On the extension of the Savary construction to spherical surfaces and line space.

by

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Özet: Küre kinematiğini özet halinde ifade eden ilk üç paragraf bu araştırmanın hem mahiyetini aydınlatmakta hem de temelini teşkil etmektedir. Biz önce Savary inşasının ve özel halinin düzlem kinematikdeki karşılıklarını küre yüzeyi üzerinde tetkik ettik. Sonra bu yapılanları duallere teşmil ederek Savary inşasının çizgiler uzayındaki karşılığını inceledik. Bu problem L. Biran'ın da ifade ettiği gibi «sabit bir K′ sistemine göre hareketli katı bir K cisminin K/K′ hareketinin P âni ekseni K′ ve K da (P′) ve (P) regle pol yüzeylerini tevlit eder. Bu iki regle yüzeyin B = (P_1, P_2, P_3) müşterek asal üçyüzlüsünün K′ ve K ya göre âni eksenleri M′ ve M olsun. M′ ve M bilindiğine göre K nın sabit bir X doğrusunun tevlit ettiği regle yüzeyin (X_1, X_2, X_3) aaal üçyüçlüsünün X′ âni eksenini çizmek» den ibarettir.

X ile P_2 bir birini dik kesiyorlarsa bu çizim artık doğru değildir. Fakat takip ettiğimiz metod bu özel halin çizimini de mümkün kılmaktadır.

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Summary: This paper is comprised of two parts. § 1—§ 3 is the summary of the essentials of the Spherical Kinematics which gives the fundamental preparations for the studies in § 4—§ 5.

In § 4 efforts have been made to demonstrate the reciprocal of the Savary construction and its special case on the surface of a sphere.

In § 5 on the basis of Study's [1] Principle of correspondence all the results of § 4 have been extended to the line space.

Although the extension of the Savary construction to the ruled surfaces has already been done by L. Biran [2] a different method has been used here and also the special case of the construction is included,

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§ 1 — Spherical motion.

Let K, K′ bet two concentric spherical surfaces which are not stationary with respect to one to another, refered to rectangular trihedrals $\{0, \vec{e_i}\}$ and $\{0, \vec{e'_i}\}$. Let the first trihedral attached to the moving sphere K and the second to the fixed

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sphere K′. The conditions for orthogonality and norming are :

$$\vec{e}_i\,\vec{e}_j = \vec{e}'_i\,\vec{e}'_j = \delta_{ij}$$

The symbol is defined as : $\delta_{ij} = \begin{cases} i = j\ , & 1 \\ i \neq j\ , & 0 \end{cases}$

In order not to make any differentiation between the two systems let $\{O, \vec{r}_i\}$ be and auxiliary system with the same conditions of the other systems and refer the spherical motions to this new system. The variation of a vector (point) with respect to K, K′ are shown with "d..„ and "d'...„. These three trihedrals are oriented in the same way, so it is possible to pass from one to another by a rototion. These relations can be written :

$$\vec{r}_i = \sum a_{ij}\,\vec{e}_j \quad , \quad \vec{r}_i = \sum a'_{ij}\,\vec{e}'_j \qquad (i = 1,2,3).$$

The matrices $\| a_{il} \|$ and $\| a'_{ij} \|$ are orthogonal. If the a_{ij} and a'_{ij} are fonctions of a variable then we have a spherical motion with one parameter. (Which are the rotations.)

