + \frac{ty^4}{\Gamma[6\mu+2]}\right)t^{6\mu},$$

$$v_7(x,y,t) = \left(\frac{x^4}{\Gamma[7\mu+1]} + \frac{ty^4}{\Gamma[7\mu+2]}\right)t^{7\mu}, \quad v_8(x,y,t) = \left(\frac{x^4}{\Gamma[8\mu+1]} + \frac{ty^4}{\Gamma[8\mu+2]}\right)t^{8\mu},$$

$$v_9(x,y,t) = \left(\frac{x^4}{\Gamma[9\mu+1]} + \frac{ty^4}{\Gamma[9\mu+2]}\right)t^{9\mu}, \quad \cdots$$

Then, we obtain

$$v(x,y,t) = \sum_{n=0}^{\infty} v_n(x,y,t) = v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + v_3(x,y,t) + \cdots$$

$$= x^4 + ty^4 + \left(\frac{x^4}{\Gamma[\mu+1]} + \frac{ty^4}{\Gamma[\mu+2]}\right)t^{\mu} + \left(\frac{x^4}{\Gamma[2\mu+1]} + \frac{ty^4}{\Gamma[2\mu+2]}\right)t^{2\mu} + \left(\frac{x^4}{\Gamma[3\mu+1]} + \frac{ty^4}{\Gamma[3\mu+2]}\right)t^{3\mu}$$

$$+ \left(\frac{x^4}{\Gamma[4\mu+1]} + \frac{ty^4}{\Gamma[4\mu+2]}\right)t^{4\mu} + \left(\frac{x^4}{\Gamma[5\mu+1]} + \frac{ty^4}{\Gamma[5\mu+2]}\right)t^{5\mu} + \left(\frac{x^4}{\Gamma[6\mu+1]} + \frac{ty^4}{\Gamma[6\mu+2]}\right)t^{6\mu}$$

$$+ \left(\frac{x^4}{\Gamma[7\mu+1]} + \frac{ty^4}{\Gamma[7\mu+2]}\right)t^{7\mu} + \left(\frac{x^4}{\Gamma[8\mu+1]} + \frac{ty^4}{\Gamma[8\mu+2]}\right)t^{8\mu} + \left(\frac{x^4}{\Gamma[9\mu+1]} + \frac{ty^4}{\Gamma[9\mu+2]}\right)t^{9\mu} + \cdots$$

The analytical solution for Equation (24) for $\mu = 2$ is

$$v(x,y,t) = x^4 \cosh(t) + y^4 \sinh(t).$$

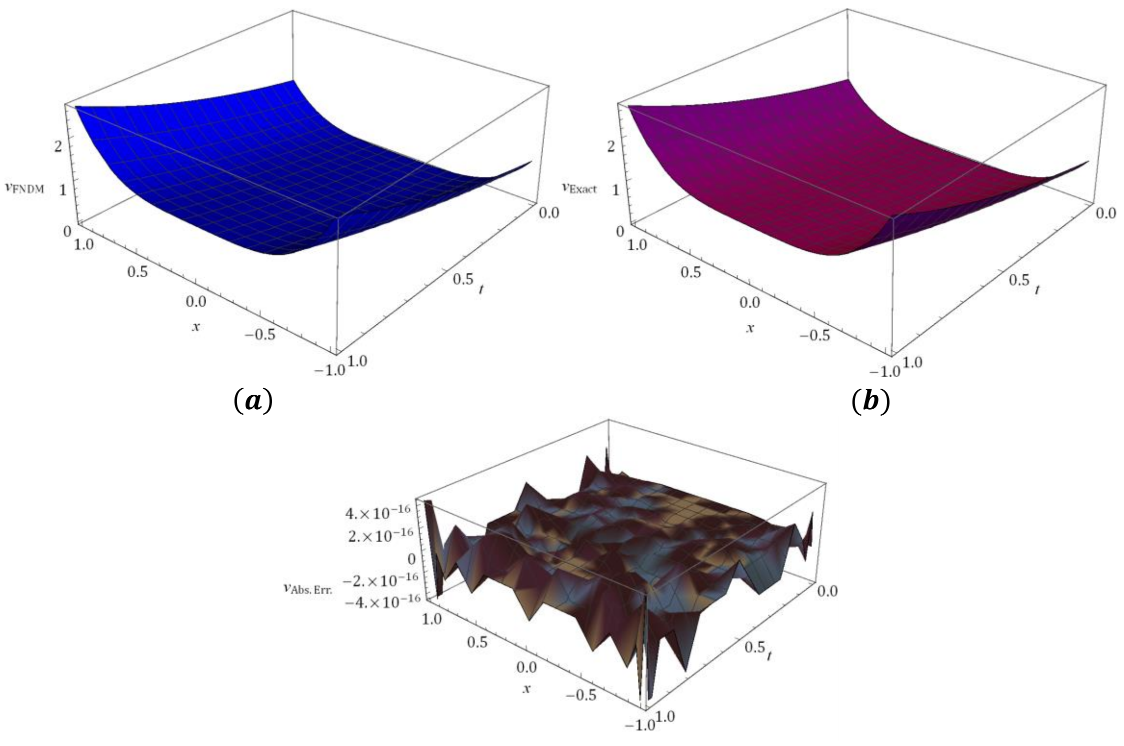


Figure 3. Behaviour of $(\boldsymbol{a})$ obtained results $(\boldsymbol{b})$ analytical solution$(\boldsymbol{c})v_{Abs.Err.} = |v_{Exact} - v_{FNDM}|$ for Ex. 4.2 at $\mu = 2$.

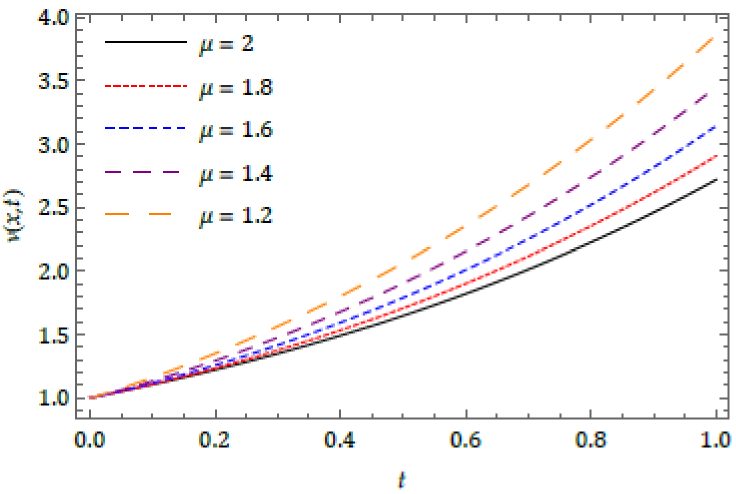


Figure 4. Nature of FNDM results for Ex. 4.2 with distinct μ at $x = 1$.

