SOME RESULTS ON SEVERAL NUMERICAL $P_{+53}$ SETS

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Abstract:

Diophantine set theory has an important role in Mathematics. In this paper, we consider prime number $p=+53$ and give some Diophantine $P_{+53}$ triples. Some of the such sets are extended but others not. We give several of them with proofs. Also, some types of elements of the Diophantine $P_{+53}$ $m$-tuples are determined. One can be work on other Diophantine $P_{+53}$- $m$-tuples and discover extendibility of them.

Keywords: Diophantine $P_{s}$ 3-Tuple, Number Theory, Pell Equations, Elements of Diophantine $P_{s}$ $m$-tuples, Quadratic Reciprocity Law.

Introduction and Preliminaries

To obtain proofs of our main results we need following definitions, lemmas, theorems and so on... All of the following informations are found in the references [1-25].

1. Let $n$ be an non-zero integer. A set of $m$ positive integers

$$\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$$

such that $\alpha_i \alpha_j + n$ is a perfect square for all $1 \leq i < j \leq m$ is called $\alpha$ Diophantine $m$-tuple with the property $D(n)$.

2. Let $p$ be an odd prime and let $a$ be an integer. The Legendre symbol of $a$ with respect to $p$ is defined by
\[
\left( \frac{\alpha}{p} \right) = \begin{cases} 
1 & \text{if } \alpha \text{ is a quadratic residue modulo } p \text{ and } \alpha \not\equiv 0 \pmod{p} \\
-1 & \text{if } \alpha \text{ is a quadratic non-residue modulo } p \\
0 & \text{if } \alpha \equiv 0 \pmod{p}.
\end{cases}
\]

(a) \( \left( \frac{-1}{p} \right) = (-1)^{\frac{p - 1}{2}} \), so it is 1 if and only if \( p \equiv 1 \pmod{4} \).

(b) \( \left( \frac{2}{p} \right) = (-1)^{\frac{p^2 - 1}{8}} \) for an odd prime \( p \), so it is 1 if and only if \( p \equiv \pm 1 \pmod{8} \).

3. **Law of Quadratic Reciprocity** is given by
\[
\left( \frac{p}{q} \right) \left( \frac{q}{p} \right) = (-1)^{\frac{p - 1}{2} \cdot \frac{q - 1}{2}},
\]
where \( p \) and \( q \) are odd prime numbers, and \( \left( \frac{p}{q} \right) \) denotes the Legendre symbol.

**Note:** (Extension of the law of quadratic reciprocity) If \( m \) and \( n \) are coprime positive odd integers,
\[
\left( \frac{m}{n} \right) \left( \frac{n}{m} \right) = (-1)^{\frac{m - 1}{2} \cdot \frac{n - 1}{2}}.
\]

**Main Results**

**Theorem 1.** \( P_{+53} = \{11, 13, 52\} \) Diophantine triple cannot be extended to Diophantine \( P_{+53} \) quadruple.

**Proof.**

Assume that \( d \) is in the set of Diophantine \( P_{+53} \) set. So, we obtain following result from the definition of Diophantine \( P_{+53} \) set.
\[
\{11, 13, 52, d\} \rightarrow (1) \quad 11d + 53 = x^2 \\
(2) \quad 13d + 53 = y^2 \\
(3) \quad 52d + 53 = z^2.
\]
These (2) and (3) equations imply that \( z^2 - 4y^2 = -159 \). Table 1 gives us integer solutions of the equation as follow:
Table 1. $z^2 - 4y^2 = -159$

<table>
<thead>
<tr>
<th>$(z, y)$</th>
<th>$(z, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\pm 40, \pm 79)$</td>
<td>$(\pm 14, \pm 25)$</td>
</tr>
</tbody>
</table>

From the (1) and (2), we get,

$$13x^2 - 11y^2 = 2.53 \implies 13x^2 - 11y^2 = 106$$

And also integers solutions of the $13x^2 - 11y^2 = 106$ can be given as Table 2.

Table 2. $13x^2 - 11y^2 = 106$

<table>
<thead>
<tr>
<th>$(x, y)$</th>
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<th>$(x, y)$</th>
<th>$(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\pm 1125, \pm 1223)$</td>
<td>$(\pm 597, \pm 649)$</td>
<td>$(\pm 47, \pm 51)$</td>
<td>$(\pm 25, \pm 27)$</td>
<td>$(\pm 3, \pm 1)$</td>
</tr>
</tbody>
</table>

If we compare Table 1 and Table 2, we obtain that there is no common integer solution for the system of pell equations. So, $\{11,13,52\}$ can not be extended to Diophantine $P_{+53}$ Quadruple.