If we refer the variations of the vectors $\vec{r}_i$ to the new system then we have the equations :

$$(1) \qquad \begin{cases} d\vec{r}_1 = \vec{r}_2\omega_3 - \vec{r}_3\omega_2 \\ d\vec{r}_2 = \vec{r}_3\omega_1 - \vec{r}_1\omega_3 \\ d\vec{r}_3 = \vec{r}_1\omega_2 - \vec{r}_2\omega_1 \end{cases}$$

$$(1') \qquad \begin{cases} d'\vec{r}_1 = \vec{r}_2\omega'_3 - \vec{r}_3\omega'_2 \\ d'\vec{r}_2 = \vec{r}_3\omega'_1 - \vec{r}_1\omega'_3 \\ d'\vec{r}_3 = \vec{r}_1\omega'_2 - \vec{r}_2\omega'_1 \end{cases}$$

The numbers ω which are in the formulas are defined by :

$$(2) \qquad \omega_i = d\vec{r}_j\,\vec{r}_k$$

$$(2') \qquad \omega'_i = d'\vec{r}_j\,\vec{r}_k \qquad (i, j, k = 1,2,3;\, 2,3,1;\, 3,1,2)$$

Let X be a point referred to the new system then :

$$\vec{OX} = \vec{X} = \vec{r}_1 x_1 + \vec{r}_2 x_2 + \vec{r}_3 x_3$$

Let us calculate the variation of X with respect to K by means of (1) ;

$$(3) \qquad d\vec{X} = \vec{r}_1 (dx_1 - x_2\omega_3 + x_3\omega_2) + \vec{r}_2 (dx_2 - x_3\omega_1 + x_1\omega_3) + \vec{r}_3 (dx_3 - x_1\omega_2 + x_2\omega_1).$$

The conditions of X being a fixed paint on K are :

(4) $dx_1 = x_2\omega_3 - x_3\omega_2,\ dx_2 = x_3\omega_1 - x_1\omega_3,\ dx_3 = x_1\omega_2 - x_2\omega_1$

The variation of X with respect to K′ is derived in the similar way :

$$(5)\quad d'\vec{X} = \vec{r_1}(dx_1 - x_2\omega'_3 + x_3\omega'_2) + \vec{r_2}(dx_2 - x_3\omega'_1 + x_1\omega'_3) + \vec{r_3}(dx_3 - x_1\omega'_2 + x_2\omega'_1)$$

The conditions (4) replaced in (5), taking

$$(6)\quad \psi_i = \omega'_i - \omega_i$$

we have :

$$(7)\quad d_f\vec{X} = \vec{r_1}(\psi_2 x_3 - \psi_3 x_2) + \vec{r_2}(\psi_3 x_1 - \psi_1 x_3) + \vec{r_3}(\psi_1 x_2 - \psi_2 x_1)$$

If we consider the vector of rotation

$$(8)\quad \vec{\psi} = \vec{r_1}\psi_1 + \vec{r_2}\psi_2 + \vec{r_3}\psi_3$$

in the calculations then instead of (7) we have :

$$(9)\quad d_f\vec{X} = \vec{\psi} \times \vec{X}$$

Let us consider a unit vector $\vec{P}$ with the direction of the vector of rotation $\vec{\psi}$. Then :

$$\vec{\psi} = \vec{P}\cdot\sqrt{\psi_1^2 + \psi_2^2 + \psi_3} = \psi\vec{P}$$

can be written. Here ψ is the infinitesimal angle of rotation. The point P is the instantaneous center. According to (9) is $d_f\,x = 0$ for this point.

Formula (9) leads us to the theorem :

Taking into consideration a spherical motion with one-parameter the point X of the moving sphere K draws a curve on the fixed sphere K′ such that the spherical normal of this curve always passes through the instantaneous center.

§ 2 — Canonical reference system.