Table 2. Numerical study for Ex. 4.2 with diverse t and x at $\mu = 2$.

x	t	$\left\|v_{Exact} - v_{FNDM}^{(3)}\right\|$	$\left\|v_{Exact} - v_{FNDM}^{(5)}\right\|$	$\left\|v_{Exact} - v_{FNDM}^{(7)}\right\|$	$\left\|v_{Exact} - v_{FNDM}^{(9)}\right\|$
0.25	0.25	1.19976×10^{-11}	0	0	5.55112×10^{-17}
	0.50	5.77404×10^{-9}	2.16493×10^{-14}	1.11022×10^{-16}	0
	0.75	2.17735×10^{-7}	4.08451×10^{-12}	0	0
	1	2.87891×10^{-6}	1.69558×10^{-10}	2.88658×10^{-15}	0
0.50	0.25	3.41873×10^{-11}	5.55112×10^{-17}	0	0
	0.50	1.14665×10^{-8}	5.15143×10^{-14}	0	0
	0.75	3.64134×10^{-7}	7.97140×10^{-12}	1.11022×10^{-16}	0
	1	4.34840×10^{-6}	2.92558×10^{-10}	5.77316×10^{-15}	2.22045×10^{-16}
0.75	0.25	1.30343×10^{-10}	0	0	1.11022×10^{-16}
	0.50	3.61337×10^{-8}	1.80966×10^{-13}	1.11022×10^{-16}	1.11022×10^{-16}
	0.75	9.98530×10^{-7}	2.48142×10^{-11}	4.44089×10^{-16}	2.22045×10^{-16}
	1	1.07162×10^{-5}	8.25556×10^{-10}	1.79856×10^{-14}	2.22045×10^{-16}
1	0.25	3.89223×10^{-10}	0	0	0
	0.50	1.02545×10^{-7}	5.29798×10^{-13}	2.22045×10^{-16}	0
	0.75	2.70652×10^{-6}	7.01603×10^{-11}	4.44089×10^{-16}	0
	1	2.78602×10^{-5}	2.26055×10^{-9}	5.01821×10^{-14}	4.44089×10^{-16}

Example 4.3. Consider the 3D fractional wave-like equation:

$$D^{\mu}v(x,y,z,t) = (x^2+y^2+z^2) + \frac{1}{2}\left(x^2 v_{xx} + y^2 v_{yy} + z^2 v_{zz}\right), \quad 1 < \mu \leq 2, \quad (31)$$

associated with

$$v(x,y,z,0) = 0 \text{and } v_t(x,y,z,0) = x^2 + y^2 - z^2 \,. \quad (32)$$

By employing NT on Equation (31), we have

$$\mathbb{N}^{+}\left[D_t^{\mu} v(x,y,z,t)\right] = \mathbb{N}^{+}\left[(x^2+y^2+z^2) + \frac{1}{2}\left(x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2} + z^2\frac{\partial^2 v}{\partial z^2}\right)\right]. \quad (33)$$

The non-linear operator is defined as

$$\frac{s^{\mu}}{w^{\mu}}\mathbb{N}^{+}[v]-\sum_{k=0}^{n-1}\frac{w^{k-\mu}}{s^{k+1-\mu}}[D^{k}v]_{t=0}=\mathbb{N}^{+}\left[(x^2+y^2+z^2)+\frac{1}{2}\left(x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}+z^2\frac{\partial^2 v}{\partial z^2}\right)\right]. \quad (34)$$

By Equations (32) and (34), we get

$$\mathbb{N}^{+}[v(x,y,z,t)]=(x^2+y^2-z^2)t+\frac{w^{\mu}}{s^{\mu}}\mathbb{N}^{+}\left[(x^2+y^2+z^2)+\frac{1}{2}\left(x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}+z^2\frac{\partial^2 v}{\partial z^2}\right)\right]. \quad (35)$$

On applying inverse NT to Equation (35), it gives

$$v(x,y,z,t)=(x^2+y^2-z^2)t+\mathbb{N}^{-1}\left[\frac{w^{\mu}}{s^{\mu}}\mathbb{N}^{+}\left[(x^2+y^2+z^2)+\frac{1}{2}\left(x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}+z^2\frac{\partial^2 v}{\partial z^2}\right)\right]\right]. \quad (36)$$

The infinite series solution for $v(x,y,z,t)$is$v(x,y,z,t)=\sum_{n=0}^{\infty}v_n(x,y,z,t)$. Now, we rewrite Equation (36) as

$$\sum_{n=0}^{\infty}v_n(x,y,z,t)=(x^2+y^2-z^2)t+\mathbb{N}^{-1}[\frac{w^{\mu}}{s^{\mu}}\mathbb{N}^{+}[(x^2+y^2+z^2)+\frac{1}{2}(x^2\sum_{n=0}^{\infty}\frac{\partial^2 v_n}{\partial x^2}+y^2\sum_{n=0}^{\infty}\frac{\partial^2 v_n}{\partial y^2}+z^2\sum_{n=0}^{\infty}\frac{\partial^2 v_n}{\partial z^2})]]. \quad (37)$$

On relating Equation (37) with two sides, we get

$$v_0(x,y,z,t)=(x^2+y^2-z^2)t,\ \ v_1(x,y,z,t)=(x^2+y^2+z^2)\frac{t^{\mu}}{\Gamma[\mu+1]}+(x^2+y^2-z^2)\frac{t^{\mu+1}}{\Gamma[\mu+2]},$$
$$v_2(x,y,z,t)=(x^2+y^2+z^2)\frac{t^{2\mu}}{\Gamma[2\mu+1]}+(x^2+y^2-z^2)\frac{t^{2\mu+1}}{\Gamma[2\mu+2]},$$
$$v_3(x,y,z,t)=(x^2+y^2+z^2)\frac{t^{3\mu}}{\Gamma[3\mu+1]}+(x^2+y^2-z^2)\frac{t^{3\mu+1}}{\Gamma[3\mu+2]},$$
$$v_4(x,y,z,t)=(x^2+y^2+z^2)\frac{t^{4\mu}}{\Gamma[4\mu+1]}+(x^2+y^2-z^2)\frac{t^{4\mu+1}}{\Gamma[4\mu+2]},\cdots$$

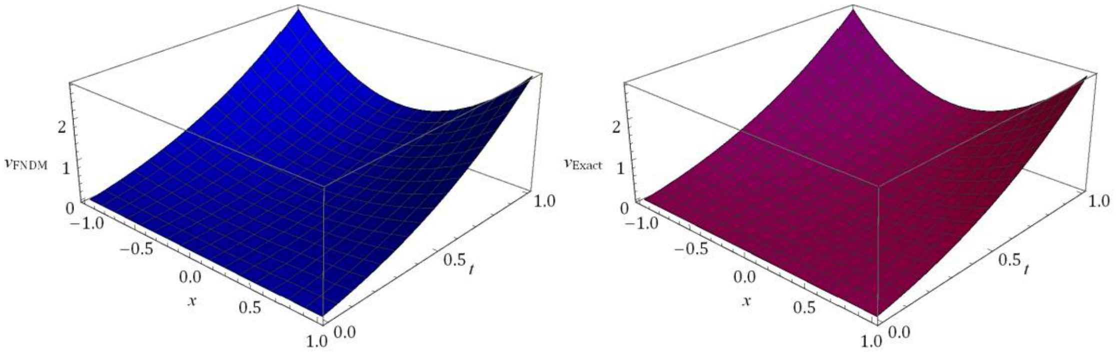
Then, we can obtain

$$v(x,y,z,t)=\sum_{n=0}^{\infty}v_n(x,t)=v_0(x,y,z,t)+v_1(xy,z,,t)+v_2(x,y,z,t)+v_3(x,y,z,t)+\cdots$$
$$=(x^2+y^2-z^2)t+(x^2+y^2+z^2)\frac{t^{\mu}}{\Gamma[\mu+1]}+(x^2+y^2-z^2)\frac{t^{\mu+1}}{\Gamma[\mu+2]}$$

