Auto correlation analysis of the light curve of Z Ursa Majoris

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Özet : Otokorelasyon metodu Z Ursa Majorisin ışık eğrisine tatbik edildi. Korelogram, peryodu 195 gün civarında olan bir sönümlü oksilâsyondur. Oksilâsyon simetrik değildir ve ışık eğrisi ikinci mertebeli bir otoregrasyon zinciri ile tarif edilemez. Işık eğrisinin kuvvet spektrumu otokorelasyon fonksiyonuna Fourier transformu yaparak elde edilir. Kuvvet spectrumu, müteakiben, 195, 100 ve 70 inci günlerde üç tepe gösterir. Bu, uyartma stochastik ise, esas tondan başka birinci ve mühtemelen üst tonların mevcut olduğunu ifade eder; böylece, ν; 2ν 3ν frekanslı bir oskilasyon serisi verir,

Peryod büyük sıhhatle tayin edilemez. Üst tonların mevcudiyetine, daha başka ışık eğrileri, ayni şekilde incelenmeden, katiyetle tesbit edilmiş olarak nazarı itibare alınmaması lâzımdır.

§ 1. Introduction.

A powerful method for the analysis of time series has been developed by Kendall (1). In their careful statistical analysis of the light curve of μ Cephei, Ashbrook, Duncombe and van Woerkom (2) have succesfully applied his autocorrelation analysis. Ashbrook, Duncombe and van Woerkom found the light curve of μ Cephei to result from stochastic rather than harmonic processes while it could be represented by a second order auto-regressive chain. At the same time from the method applied by them, they were able to show that the four periods, which on the basis of a Schuster periodogram had been assigned to μ Cephei, were illusory.

The light curve is not explainable by a simple pulsation; instead it may be interpreted as arising from temporary, random surface disturbances on the star. These same authors finally call attention to the interesting possibility of applying the autocorrelation method to the analysis of light curves of Mira variables.

The result might be the detection of harmonic terms. Kendall (1) shows that the correlogram of a series, generated by moving averages, may oscillate but will vanish after a certain point; that of a series of harmonic terms will oscillate, but will not vanish or be damped ; that of the autoregressive scheme will oscillate and will not vanish, but it will be damped. Consequently, although the three types of series he considers are very similar to the eye, the correlogram offers a theoretical basis for discriminating between the three types of oscillatory series. As a result of the suggestions made by Ashbrook, Duncombe and van Woerkom, in the present paper the autocorrelation method is applied to the light curve of Z Ursa Majoris.

§ 2. The light curve of Z Ursa Majoris.

The requirement of extent, continuity and reasonable homogenity for the light curve to be analysed led to the selection of Z Ursa Majoris. Since the autumn of 1931 this semiregular variable has been under continuous observations by the members of the section of variable star observers of the Scandinavian Astronomical Society. The total number of observations made prior to 1950.5 amounts to about 6000. A summary of these observations is given by Axel Nielsen (3) who derives normal points through which the light curve is drawn. Nielsen finds great differences in the course of the light variations He finds that during the interval of time 1931.5 - 42.0 the range of variation repeatedly exceeded 1.0^m5 or even $2.^m0$, but during the first interval the variation was a regular one or at any rate semiregular, but during the second interval any trace of regularity disappeared. Finally during the interval 1949.0 - 50.5 the range again increased and a trace of the old periodicity appeared.

The autocorrelation method to be used requires equidistant observations. Therefore I have derived by interpolation 475 magnitudes of Z UMa at 10 days intervals from J.D 2428720 to J.D 2433460. This interval of length was chosen because it is conveniently short in comparison with the period of 198.0 days indicated by Nielsen.

In the computations it is convenient to use the deviations from the overall mean

$$u_t = m_t - 7.72$$

which are tabulated in table 1. There is no clear-cut secular trend in the series as may be seen from the consecutive groups given in table 2.

§ 3. The correlogram.