Theorem 2. Diophantine $P_{+53} = \{4,119,169\}$ Triple can not be extended to $P_{+53}$ Quadruple.

**Proof:** Let us consider Diophantine $P_{+53} = \{4,119,169\}$. If $d$ is an element of the such property set, then it is written by Diophantine $\{4,119,169, d\}$ 4-tuples. Then we obtain following results

$$\left\{
\begin{array}{l}
(1) 4d + 53 = A^2 \\
(2) 119 + 53 = B^2 \\
(3) 169d + 53 = C^2 \\
\end{array}
\right.$$

From (1) and (3), it is obtained that

$$169 / 4d + 53 = A^2$$
$$-4 / 169d + 53 = C^2$$
$$\Rightarrow 169A^2 - 4C^2 = 165.53$$
\[ 169A^2 - 4C^2 = 8745 \]  
\[ 119A^2 - 4B^2 = 6095 \]

From (1) and (2), we get:
\[ 119A^2 - 4B^2 = 115.53 \Rightarrow 119A^2 - 4B^2 = 6095 \]

For (4) and (5), we have Table 3 and Table 4 include integer solutions.

**Table 3.** $169A^2 - 4C^2 = 8745$

<table>
<thead>
<tr>
<th>$(A, C)$</th>
<th>$(A, C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\pm 31, \pm 196)$</td>
<td>$(\pm 23, \pm 142)$</td>
</tr>
</tbody>
</table>

**Table 4.** $119A^2 - 4B^2 = 6095$

<table>
<thead>
<tr>
<th>$(A, B)$</th>
<th>$(A, B)$</th>
<th>$(A, B)$</th>
<th>$(A, B)$</th>
<th>$(A, B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\pm 1389, \pm 7576)$</td>
<td>$(\pm 531, \pm 2896)$</td>
<td>$(\pm 37, \pm 198)$</td>
<td>$(\pm 27, \pm 142)$</td>
<td>$(\pm 19, \pm 96)$</td>
</tr>
</tbody>
</table>

From Table 3 and Table 4, we can not get common integer solutions for (3) and (4). So, \{4, 119, 169\} can not be extended.

**Theorem 3.** $P_{+53}^+ = \{4, 169, 227\}$ cannot be extendable to Diophantine $P_{+53}^+$ quadruple.

**Proof.** It is proven like previous proofs of the theorems.

**Theorem 4.** There is no elements in the set of Diophantine $P_{+53}^m$-tuples if they are written by three fold or five fold or thirtyone fold or fortyone fold or thirtynine fold, so on...

**Proof.**

(a) Assume that $3k$ ($k \in \mathbb{Z}^+$) is in the set of Diophantine $P_{+53}$ m-tuples. So, following equation have solution:

\[ 3k \cdot s + 53 = x^2 \]

for $s \in P_{+53}$ m-tuples. It implies that

\[ x^2 = 2 \mod 3. \]

This congruents can solvable if \( \left( \frac{2}{3} \right) = +1 \) but \( \left( \frac{2}{3} \right) = (-1)^{\frac{a-1}{8}} = (-1) \).

This implies that $3 \notin$ Diophantine $P_{+53}$ m-tuples.
(b) Suppose that \(31r (r \in \mathbb{Z}^+)\) is an element of the Diophantine \(P_{+53}\) m-tuples. Then, we obtain following equation from the definition of the Diophantine \(P_{+53}\) m-tuples.

\[
31r. u + 53 = A^2 \quad \exists \quad u \in \text{Diophantine } P_{+53} \text{ m-tuples.}
\]

It implies that

\[
A^2 \equiv 22 \pmod{31} \text{ solvable } \iff \left(\frac{22}{31}\right) = 1 \quad (?)
\]

\[
\left(\frac{22}{31}\right) = \left(\frac{2}{31}\right) \cdot \left(\frac{11}{31}\right) \quad \text{and from Quadratic reciprocity;}
\]

\[
\left(\frac{11}{31}\right) \cdot \left(\frac{31}{11}\right) = (-1)^{\frac{11-1}{2}} \cdot \frac{31-1}{2} \quad \Rightarrow \quad \left(\frac{11}{31}\right) = -1
\]

\[
\left(\frac{2}{31}\right) = (-1)^{\frac{31^2-1}{8}} = (-1)^{120} = +1 \quad \text{then } 31r \notin \text{ Diophantine } P_{+53} \text{ m-tuples.}
\]

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References


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