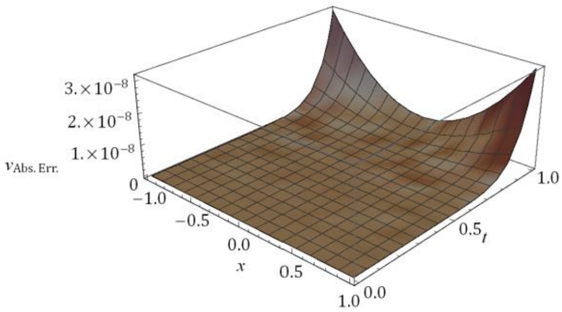
$$+(x^2+y^2+z^2)\frac{t^{2\mu}}{\Gamma[2\mu+1]}+(x^2+y^2-z^2)\frac{t^{2\mu+1}}{\Gamma[2\mu+2]}+(x^2+y^2+z^2)\frac{t^{3\mu}}{\Gamma[3\mu+1]}$$
$$+(x^2+y^2-z^2)\frac{t^{3\mu+1}}{\Gamma[3\mu+2]}+(x^2+y^2+z^2)\frac{t^{4\mu}}{\Gamma[4\mu+1]}+(x^2+y^2-z^2)\frac{t^{4\mu+1}}{\Gamma[4\mu+2]}+\cdots$$

The analytical solution for Eq. (31) for $\mu = 2$ is

$$v(x,y,z,t) = (x^2+y^2)(e^t-1)+z^2(e^{-t}-1).$$



(***a***) (***b***)



(***c***)

Figure 5. Behaviour of (***a***) obtained results (***b***) analytical solution(***c***)$v_{Abs.Err.} = |v_{Exact} - v_{FNDM}|$ for Ex. 4.3 at $\mu = 2$.

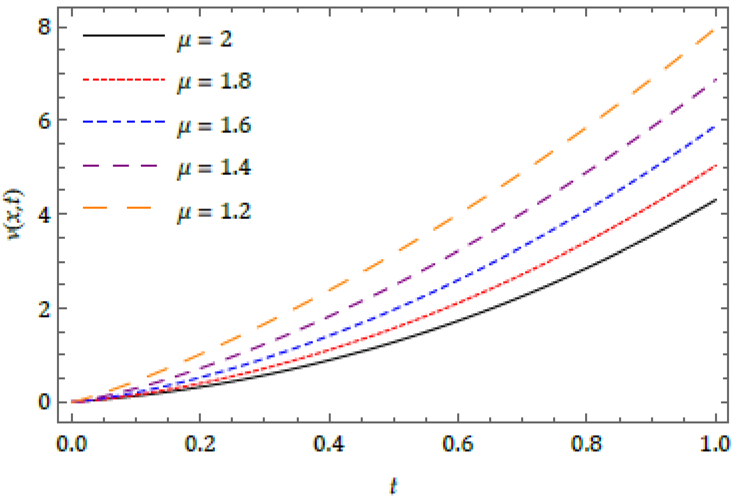


Figure 6. Nature of FNDM results for Ex. 4.3 with distinct μ at $x = 1$.

Table 3. Numerical study for Ex. 4.3 with diverse t and x at $\mu = 2$.

x	t	$\left\lvert v_{Exact} - v_{FNDM}^{(3)} \right\rvert$	$\left\lvert v_{Exact} - v_{FNDM}^{(5)} \right\rvert$	$\left\lvert v_{Exact} - v_{FNDM}^{(7)} \right\rvert$	$\left\lvert v_{Exact} - v_{FNDM}^{(9)} \right\rvert$
0.25	0.25	3.36903×10^{-4}	7.00901×10^{-7}	7.81737×10^{-10}	5.42546×10^{-13}
	0.50	5.43243×10^{-3}	4.50567×10^{-5}	2.00711×10^{-7}	5.56867×10^{-10}
	0.75	2.78314×10^{-2}	5.16656×10^{-4}	5.16625×10^{-6}	3.22101×10^{-8}
	1	8.93872×10^{-2}	2.92888×10^{-3}	5.18998×10^{-5}	5.74275×10^{-7}
0.50	0.25	3.69012×10^{-4}	7.66822×10^{-7}	8.54716×10^{-10}	5.92895×10^{-13}
	0.50	5.97391×10^{-3}	4.94356×10^{-5}	2.19938×10^{-7}	6.09719×10^{-10}
	0.75	3.07259×10^{-2}	5.68478×10^{-4}	5.67372×10^{-6}	3.53313×10^{-8}
	1	9.90651×10^{-2}	3.23173×10^{-3}	5.71236×10^{-5}	6.31066×10^{-7}
0.75	0.25	4.22528×10^{-4}	8.76692×10^{-7}	9.76348×10^{-10}	6.76939×10^{-13}
	0.50	6.87640×10^{-3}	5.67337×10^{-5}	2.51983×10^{-7}	6.97805×10^{-10}
	0.75	3.55502×10^{-2}	6.54848×10^{-4}	6.51951×10^{-6}	4.05334×10^{-8}
	1	1.15195×10^{-1}	3.73646×10^{-3}	6.58299×10^{-5}	7.25718×10^{-7}
1	0.25	4.97450×10^{-4}	1.03051×10^{-6}	1.14663×10^{-9}	7.94511×10^{-13}
	0.50	8.13987×10^{-3}	6.69511×10^{-5}	2.96847×10^{-7}	8.21127×10^{-10}
	0.75	4.23041×10^{-2}	7.75766×10^{-3}	7.70361×10^{-6}	4.78164×10^{-8}
	1	1.37776×10^{-1}	7.00901×10^{-7}	7.80187×10^{-5}	8.58231×10^{-7}

Example 4.4. Consider the 1D nonlinear wave-like equation having fractional order [36]:

$$D^{\mu}v(x,t) = x^2 \frac{\partial}{\partial x}(v_x v_{xx}) - x^2 (v_{xx})^2 - v, \quad 1 < \mu \leq 2, \tag{38}$$

associated with initial conditions

$$v(x,0) = 0 \text{and } v_t(x,0) = x^2 . \tag{39}$$

By employing NT on Equation (38), we have

$$\mathbb{N}^{+}\left[D_t^{\mu} v(x,t)\right] = \mathbb{N}^{+}\left[x^2 \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\right) - x^2\left(\frac{\partial^2 v}{\partial x^2}\right)^2 - v\right]. \tag{40}$$

The non-linear operator is defined as

$$\frac{s^{\mu}}{w^{\mu}}\mathbb{N}^{+}[v(x,t)] - \sum_{k=0}^{n-1}\frac{w^{k-\mu}}{s^{k+1-\mu}}[D^k v]_{t=0} = \mathbb{N}^{+}\left[x^2 \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\right) - x^2\left(\frac{\partial^2 v}{\partial x^2}\right)^2 - v\right]. \tag{41}$$

By Equations (39) and (41), we get

$$\mathbb{N}^{+}[v(x,t)] = x^2 t + \frac{w^\mu}{s^\mu}\mathbb{N}^{+}\left[x^2 \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\right) - x^2\left(\frac{\partial^2 v}{\partial x^2}\right)^2 - v\right]. \quad (42)$$