For a time series with equidistant terms $u_1, u_2, n_3, \ldots u_N$ for which $\Sigma u_t = 0$; the autocorrelation coefficient r_K of the series is a measure of the correlation of the series with itself, after a displacement of k steps in time. It is defined by

$$r_K = \frac{\Sigma u_i u_{i+K}}{(\Sigma u_i^2 \cdot \Sigma u_{i+K}^2)^{\frac{1}{2}}} \quad \cdots\cdots (1)$$

where all summations are from 1 to N — k. For practical computations (4) the convenient approximation

$$r_K = \frac{\Sigma u_i \cdot u_{i+K}}{\Sigma u_i^2 - k\overline{u_i^2}} \quad \cdots\cdots (2)$$

was used. This approximation is adequate if $N \gg K$. The correlogram of the series is the plot of r_K against the lag of k and from examination of this plot the existance of a periodicity P in the time series can be detected. When the lag of k is zero, eq. 2 obviously reduces to $r_0 = 1$,

From the values u_t, observed for Z UMa and as enumerated in table 1, the correlation coefficients r_K were computed. Up till $k = 30$, table 3 gives the numerical value of r_K for all consecutive values k. Between $k = 30$ and $k = 60$ only coefficients r_K for even values of k were computed, except near maxima of the correlogram, where a few additional values r_K have been determined. Between $k = 60$ and $k = 82$ only some significant values of r_K were computed so as to determine the general shape of the correlogram between $k = 60 - 82$. In this interval also near the maxima a few extra coefficients r_K were computed. The numerical results of table 3 are graphically represented in fig. 1, where the values r_K are plotted againt k.

From figure 1 it is evident that the correlogram is a damped oscillation. The maxima are around $k = 19$; $k = 39$; $k = 59$ and $k = 79$ indicating a period of oscillation of about $19.5 \times 10 = 195$ days. This is very near the value indicated by Nielsen.

TABLE I.

Values of $u_t = m_t - 7{\cdot}72$ for Z Ursa Maj., at 10 day intervals beginning with J. D 1428720 (unit 0.01 magnitudes)