Let us choose the reference system so that :

$$(10)\quad \vec{P} = \vec{P_1} = \vec{r_3}$$

This condition makes it necessary for $\vec{P}$ to be perpendicular to $\vec{r_1}$ and $\vec{r_2}$ which means by (8)

$$\psi_1 = \psi_2 = 0$$

These results corresponds to:

(11) $$\omega'_1 = \omega_1 \quad , \quad \omega'_2 = \omega_2$$

In this case the vector of instantaneous rotation becomes:

(12) $$\vec{\psi} = \psi_3 \vec{r_3} = (\omega_2' - \omega_3)\ \vec{r_3}$$

Condition (10) does not determine the reference system uniquely for this system can still rotate around $\vec{r_3} = \vec{P_1}$. Let us try to fix $\vec{r_1}$ and $\vec{r_2}$. If the system is rotated around P angle λ we have :

$$\vec{P_1} = \vec{r_3}$$
$$\vec{P_2} = \vec{r_1} \cos\lambda + \vec{r_2} \sin\lambda$$
$$\vec{P_3} = -\vec{r_1} \sin\lambda + \vec{r_2} \cos\lambda$$

The rotated system is shown by $\{0, \vec{P_i}\}$, the derived equations which are similar to (1) and (1') are :

$$d\vec{P_1} = P\vec{P_2} - R\vec{P_3} \qquad d'\vec{P_1} = P'\vec{P_2} - R'\vec{P_3}$$
$$d\vec{P_2} = Q\vec{P_3} - P\vec{P_1} \qquad d'\vec{P_2} = Q'\vec{P_3} - P'\vec{P_1}$$
$$d\vec{P_3} = R\vec{P_1} - Q\vec{P_2} \qquad d'\vec{P_3} = R'\vec{P_1} - Q'\vec{P_2}$$

and after simple calculations we derive to these equations:

$$P = \omega_2 \cos\lambda - \omega_1 \sin\lambda \qquad P' = \omega_2 \cos\lambda - \omega_1 \sin\lambda$$
$$Q = \omega_3 + d\lambda \qquad Q' = \omega'_3 + d\lambda$$
$$R = \omega_2 \sin\lambda + \omega_1 \cos\lambda \qquad R' = \omega_2 \sin\lambda + \omega_1 \cos\lambda$$

To norm the auxiliary system, λ can be chosen such that always $R = R' = 0$, According (11) the equations for the fixed new system become :

(13)
$$d\vec{P_1} = P\vec{P_2}$$
$$d\vec{P_2} = -P\vec{P_1} + Q\vec{P_3}$$
$$d\vec{P_3} = -Q\vec{P_2}$$

(13')
$$d'\vec{P_1} = P\vec{P_3}$$
$$d'\vec{P_2} = -P\vec{P_1} + Q'\vec{P_3}$$
$$d'\vec{P_3} = -Q'\vec{P_2}$$

The vector $\vec{P_1} = \vec{r_3}$ draws a curve (P) on the moving sphere K we will call this curve the moving polar curve of the motion D. According to (13) the vector $\vec{P_2}$ is the unit tangent of (P) at the point P. The vector $\vec{P_1}$ on the fixed sphere K′ draws the fixed polar curve (P′) which also has the same tangent $\vec{P_2}$ at the point P. If the equations (13) are considered as the equations for the strip of curvature along the curve (P), then we have $dS = P$ where dS is the element of arc for (P). It can be derived that the geodesic curvature element of (P) is Q. Thus the ratio Q : P gives the geodesic curvature of (P) at P.

In a similar way $P' = P = dS = dS'$, and the ratio $Q' : P$ are respectively the element of arc and the geodesic curvature of (P′) at P.

If a spherical surface B is considered fastened to the system $\{O, P_1, P_2, P_3\}$ then the three spheres which are not stationary one to another is taken.

The rotations B/K and B/K′ are given by the system of equations (13) and (13′). Then the vectors of instantaneous rotations $\vec{\omega}$ and $\vec{\omega}'$ can be written as :

$$\vec{\omega} = \Delta_1\vec{P_1} + \Delta_2\vec{P_2} + \Delta_3\vec{P_3}$$
$$\vec{\omega}' = \Delta_1'\vec{P} + \Delta_2'\vec{P_2} + \Delta_3'\vec{P_3}$$