On plugging inverse NT to Equation. (42), we obtain

$$v(x,t) = x^2 t + \mathbb{N}^{-1}\left[x^2 \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\right) - x^2\left(\frac{\partial^2 v}{\partial x^2}\right)^2 - v\right]. \quad (43)$$

Let $v(x,t) = \sum_{n=0}^{\infty} v_n(x,t)$ be the infinite series solution for $v(x,t)$. Note that, $v_x v_{xx} = \sum_{n=0}^{\infty} A_n$ and$(v_{xx})^2 = \sum_{n=0}^{\infty} B_n$, are the Adomian polynomials. Then, Equation (43) becomes

$$\sum_{n=0}^{\infty} v_n(x,t) = x^2 t + \mathbb{N}^{-1}\left[\frac{w^\mu}{s^\mu}\mathbb{N}^{+}\left[x^2 \frac{\partial}{\partial x}\sum_{n=0}^{\infty} A_n - x^2 \sum_{n=0}^{\infty} B_n - \sum_{n=0}^{\infty} v_n\right]\right]. \quad (44)$$

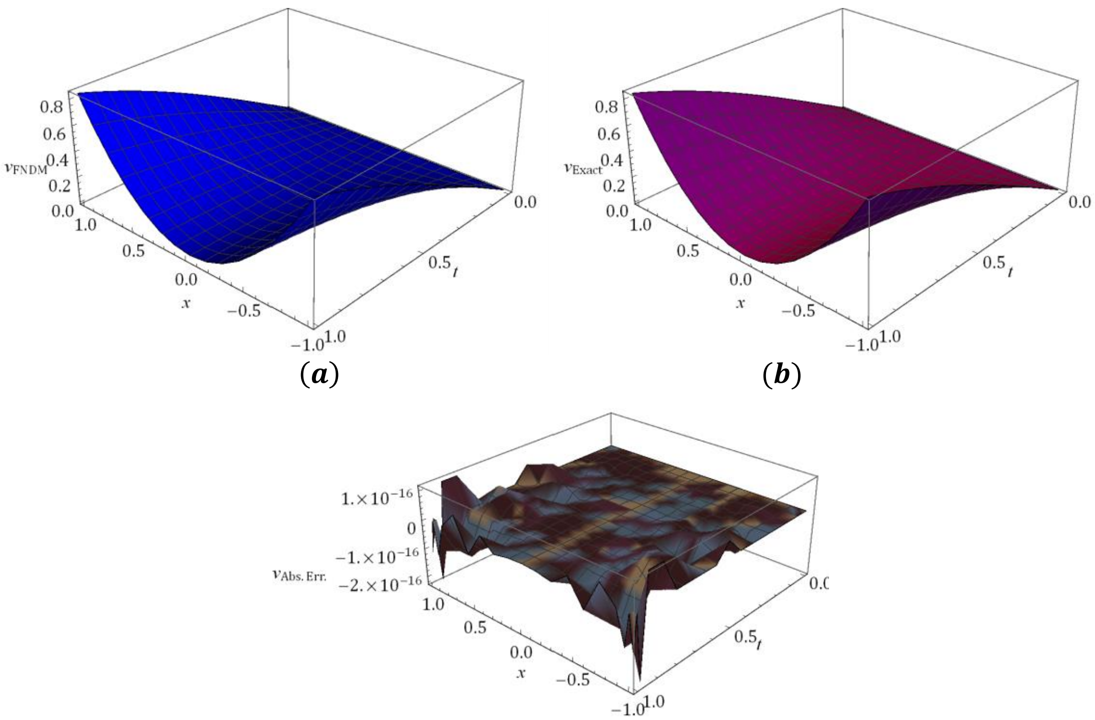
On relating Equation (44) with two sides, we get

$$v_0(x,t) = x^2 t, \quad v_1(x,t) = -\frac{x^2 t^{\mu+1}}{\Gamma[\mu+2]}, \quad v_2(x,t) = \frac{x^2 t^{2\mu+1}}{\Gamma[2\mu+2]},$$
$$v_3(x,t) = -\frac{x^2 t^{3\mu+1}}{\Gamma[3\mu+2]}, \quad v_4(x,t) = \frac{x^2 t^{4\mu+1}}{\Gamma[4\mu+2]}, \quad v_5(x,t) = -\frac{x^2 t^{5\mu+1}}{\Gamma[5\mu+2]},$$
$$v_6(x,t) = \frac{x^2 t^{6\mu+1}}{\Gamma[6\mu+2]}, \quad v_7(x,t) = -\frac{x^2 t^{7\mu+1}}{\Gamma[7\mu+2]}, \quad v_8(x,t) = \frac{x^2 t^{8\mu+1}}{\Gamma[8\mu+2]},$$
$$v_9(x,t) = -\frac{x^2 t^{9\mu+1}}{\Gamma[9\mu+2]}, \quad \cdots$$

Similarly, the remaining terms of $v_n (n \geq 10)$ can be attained. Then, we have

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots$$
$$= x^2 t - \frac{x^2 t^{\mu+1}}{\Gamma[\mu+2]} + \frac{x^2 t^{2\mu+1}}{\Gamma[2\mu+2]} - \frac{x^2 t^{3\mu+1}}{\Gamma[3\mu+2]} + \frac{x^2 t^{4\mu+1}}{\Gamma[4\mu+2]} - \frac{x^2 t^{5\mu+1}}{\Gamma[5\mu+2]} + \frac{x^2 t^{6\mu+1}}{\Gamma[6\mu+2]} - \frac{x^2 t^{7\mu+1}}{\Gamma[7\mu+2]} + \cdots$$

The analytical solution for Equation (38) at $\mu = 2$ is $v(x,t) = x^2 \sin(t)$.



($\boldsymbol{c}$)

Figure 7. Behaviour of ($\boldsymbol{a}$) obtained results ($\boldsymbol{b}$) analytical solution($\boldsymbol{c}$)$v_{Abs.Err.} = |v_{Exact} - v_{FNDM}|$ for Ex. 4.4 at $\mu = 2$.

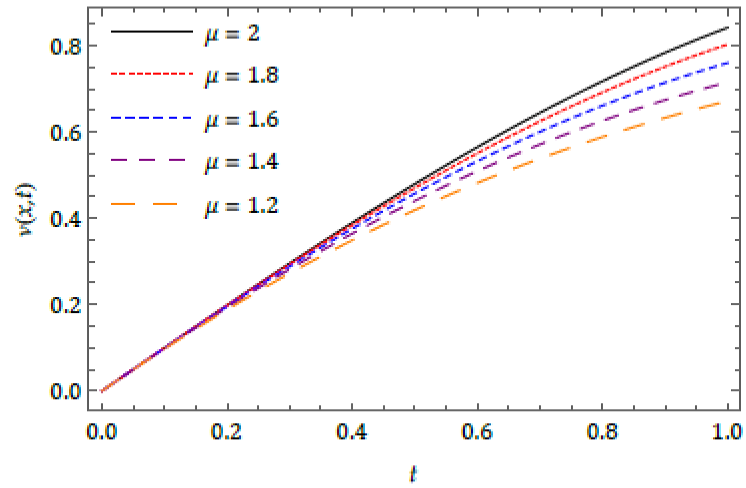


Figure 8. Nature of FNDM results for Ex. 4.4 with distinct μat $x = 1$.