+ 92	+ 13	+ 31	+ 53	− 7	+ 15	+ 10	− 1	+ 4	− 22
+103	0	+ 14	+ 28	+ 8	+ 9	+ 4	+ 6	− 12	− 4
+ 93	+ 5	− 37	+ 1	+ 13	+ 8	− 7	+ 13	− 26	+ 10
+ 51	+ 25	− 77	− 20	− 11	+ 31	− 7	+ 15	− 29	+ 18
+ 9	+ 44	− 80	− 35	− 37	0	− 4	+ 13	− 28	+ 28
− 22	+ 56	− 77	− 40	− 41	− 14	+ 3	+ 2	− 32	+ 46
− 42	+ 62	− 39	− 41	− 37	− 23	− 9	− 7	− 38	+ 63
− 35	+ 61	+ 23	− 34	− 17	− 35	− 20	− 6	− 37	+ 75
− 10	+ 62	+ 59	− 30	− 2	− 40	− 31	+ 5	− 31	+ 79
+ 11	+ 66	+ 76	− 26	− 9	− 35	− 27	+ 5	− 27	+ 62
+ 16	+ 61	+ 85	− 23	− 11	− 26	− 8	+ 5	− 32	+ 38
0	+ 29	+ 68	− 27	− 12	− 25	+ 13	+ 4	− 33	+ 13
− 17	− 8	+ 30	− 45	− 18	− 19	+ 25	− 2	− 19	− 48
− 20	− 36	− 1	− 42	− 22	− 8	+ 30	− 19	+ 8	− 70
+ 3	− 60	− 10	− 31	− 23	0	+ 24	− 35	+ 19	− 71
0	− 58	+ 4	− 7	− 36	+ 9	+ 3	− 40	+ 16	− 66
+ 15	− 52	+ 32	+ 45	− 42	+ 33	− 19	− 26	+ 6	− 50
+ 46	− 35	+ 45	+ 59	− 55	+ 37	− 21	− 12	+ 7	− 27
+ 70	+ 6	+ 39	+ 66	− 42	+ 24	− 15	− 6	+ 32	− 17
+ 85	+ 35	+ 28	+ 59	− 23	+ 15	− 16	− 15	+ 42	− 12
+ 87	+ 46	− 8	+ 38	− 4	+ 7	− 23	− 27	+ 46	− 4
+ 85	+ 55	− 47	− 11	+ 16	+ 7	− 30	− 16	+ 53	+ 15
+ 41	+ 49	− 61	− 33	+ 36	+ 14	− 31	+ 9	+ 41	+ 30
− 31	+ 38	− 56	− 38	+ 33	+ 6	− 29	+ 26	+ 26	+ 67
− 80	+ 27	− 31	− 32	+ 33	+ 5	− 26	+ 32	+ 18	+ 83
− 69	+ 20	− 10	− 34	+ 32	− 3	− 22	+ 19	+ 9	
− 58	+ 32	+ 21	− 47	+ 31	− 7	− 16	− 3	− 22	
− 40	+ 53	− 4	− 57	+ 17	− 4	− 12	+ 1	− 52	
+ 7	+ 66	+ 89	− 45	+ 3	+ 4	+ 3	+ 8	− 53	
+ 21	+ 61	+ 82	− 36	+ 3	+ 15	+ 14	+ 5	− 49	
+ 20	+ 39	+ 59	− 28	+ 16	+ 14	+ 15	− 16	− 45	
+ 13	+ 11	− 1	− 25	+ 19	+ 3	+ 14	− 20	− 26	
− 1	− 44	− 30	− 20	+ 13	− 4	+ 13	− 20	+ 4	
− 8	− 74	− 38	− 23	− 5	− 13	+ 14	− 19	+ 19	
− 11	− 86	− 45	− 25	− 31	− 12	+ 16	− 18	+ 20	
0	− 84	− 38	− 20	− 60	− 2	+ 8	− 16	+ 23	
+ 21	− 53	− 21	− 3	− 62	+ 8	− 7	− 17	+ 30	
+ 38	− 5	− 25	+ 9	− 64	+ 20	− 16	− 22	+ 30	
+ 63	+ 32	− 41	+ 7	− 64	+ 10	− 15	− 26	+ 25	
+ 88	+ 58	− 44	− 13	− 51	+ 2	− 5	− 34	+ 28	
+ 76	+ 77	− 63	− 37	− 20	− 5	− 4	− 20	+ 36	
+ 47	+ 82	− 72	− 46	− 4	− 3	− 4	− 3	+ 23	
− 12	+ 79	− 64	− 41	+ 38	+ 4	+ 3	0	− 2	
− 47	+ 39	− 60	− 41	+ 45	+ 14	+ 2	+ 15	− 35	
− 50	+ 3	− 8	− 40	+ 47	+ 18	− 4	− 22	− 54	
− 40	− 34	+ 23	− 40	+ 46	+ 23	− 11	− 35	− 66	
− 27	+ 16	+ 48	− 37	+ 52	+ 34	− 7	− 29	− 70	
+ 6	+ 29	+ 62	− 39	+ 64	+ 33	+ 1	− 30	− 66	
+ 21	+ 44	+ 92	− 30	+ 38	+ 25	− 7	+ 8	− 53	
+ 22	+ 43	+ 86	− 18	+ 24	+ 19	− 9	+ 11	− 36	

The period of the correlogram is not a period in a strict sense (5). The lengths from peak to peak vary in a characteristic way. The distribution of the distances from peak to peak is of the unimodal type with a central value somewhere near the mean distance between peaks. With the correlogram of Z UMa however the distances from peak to peak are nearly equal.

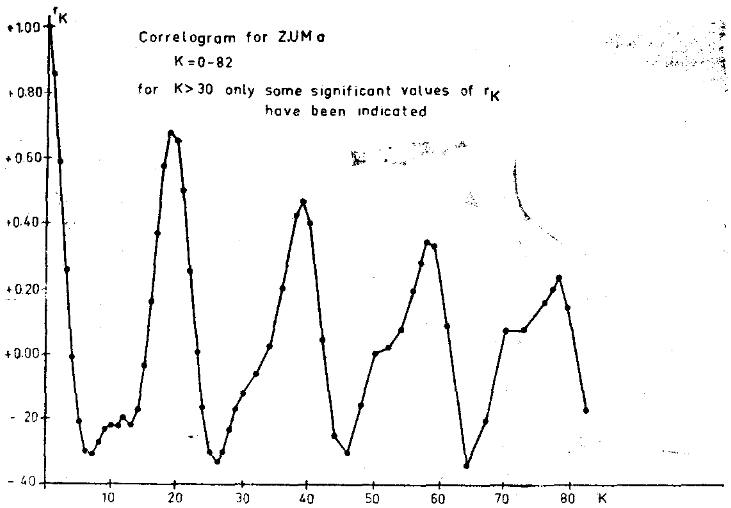


Fig. 1.