By comparing the coefficients Δ with (2) and (2′) we get the equations :

$$\vec{\omega} = Q\vec{P_1} + P\vec{P_3} \tag{14}$$
$$\vec{\omega}' = Q'\vec{P_1} + P\vec{P_3} \tag{14'}$$

If the unit vectors which have the same directions with $\vec{\omega}$ and $\vec{\omega}'$ are $\vec{OM} = \vec{M}$ and $\vec{OM'} = \vec{M'}$, then the points M and M′ are the instantaneous centers of the motions B/K and B/K′. Then according to (12) the vector of instantaneous rotation $\vec{\psi}$ of the motion K/K′ is :

$$\vec{\psi} = \vec{P_1}\,(Q' - Q). \tag{15}$$

Thus :

(16) $$\psi = Q' - Q$$

is derived.

§ 3 — The reciprocal of the Euler-Savary formula for the center of curvature of the orbit.

A fixed point X on the moving sphere K while moving draws an orbit (X) on K′. Let us investigate the center of the spherical curvature X′ of the orbit (X). Let the motion be referred to the canonical auxiliary system B. The spherical distance of the point X from P is ϑ.

The orbit (X) is a strip of curvature. A reference system is attached to this strip satisfyin the following conditions :

1) The normal of the surface is given by the position vector $\overrightarrow{OX} = \vec{X} = \vec{X}_1$.

2) Let the unit vector $\vec{X}_2$ be the tangent of the orbit at the point $\vec{X}_1$.

3) Let the unit vector $\vec{X}_3$ be the normal of the orbit which lies in the plane of the strip. Which means :

$$\vec{X}_3 = \vec{X}_1 \times \vec{X}_2$$

Then we have the equations :

$$\begin{aligned} d'\vec{X}_1 &= P_x\vec{X}_2 \\ d'\vec{X}_2 &= -P_x\vec{X}_2 + Q_x\vec{X}_3 \\ d'\vec{X}_3 &= -Q_x\vec{X}_2 \end{aligned}$$

According this definition the vectors $\vec{X}_1$ and $\vec{X}_3$ are situated in the normal plane of the curve. The intersection of this plane and the sphere is the spherical normal of the curve. The instantaneous center $\vec{P}$ and the center of spherical curvature $\vec{X}'$ are situated on this great circle of the sphere. Let ϑ' show the spherical distance of $\vec{X}'$ from $\vec{P}$, defining the angle between $\vec{X}_2$ and $\vec{P}_2$ as :

(17) $$\vec{X}_2\,\vec{P}_2 = -\sin\alpha$$

we have the formula :

$$(18) \qquad (\text{cotg}\, \vartheta' - \text{cotg}\, \vartheta) \sin \alpha = \frac{Q'}{P} - \frac{Q}{P} = \frac{\psi}{P}$$

§ 4 — The extension of the Savary construction to the spherical surface.

The relation in between the fixed point X of K and the center of spherical curvature X′ of its orbit on K′ is given by (18). Our aim is to give a rule to have a construction on the sphere (according to this formula) which enables us to find by construction one of the points if the position of the other point on the sphere is known. This construction is shown in fig (1). We will give the details and then prove it.

Construction : At first the spherical polar ray $\widehat{PX}$ (its plane is denoted by f_1) and second its spherical normal at P (this great circle is f_2) is constructed. Let f_2 and the great circle $\widehat{MX}$ (its plane is h) intersect at Q. The great circle which passes through Q and M′ (its plane is h') intersects $\widehat{PX}$ at the desired point X′.

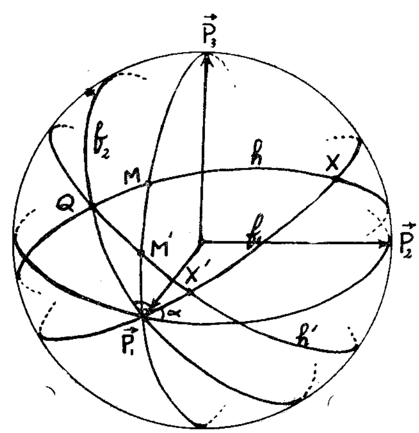

Fig. 1

To prove the construction let us denote by $\vec{D}$ the intersection of the great circles $\widehat{P_1X_1}$ and $\widehat{P_2P_3}$, fig (2). From (17):

$$\vec{P}_2\,\vec{D} = \cos \alpha$$

can be written.