Table 4. Numerical study of obtained results for Ex. 4.4 with diverse t and $x$at $\mu = 2$.

x	t	$\left\|v_{Exact} - v^{(3)}_{FNDM}\right\|$	$\left\|v_{Exact} - v^{(5)}_{FNDM}\right\|$	$\left\|v_{Exact} - v^{(7)}_{FNDM}\right\|$	$\left\|v_{Exact} - v^{(9)}_{FNDM}\right\|$
0.25	0.25	6.56645×10^{-13}	0	0	0
	0.50	3.35630×10^{-10}	1.22125×10^{-15}	0	0
	0.75	1.28662×10^{-8}	2.37810×10^{-13}	0	0
	1	1.70677×10^{-7}	9.98928×10^{-12}	1.73472×10^{-16}	0
0.50	0.25	2.62658×10^{-12}	0	0	0
	0.50	1.34252×10^{-9}	4.88498×10^{-15}	0	0
	0.75	5.14647×10^{-8}	9.51239×10^{-13}	0	0
	1	6.82710×10^{-7}	3.99571×10^{-11}	6.93889×10^{-16}	0
0.75	0.25	5.90983×10^{-12}	0	2.77556×10^{-17}	2.77556×10^{-17}
	0.50	3.02067×10^{-9}	1.10467×10^{-14}	0	5.55112×10^{-17}
	0.75	1.15796×10^{-7}	2.14034×10^{-12}	0	0
	1	1.53610×10^{-6}	8.99035×10^{-11}	1.60982×10^{-15}	0
1	0.25	1.05063×10^{-11}	0	0	0
	0.50	5.37008×10^{-9}	1.95399×10^{-14}	0	0
	0.75	2.05859×10^{-7}	3.80496×10^{-12}	0	0
	1	2.73084×10^{-6}	1.59828×10^{-10}	2.77556×10^{-15}	0

Example 4.5. Consider the 2D nonlinear fractional wave-like equation [36]:

$$D^{\mu}v(x,t) = \frac{\partial^2}{\partial x \partial y}\left(v_{xx}v_{yy}\right) - \frac{\partial^2}{\partial x \partial y}\left(xyv_x v_y\right) - v, \quad 1 < \mu \leq 2, \tag{45}$$

associated with initial conditions

$$v(x,y,0) = e^{xy} \text{and } v_t(x,y,0) = e^{xy}. \tag{46}$$

By employing NT on Equation (45), we have

$$\mathbb{N}^{+}\left[D_t^{\mu}v(x,y,t)\right] = \mathbb{N}^{+}\left[\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial^2}{\partial x \partial y}\left(xy\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) - v\right]. \tag{47}$$

The non-linear operator is defined as

$$\frac{s^{\mu}}{w^{\mu}}\mathbb{N}^{+}[v(x,y,t)] - \sum_{k=0}^{n-1}\frac{w^{k-\mu}}{s^{k+1-\mu}}[D^k v]_{t=0} = \mathbb{N}^{+}\left[\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial^2}{\partial x \partial y}\left(xy\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) - v\right]. \tag{48}$$

By Equations (46) and (48), we get

$$\mathbb{N}^{+}[v(x,y,t)] = (1+t)e^{xy} + \frac{w^{\mu}}{s^{\mu}}\mathbb{N}^{+}\left[\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial^2}{\partial x \partial y}\left(xy\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) - v\right]. \tag{49}$$

On plugging inverse NT to Equation (49), we obtain

$$v(x,y,t) = (1+t)e^{xy} + \mathbb{N}^{-1}\left[\frac{w^{\mu}}{s^{\mu}}\mathbb{N}^{+}\left[\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial^2}{\partial x\partial y}\left(xy\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) - v\right]\right]. \tag{50}$$

Let $v(x,y,t) = \sum_{n=0}^{\infty} v_n(x,y,t)$ be the infinite series solution of $v(x,y,t)$. Note that, $v_{xx}v_{yy} = \sum_{n=0}^{\infty} A_n$ and $v_x v_y = \sum_{n=0}^{\infty} B_n$ are the Adomian polynomials. Then, Equation (51) becomes

$$\sum_{n=0}^{\infty} v_n(x,y,t) = (1+t)e^{xy} + \mathbb{N}^{-1}\left[\frac{w^{\mu}}{s^{\mu}}\mathbb{N}^{+}\left[\frac{\partial^2}{\partial x\partial y}\sum_{n=0}^{\infty} A_n - \frac{\partial^2}{\partial x\partial y}\left(xy\sum_{n=0}^{\infty} B_n\right) - \sum_{n=0}^{\infty} v_n\right]\right]. \tag{52}$$

On relating Equation (52) with two sides, we get

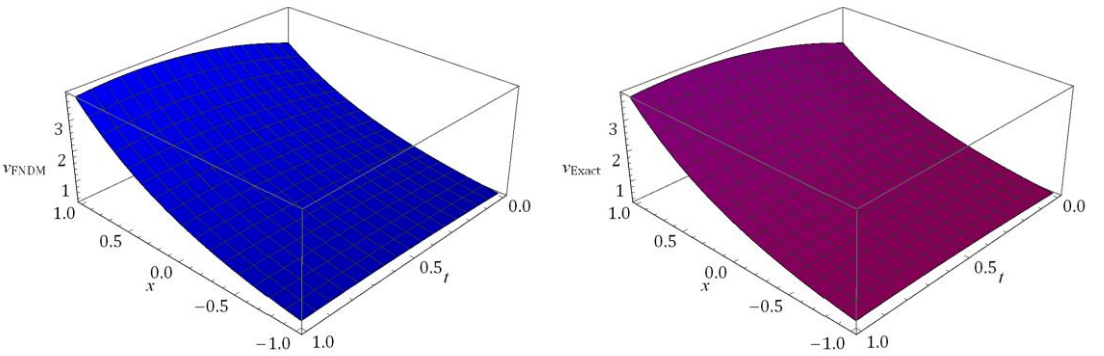
$$v_0(x,y,t) = (1+t)e^{xy},$$
$$v_1(x,y,t) = -\left(\frac{t^{\mu}}{\Gamma[\mu+1]} + \frac{t^{\mu+1}}{\Gamma[\mu+2]}\right)e^{xy},$$
$$v_2(x,y,t) = \left(\frac{t^{2\mu}}{\Gamma[2\mu+1]} + \frac{t^{2\mu+1}}{\Gamma[2\mu+2]}\right)e^{xy},$$
$$v_3(x,y,t) = -\left(\frac{t^{3\mu}}{\Gamma[3\mu+1]} + \frac{t^{3\mu+1}}{\Gamma[3\mu+2]}\right)e^{xy}$$
$$v_4(x,y,t) = \left(\frac{t^{4\mu}}{\Gamma[4\mu+1]} + \frac{t^{4\mu+1}}{\Gamma[4\mu+2]}\right)e^{xy},$$
$$v_5(x,y,t) = -\left(\frac{t^{5\mu}}{\Gamma[5\mu+1]} + \frac{t^{5\mu+1}}{\Gamma[5\mu+2]}\right)e^{xy},$$
$$v_6(x,y,t) = \left(\frac{t^{6\mu}}{\Gamma[6\mu+1]} + \frac{t^{6\mu+1}}{\Gamma[6\mu+2]}\right)e^{xy},$$
$$v_7(x,y,t) = -\left(\frac{t^{7\mu}}{\Gamma[7\mu+1]} + \frac{t^{7\mu+1}}{\Gamma[7\mu+2]}\right)e^{xy},$$
$$v_8(x,y,t) = \left(\frac{t^{8\mu}}{\Gamma[8\mu+1]} + \frac{t^{8\mu+1}}{\Gamma[8\mu+2]}\right)e^{xy},$$
$$v_9(x,y,t) = -\left(\frac{t^{9\mu}}{\Gamma[9\mu+1]} + \frac{t^{9\mu+1}}{\Gamma[9\mu+2]}\right)e^{xy}, \quad \cdots$$