Quite obviously the oscillations of the correlogram of Z UMa are damped, though the damping is considerably less than that which Ashbrook, Duncombe and van Woerkom found from the light curve of μ Cephei. Therefore it is tempting to try in this case also to represent the light curve by a second order autoregression equation for which the recurrence formula is

$$u_{t+2} = au_{t+1} + bu_t + \varepsilon_{t+2} \qquad \dots (3)$$

The correlogram then will be of the shape

$$r_K = \frac{p^K \sin(k\theta + \psi)}{\sin \psi} \qquad \dots (4)$$

It will oscillate with period $2\pi/\theta$ but owing to the factor p^K it will be damped. However, the correlogram in figure 1 is not a damped sinus curve. There is some resemblance to a sinus curve but the oscillations are non symmetrical.

On the ascending part of the oscillations a hump occurs of which the position seems to be variable. This gives rise to the suspicion that a second oscillation of lesser amplitude may be present. A disadvantage of the use of correlograms is that a secondary periodicity may be masked, if shorter than the dominant periodicity.

§ 4. The correlogram from $k = 175 - 200$.

I have considered the possibility that apart from a stochastic there also is a harmonic disturbance. If a harmonic term is present, it would persist in the r_K coefficients through the higher values of k, while the stochastic disturbance would be damped out of existence. The disadvantage is that with the higher values of k the numerical values of r_K are largely determined by sampling errors. Nevertheless I have thought it worth while to compute some of the advanced values of r_K. In table 4 the autocorrelation coefficients r_K for $k = 175$—200 are given and in figure 2 these values r_K are plotted against the corresponding values k. The result however is negative. The oscillation with a period of about 19.5×10 days, though much damped, seems to persist even through these higher values of r_K. There even seems to be an indication that here also the oscillation is not quite symmetrical, but hardly any weight can be attached to this. Anyhow, the extension of the correlogram does not lead to the detection of any harmonic terms. At the same time in this range the weight of the coefficients r_K has diminished to such an extent that from this we may not conclude that no harmonic terms are present.

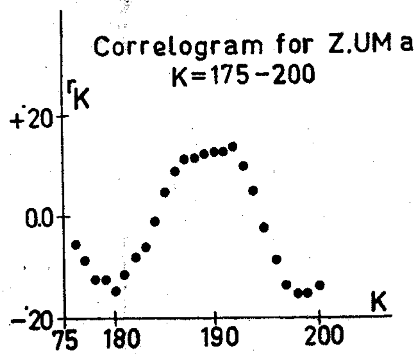


Fig. 2

§ 5. The power spectrum of Z UMa.

While the disadvantage of the correlogram is to mask a secondary periodicity, this disability can be overcome by transfor-

TABLE 2.

Mean values of U_t

t	
1 — 95	+ .15
96 — 190	— .04
191 — 285	— .06
286 — 380	— .01
381 — 475	— .05
1 — 475	.00

TABLE 3.

Autocorrelation coefficients from the light curve of Z UMa (unit of k is 10 days)

k	r_K	k	r_K	k	r_K
0	+ 1.000	20	+ .662	48	— .148
1	+ .865	21	+ .498	50	+ .012
2	+ .592	22	+ .264	52	+ .027
3	+ .264	23	+ .011	54	+ .079
4	— .011	24	— .156	56	+ .205
5	— .214	25	— .302	57	+ .288
6	— .301	26	— .332	58	+ .346
7	— .315	27	— .306	59	+ .342
8	— .270	28	— .234	61	+ .093
9	— .233	29	— .167	64	— .341
10	— .222	30	— .120	67	— .206
11	— .215	32	— .058	70	+ .075
12	— .188	34	+ .031	73	+ .075
13	— .224	36	+ .212	76	+ .157
14	— .174	38	+ .433	77	+ .202
15	— .032	39	+ .472	78	+ .237
16	+ .164	40	+ 404	79	+ .144
17	+ .385	42	+ .051	82	— .167
18	+ .580	44	+ .248		
19	+ .685	46	— .302		

TABLE 4.