Then, we have

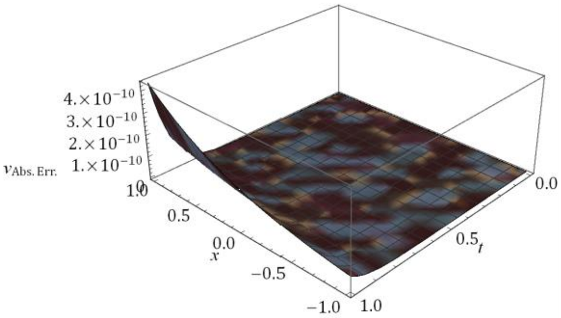
$$v(x,y,t) = \sum_{n=0}^{\infty} v_n(x,y,t) = v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + v_3(x,y,t) + \cdots$$
$$= x^4 + ty^4 + \left(\frac{x^4}{\Gamma[\mu+1]} + \frac{ty^4}{\Gamma[\mu+2]}\right)t^{\mu} + \left(\frac{x^4}{\Gamma[2\mu+1]} + \frac{ty^4}{\Gamma[2\mu+2]}\right)t^{2\mu} + \left(\frac{x^4}{\Gamma[3\mu+1]} + \frac{ty^4}{\Gamma[3\mu+2]}\right)t^{3\mu}$$

$$+\left(\frac{x^4}{\Gamma[4\mu+1]}+\frac{ty^4}{\Gamma[4\mu+2]}\right)t^{4\mu}+\left(\frac{x^4}{\Gamma[5\mu+1]}+\frac{ty^4}{\Gamma[5\mu+2]}\right)t^{5\mu}$$
$$+\left(\frac{x^4}{\Gamma[6\mu+1]}+\frac{ty^4}{\Gamma[6\mu+2]}\right)t^{6\mu}$$
$$+\left(\frac{x^4}{\Gamma[7\mu+1]}+\frac{ty^4}{\Gamma[7\mu+2]}\right)t^{7\mu}+\left(\frac{x^4}{\Gamma[8\mu+1]}+\frac{ty^4}{\Gamma[8\mu+2]}\right)t^{8\mu}$$
$$+\left(\frac{x^4}{\Gamma[9\mu+1]}+\frac{ty^4}{\Gamma[9\mu+2]}\right)t^{9\mu}+\cdots$$

The analytical solution for Equation (45) for $\mu = 2$ is $v(x, y, t) = x^4 \cosh(t) + y^4 \sinh(t)$.



$(\boldsymbol{a})$ $(\boldsymbol{b})$



$(\boldsymbol{c})$

Figure 9. Behaviour of $(\boldsymbol{a})$ obtained results $(\boldsymbol{b})$ analytical solution$(\boldsymbol{c})v_{Abs.Err.} = |v_{Exact} - v_{FNDM}|$ for Ex. 4.5 at $\mu = 2$.

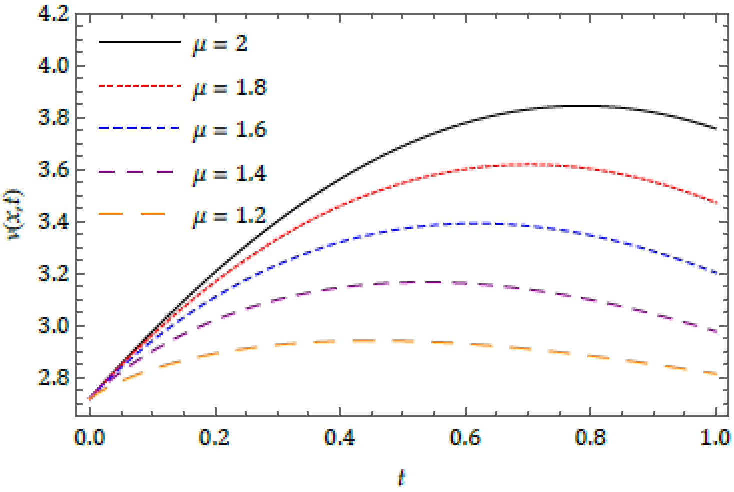


Figure 8. Nature of FNDM results for Ex. 4.5 with distinct μat $x = 1$.

Table 5. Numerical analysis for Ex. 4.5 with diverse t and $x$at $\mu = 2$.

x	t	$\lvert v_{Exact} - v_{FNDM}^{(3)} \rvert$	$\lvert v_{Exact} - v_{FNDM}^{(5)} \rvert$	$\lvert v_{Exact} - v_{FNDM}^{(7)} \rvert$	$\lvert v_{Exact} - v_{FNDM}^{(9)} \rvert$
0.25	0.25	4.99082×10^{-10}	0	2.22045×10^{-16}	2.22045×10^{-16}
	0.50	1.30948×10^{-7}	6.79012×10^{-13}	3.30846×10^{-14}	3.30846×10^{-14}
	0.75	3.43267×10^{-6}	8.95364×10^{-11}	4.99401×10^{-12}	4.99334×10^{-12}
	1	3.50012×10^{-5}	2.87118×10^{-9}	2.05285×10^{-10}	2.05220×10^{-10}
0.50	0.25	6.40834×10^{-10}	4.44089×10^{-16}	8.88178×10^{-16}	4.44089×10^{-16}
	0.50	1.68141×10^{-7}	8.71303×10^{-13}	4.17444×10^{-14}	4.17444×10^{-14}
	0.75	4.40763×10^{-6}	1.14967×10^{-10}	6.41309×10^{-12}	6.41176×10^{-12}
	1	4.49424×10^{-5}	3.68667×10^{-9}	2.63591×10^{-10}	2.63508×10^{-10}
0.75	0.25	8.22847×10^{-10}	0	4.44089×10^{-16}	4.44089×10^{-16}
	0.50	2.15897×10^{-7}	1.11910×10^{-12}	5.41789×10^{-14}	5.37348×10^{-14}
	0.75	5.65952×10^{-6}	1.47620×10^{-10}	8.23341×10^{-12}	8.23208×10^{-12}
	1	5.77072×10^{-5}	4.73378×10^{-9}	3.38458×10^{-10}	3.38351×10^{-10}
1	0.25	1.05656×10^{-9}	4.44089×10^{-16}	4.44089×10^{-16}	4.44089×10^{-16}
	0.50	2.77218×10^{-7}	1.43663×10^{-12}	6.92779×10^{-14}	6.83897×10^{-14}
	0.75	7.26696×10^{-6}	1.89549×10^{-10}	1.05724×10^{-11}	1.05707×10^{-11}
	1	7.40975×10^{-5}	6.07830×10^{-9}	4.34588×10^{-10}	4.34452×10^{-10}

5. Numerical results and discussion

Here, the multi-dimensional time-fractional linear and nonlinear wave-like equations with variable coefficients have been examined with the aid of FNDM. To illustrate the capability and exactness of the projected procedure, the numerical study presented with a change in the order of the solution for five cases considered and presented in Tables 1-5 in terms of absolute error for change in x and t. From the demonstrated results shows for the higher value of order the obtained solution closure to analytical results of the corresponding equations and also authorize the accuracy of FNDM. Moreover, for some cases, the obtained results are identical to the exact one and hence it shows the absolute error is zero. These types of analysis may aid researchers about the reliability and capability of the scheme while investigating complex systems.

As much as important of formulating the problem and finding the solution for the corresponding equations, it is very important to capture nature also. In this connection, we presented the obtained results behaviour for each case related to the exact result to both exemplify exactness and nature of results achieved. For Example 4.1, the response of FNDM results and the absolute surface is cited in Figure 1 with the exact solution. In the same manner, we presented for Example 4.2, 4.3, 4.4 and 4.5 respectively in the Figures 3, 5, 7 and 9. Moreover, by incorporating the concept of the classical derivative by non-integer derivative we get more degree of freedom and aid us to illustrate more hide future of the model related to history and hereditary based results. In regard to this, with respect to different arbitrary order, the response with time is illustrated respectively for Examples 1-5 in Figures 2, 4, 6, 8 and 10. With the assist of tables and figures, we can observe the hired solution procedure is very effective and more accurate to analyse the projected wave-like equations with fractional order. Further, this investigation can capture or limn more behaviour and it can help the researchers to analyse diverse applications of projected problems.

6. Conclusion

In the present framework, the multi-dimensional wave-like equations with variable coefficients are effectively and accurately analysed with the assist of FNDM within the frame of fractional calculus. The projected solution procedure assesses the solution for the differential equations without employing any conversion, discretization or perturbation. The plots authorize the reliability of the hired algorithm and the effect of fractional order while we are analysing wave equations. Related to significances available in the literature, the attained results with the projected scheme are more stimulating and interesting. The nature of the obtained results are exemplified with the aid of figures and tables, and also which helps understand the behaviour of the projected models with the specific values. Moreover, we authorize the achieved results to get closure to the analytical solution as the number of iterative terms increases. Lastly, we can say that the illustrated solution procedure accurate and highly systematic and it can be employed to analyse the numerous families of linear and nonlinear equations arisen in science and engineering.

References

[1] Liouville, J., Memoire surquelques questions de geometrieet de mecanique, etsur un nouveau genre de calcul pour resoudreces questions, **J. Ecole. Polytech**., 13, 1-69, (1832).

[2] Riemann, G.F.B., VersuchEinerAllgemeinenAuffassung der Integration und Differentiation, GesammelteMathematischeWerke, Leipzig, (1896).

[3] Caputo, M., Elasticita e Dissipazione, Zanichelli, Bologna, (1969).

[4] Miller, K.S. and Ross, B., An introduction to fractional calculus and fractional differential equations, A Wiley, New York, (1993).

[5] Podlubny, I., Fractional Differential Equations, Academic Press, New York, (1999).

[6] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J., Theory and applications of fractional differential equations, Elsevier, Amsterdam, (2006).

[7] Baleanu, D., Guvenc, Z.B. and Machado, T.J.A., New trends in nanotechnology and fractional calculus applications, Springer Dordrecht Heidelberg, London New York, (2010).

[8] Esen, A.,Sulaiman, T.A.,Bulut, H. and Baskonus, H.M., Optical solitons and other solutions to the conformable space-time fractional Fokas-Lenells equation, **Optik,**167, 150-156, (2018).

[9] Prakasha, D. G. and Veeresha, P., Analysis of Lakes pollution model with Mittag-Leffler kernel, **J. Ocean Eng. Sci.,** 1-13, (2020), DOI: 10.1016/j.joes.2020.01.004.

[10] Baleanu, D., Wu, G.C. and Zeng, S.D., Chaos analysis and asymptotic stability of generalized Caputo fractional differential equations, **Chaos Solitons Fractals**, 102, 99-105, (2017).

[11] Veeresha, P., Prakasha, D.G. and Baskonus, H.M., New numerical surfaces to the mathematical model of cancer chemotherapy effect in Caputo fractional derivatives, **Chaos,** 29, (013119), (2019), DOI: 10.1063/1.5074099.

[12] Baskonus, H.M., Sulaiman, T.A. and Bulut, H., On the new wave behavior to the Klein-Gordon-Zakharov equations in plasma physics, **Indian J. Phys.,** 93, (3), 393-399, (2019).

[13] Veeresha, P. and Prakasha, D.G., Solution for fractional generalized Zakharov equations with Mittag-Leffler function, **Results Eng.,** 5, 1-12, (2020), DOI: 10.1016/j.rineng.2019.100085.

[14] Prakasha, D.G., Malagi, N.S. and Veeresha, P., New approach for fractional Schrödinger–Boussinesq equations with Mittag-Leffler kernel, **Math. Meth. Appl. Sci.,**(2020), DOI: 10.1002/mma.6635.

[15] Gao, W., Baskonus, H.M. and Shi, L., New investigation of Bats-Hosts-Reservoir-People coronavirus model and apply to 2019-nCoV system, **Adv. Differ. Equ.,** 391, (2020), DOI: 10.1186/s13662-020-02831-6

[16] Cattani, C. and Pierro, G., On the fractal geometry of DNA by the binary image analysis, **Bull. Math. Biol.,** 75, (9), 1544-1570, (2013).

[17] Gao, W., Veeresha, P., Prakasha, D. G. and Baskonus, H.M., Novel dynamical structures of 2019-nCoV with nonlocal operator via powerful computational technique, **Biology,** 9, (5), (2020), DOI: 10.3390/biology9050107.

[18] Gao, W., Veeresha, P., Baskonus, H.M., Prakasha, D.G. and Kumar, P., A new study of unreported cases of 2019-nCOV epidemic outbreaks, **Chaos Solitons Fractals**, 138, (2020), DOI: 10.1016/j.chaos.2020.109929.

[19] Cattani, C., Haar wavelet-based technique for sharp jumps classification, **Math. Comput. Model.,** 39, (2-3), 255-278, (2004).

[20] Gao, W., Yel, G., Baskonus, H.M. and Cattani, C., Complex solitons in the conformable (2+1)-dimensional Ablowitz-Kaup-Newell-Segur equation, **AIMS Math.,** 5, (1), 507–521, (2020).

[21] Al-Ghafri, K. S. and Rezazadeh, H., Solitons and other solutions of (3+1)-dimensional space–time fractional modified KdV–Zakharov–Kuznetsov equation, **Appl. Math. Nonlinear Sci.,** 4, (2), 289-304, (2019).

[22] Atangana, A., Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system, **Chaos Solitons Fractals,** 102, 396-406, (2017).

[23] Veeresha, P. and Prakasha, D. G., Novel approach for modified forms of Camassa–Holm and Degasperis–Procesiequations using fractional operator, **Commun. Theor. Phys.** 72, 105002, (2020).