Autocerrelation coefficients r_K (K = 175 — 200)

k	r_K		k	r_K
176	— .056		189	+ .125
177	— .095		190	+ .131
178	— .126		191	+ .132
179	— .130		192	+ .134
180	— .151		193	+ .107
181	— .118		194	+ .050
182	— .100		195	— .019
183	— .061		196	— .089
184	— .012		197	— .135
185	+ .046		198	— .146
186	+ .086		199	— .151
187	+ .107		200	— .141
188	+ .115			

ming the correlogram into the corresponding power spectrum (6). By Kintchine's theorem the power spectrum is the Fourier transform of the autocorrelation function (7). The power spectrum gives the mean squared amplitude of each frequency. In the present case it has been computed from the relation :

$$\Pi(f) = \frac{2}{N} \sum_{k=0}^{k=25} r_K \cos 2\pi f k \qquad \cdots\cdots (5)$$

where f is the reciprocal of the trial period The computations can be limited to $k = 1$ .... 25 because of the rapidly diminishing weight for higher k's. The values $\Pi(f)$ as computed from eq. (5) appear in table 5 while the graph appears in figure 3.

The peak of the power spectrum indicates a mean cycle of approximately $19.5 \times 10 = 195$ days, but it is now at once apparent that a secondary period is also present with a mean cycle of approximately $10 \times 10 = 100$ days, probably slightly shorter than 100 days.

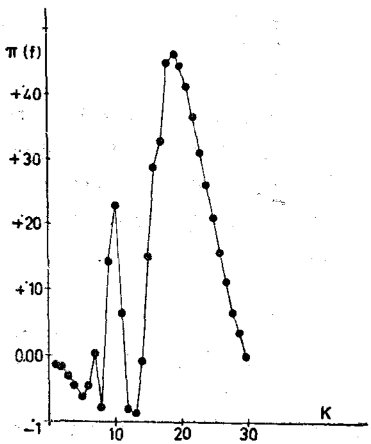
The way in which we have determined the length of the

TABLE 5.

The power spectrum of Z UMa. as computed from eq (5). The unit of trial period f is 10 days.

Trial period f	$\Pi(f)$	Trial period f	$\Pi(f)$	Trial period f	$\Pi(f)$
1	— .016	11	+ .064	21	+ .415
2	— .018	12	— .081	22	+ .368
3	— .034	13	— .089	23	+ 316
4	— .043	14	— .004	24	+ .262
5	— .065	15	+ .152	25	+ .217
6	— .042	16	+ .290	26	+ .159
7	+ .008	17	+ .330	27	+ .116
8	— .081	18	+ .451	28	+ .074
9	+ .147	19	+ .464	29	+ .039
10	+ .235	20	+ .445	30	+ .003

mean cycles is not a very refined one, but it is tempting to describe this power spectrum as being generated by a stochastic disturbance with a mean cycle of approximately 195 days and its first and perhaps even its second overtone (the third peak at approximately $f = 7$), the frequencies being ν, 2ν and 3ν. From a purely physical point of view such an explanation does not seem impossible, but such a far reaching conclusion must be considered as provisional until some additional light curves have been inspected.



Power Spedrum of Z.UM a light curve
Abcissae K, ordinates π(f).

Fig. 8

Also it will be necessary more closely to inspect the theory. The series r_K had to be broken off at a rather small value k. Owing to this the power spectrum may throw up some spurious periodicities. Anyhow, these results bear out the conclusion reached by Ashbrook, Duncombe and van Woerkom, viz, that interesting results may be expected if the autocorrelation method is applied to the Mira variables and we may add perhaps to other stars also.

To explain the idea underlying the autoregressive scheme, Kendall (8) gives the following analogy. Imagine a motorcar proceeding along a horizontal road with an irregular surface. The springs permit the car to oscillate to some extent, but are designed to damp out the oscillations If the car strikes a bump or a pothole in the road, the body will oscillate for a time but will soon come to rest so far as vertical motion is concerned. If, however, the car proceeds over a continual succession of bumps, there will be a continual oscillation of varying amplitude and distance between peaks The oscillations are continually renewed by disturbances, though the distribution of the latter may be quite random. The *regularity* of the motion is determined by the internal structure of the car, but the existence of the motion is determined by external impulses.