[24] Dananea, J., Allalia, K. and Hammouch, Z., Mathematical analysis of a fractional differential model of HBV infection with antibody immune response, **Chaos Solitons Fractals**, 136, 109787, (2020).

[25] Kiran, M.S., et al., A mathematical analysis of ongoing outbreak COVID-19 in India through nonsingular derivative, Numer. Methods Partial Differ. Equ., (2020), DOI: 10.1002/num.22579.

[26] Ali, K.K., Osman, M.S., Baskonus, H.M., Elazab, N.S. and Ilhan, E., Analytical and numerical study of the HIV-1 infection of CD4+T-cells conformable fractional mathematical model that causes acquired immunodeficiency syndrome (AIDS) with the effect of antiviral drug therapy, **Math. Meth. Appl. Sci.,** (2020), DOI:10.1002/mma.7022, 2021.

[27] Gao, W., Veeresha, P., Prakasha, D.G. and Baskonus, H.M., New numerical simulation for fractional Benney–Lin equation arising in falling film problems using two novel techniques, **Numer Methods Partial Differ Equ.**, 37, (1), 210-243, (2021).

[28] Veeresha, P., Prakasha, D.G., Kumar, D., Fractional SIR epidemic model of childhood disease with Mittag-Leffler memory, **Fractional Calculus in Medical and Health Science**, 229-248, (2020).

[29] Jothimani, K., Valliammal, N. and Ravichandran, C., Existence result for a neutral fractional integro-differential equation with state dependent delay, **J. Appl. Nonlinear Dyn.,** 7, (4), 371-381, (2018).

[30] Ravichandran, C., Jothimani, K., Baskonus, H.M. and Valliammal, N., New results on nondensely characterized integrodifferential equations with fractional order, **Eur. Phys. J. Plus,** 133, (2018), DOI: 10.1140/epjp/i2018-11966-3.

[31] Veeresha, P., Prakasha, D.G. and Baskonus, H.M., An efficient technique for coupled fractional Whitham-Broer-Kaup equations describing the propagation of shallow water waves, **Advances in Intelligent Systems and Computing**, 49-75, (2020).

[32] Subashini, R., Jothimani, K., Nisar, K.S. and Ravichandran, C., New results on nonlocal functional integro-differential equations via Hilfer fractional derivative, **Alexandria Eng. J.,** 59, (5), 2891-2899, (2020).

[33] Ismael, H.F., Bulut, H. and Baskonus, H.M., Optical soliton solutions to the Fokas–Lenells equation via sine-Gordon expansion method and $(m+(G'/G))$-expansion method, **Pramana - J Phys.,** 94, 35, (2020), DOI: 10.1007/s12043-019-1897-x.

[34] Veeresha, P., Prakasha, D.G., Singh, J., A novel approach for nonlinear equations occurs in ion acoustic waves in plasma with Mittag-Leffler law, **Eng. Comput.,** 37, (6), 1865-1897, (2019).

[35] Ravichandran, C., Logeswari, K., Panda S.K., Nisar, N.S., On new approach of fractional derivative by Mittag-Leffler kernel to neutral integro-differential systems with impulsive conditions, **Chaos Solitons Fractals**, 139, (2020), DOI: 10.1016/j.chaos.2020.110012.

[36] Silambarasan, R, et al., Longitudinal strain waves propagating in an infinitely long cylindrical rod composed of generally incompressible materials and it’s Jacobi elliptic function solutions, **Math. Comput. Simul.,** 182, 566-602, (2021).

[37] Merlani, A. L., Natale, G. and Salusti, E., On the theory of pressure and temperature nonlinear waves in compressible fluid-saturated porous rocks, **Geophys. Fluid Dyn.**, 85, 97-128, (1997).

[38] Akhmetov, A.A., Long current loops as regular solutions of the equation for coupling currents in a flat two-layer superconducting cable, **Cryogenics**, 43, 317-322, (2003).

[39] Manolis, G.D. and Rangelov, T.V., Non-homogeneous elastic waves in solid: notes on the vector decomposition technique, **Soil Dynam. Earthquake Engrg.,** 26, 952-959, (2016).

[40] Holliday, J.R., Rundle,J. B., Tiampo, K.F., Klein, W. and Donnellan, A., Modification of the pattern informatics method for forecasting large earthquake events using complex eigenfactors, **Tectonophysics**, 413, 87-91, (2006).

[41] Momani, S., Analytical approximate solution for fractional heat-like and wave-like equations with variable coefficients using the decomposition method, **Appl. Math. Comput.,** 165, 459-472, (2005).

[42] Singh, J. and Kumar, D., An application of homotopy perturbation transform method to fractional heat and wave-like equations, **J. Fract. Calc. Appl.,** 4, (2), 290-302, (2013).

[43] Ozis, T. and Agirseven, D., He's homotopy perturbation method for solving heat-like and wave-like equations with variable coefficients, **Phys. Letters A,** 372, 5944-5950, (2008).

[44] Wazwaz, A.M. and Gorguis, A., Exact solutions for heat-like and wave-like equations with variable coefficients, **Appl. Math. Comput.,** 149, 15-29, (2004).

[45] Adomian, G., A new approach to nonlinear partial differential equations, **J. Math. Anal. Appl.,** 102 (1984), 420-434.

[46] Khan, Z. H. and Khan, W. A., N-Transform-properties and applications, **NUST J. Engg. Sci.,** 1, (1), 127-133, (2008).

[47] Rawashdeh, M.S., The fractional natural decomposition method: theories and applications, **Math. Meth. Appl. Sci.,** 40, 2362-2376, (2017).

[48] Rawashdeh, M.S. and Maitama, S., Finding exact solutions of nonlinear PDEs using the natural decomposition method, **Math. Meth. Appl. Sci.,** 40, 223-236, (2017).

[49] Prakasha, D.G., Veeresha, P. and Rawashdeh, M.S., Numerical solution for $(2+1)$-dimensional time-fractional coupled Burger equations using fractional natural decomposition method, **Math. Meth. Appl. Sci.,** 42, (10), 3409-3427, (2019).

[50] Veeresha, P., Prakasha, D.G. and Singh, J., Solution for fractional forced KdV equation using fractional natural decomposition method, **AIMS Math.**, 5, (2), 798-810, (2019).

[51] Rawashdeh, M.S., Solving fractional ordinary differential equations using FNDM, **Thai J. Math.,** 17, (1), 239–251, (2019).

[52] Prakasha, D.G., Veeresha, P. and Baskonus, H.M., Two novel computational techniques for fractional Gardner and Cahn-Hilliard equations, **Comp. Math. Meth.,** 1, (2), 1-19, (2019), DOI: 10.1002/cmm4.1021.

[53] Veeresha, P. and Prakasha, D.G. An efficient technique for two-dimensional fractional order biological population model, **Int. J. Model. Simul. Sci. Comput.,** (2050005), 1-17, (2020), DOI: 10.1142/S1793962320500051.

[54] Mittag-Leffler, G. M., Sur la nouvelle fonction $E_\alpha(x)$, **C. R. Acad. Sci. Paris**, 137, 554-558, (1903).

[55] Loonker, D. and Banerji, P. K., Solution of fractional ordinary differential equations by natural transform, **Int. J. Math. Eng. Sci.,** 12, (2), 1-7, (2013).