In a similar way we can imagine that the existence of oscillations in the atmosphere of the star is determined by impulses from the interior. The regularity of the oscillations is determined by the structure of the atmosphere and in this case especially this structure might be such that the first and second overtones of the oscillation are not negligible.

§ 6. The generating function.

The general solution (9) of a second order regressive equation, that is if the difference equation (3) is

$$u_t = p^t \frac{\sin(\theta t + \Psi)}{\sin \Psi} + \sum_{j=0}^{\infty} \xi_\gamma \varepsilon_{t-\gamma+1} \qquad \ldots . \ (6)$$

where

$$\xi_\gamma = \frac{2}{\sqrt{(4b - a^2)}} p^t \sin \theta t$$

The second term in eq. (6) represents the cumulated effect of the random terms.

If overtones are present, the series cannot longer be represented by a second order autoregressive scheme. This can be ascertained by determining the numerical values of a and b from a least squares solution of the first 20 conditional equations of the form

$$a\, r_{K+1} + b\, r_K = r_{K+2}$$

Writing the solution 6 in the form

$$u_t = p^K(A \cos \theta K + B \sin \theta K) \; ; \quad p = \sqrt{-b} \; ; \; \cos \theta = \frac{a}{2\sqrt{-b}} \quad (7)$$

we find

$a = + 1,73$; $b = - 0,85$; $\theta = 19°35'$; $P = 2\pi/\theta = 18,3$ and $p = 0,92$

The adequacy of the description can be assessed by a χ^2 test (10). For χ^2 the value 3,14 is obtained and the probability that the residuals of r_K are due to chance is only 0,074.

The simple second order scheme is therefore quite inadequate. When overtones are present, the general form of the term u_t would be

$$u_t = A_1 p_1{}^t \frac{\sin(\theta t + \Psi_1)}{\sin \Psi_1} + A_2 p_2{}^t \frac{\sin(\frac{1}{2}\theta t + \Psi_2)}{\sin \Psi_2} +$$

$$+ A_3 p_3{}^t \frac{\sin(\frac{1}{3}\theta t + \Psi_3)}{\sin \Psi_3} + \varepsilon'_t = \sum_1^n A_n p_n{}^t \frac{\sin(\frac{1}{n}\theta t + \Psi_n)}{\sin \Psi_n} + \varepsilon'_t \quad (8)$$

where ε'_t represents the cumulated effect of the random terms. The coefficients A may be expected to decrease rapidly with n increasing.

Still the expression will contain a large number of terms which are not negligible and therefore also a correponding large number of coefficients. It certainly will be possible to represent the observed series u_t by a series of the form 8, but it would be difficult accurately to determine the various coefficients. It even seems possible that several series might be constructed which represent the observed series u_t equally well. Such a procedure therefore would be purely formal and no physical meaning could be attached to the coefficients. Consequently, for the present at least I have made no further attempt to determine the shape of the generating function.

§ 7. Summary.

1. A correlogram of the light curve of Z Ursa Majoris has been obtained. The correlogram is a damped oscillation with a period of approximately 195 days between the peaks.

2. The damping is sufficiently small to allow several peaks to be observed. We conclude that the curve results from sto-

chastic rather than harmonic processes, but cannot be described by a second order autoregressive chain.

3. By applying a Fourier transform to the correlogram, the power spectrum is obtained. Apart from a peak at approximately 195 days, there are secondary oscillations of smaller amplitudes with periods of approximately 100 and 70 days respectively. The peak at 70 days is poorly determined.

4. The power spectrum suggests that the oscillations in the atmosphere of the star are determined by impulses from the interior which eventually may be at random. Apart from the fundamental oscillation in the atmosphere of Z Ursa Majoris also tones of frequencies ν, 2ν and 3ν appear to be present.

5. Although for the existence of overtones the evidence seems rather conclusive, these results should be considered as tentative ones until additional light curves have been analysed in a similar way and evidence has been obtained that such overtones cannot result from the breaking off of the series r_K at a low value K.

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