Now we can refer the vectors $\vec{X}_1$ and $\vec{X}'$ to the system $\{0\,;\vec{P}_i\}$. We have :

$$(19) \qquad \vec{X} = \cos \vartheta\, \vec{P}_1 + \sin \vartheta \cos \alpha\, \vec{P}_2 + \sin \vartheta \sin \alpha\, \vec{P}_3$$

$$(20) \qquad \vec{X}' = \cos \vartheta' \vec{P}_1 + \sin \vartheta' \cos \alpha\, \vec{P}_2 + \sin \vartheta' \sin \alpha\, \vec{P}_3$$

If we denote the spherical distance of a point $\vec{Q}$ of the great circle f_2 from $\vec{P}$, φ, it is clear that :

(12) $$\vec{Q} = \cos\varphi\,\vec{P_1} - \sin\varphi\sin\alpha\,\vec{P_2} + \sin\varphi\cos\aleph\,\vec{P_3}$$

can be written.

The equation of the plane h is :

$$(\vec{X_1},\ \vec{M},\ \vec{Z}) = 0$$

Here $\vec{Z}$,

$$\vec{Z} = z_1\vec{P_1} + z_2\vec{P_2} + z_3\vec{P_3}$$

is any point in the plane h.

If we calculate the determinant by means of (14) and (19) we get to :

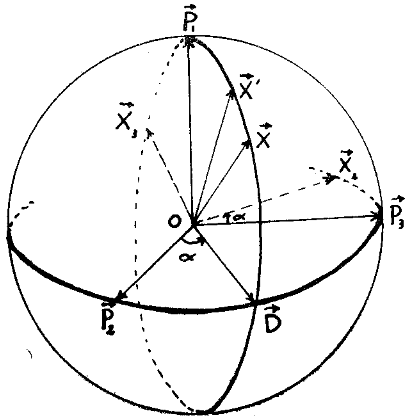

Fig. 2

$$(\cos\vartheta\,\vec{P_1} + \sin\vartheta\cos\alpha\,\vec{P_2} + \sin\vartheta\sin\alpha\,\vec{P}\ ;\ Q\vec{P_1} + P\vec{P_3}\ ;\ z_1\vec{P_1} + z_2\vec{P_2} + z_3\vec{P_3}) = 0$$

or to :

(22) $$Pz_1 - Qz_3 = \left(\frac{P}{\cos\alpha}\operatorname{cotg}\vartheta - Q\operatorname{tg}\alpha\right) z_2$$

If Q, shown in (21) is the point of intersection of h and f_2, then :

$$\vec{Z} = \vec{Q}.$$

From (22) :

(23) $$\frac{P}{\sin\alpha}\operatorname{cotg}\varphi - Q\operatorname{cotg}\alpha = Q\operatorname{tg}\alpha - \frac{P}{\cos\alpha}\operatorname{cotg}\vartheta$$

must be satisfied

In a similar way for the intersection point $\vec{Q'}$ of h' and f_2, the equation,

$$(23') \quad \frac{P}{\sin\alpha} \operatorname{cotg}\varphi' - Q' \operatorname{cotg}\alpha = Q' \operatorname{tg}\alpha - \frac{P}{\cos\alpha} \operatorname{cotg}\vartheta'$$

must hold.

Subtracting (23′) from (23) we have :

$$\frac{P}{\sin\alpha}(\operatorname{cotg}\varphi - \operatorname{cotg}\varphi') + (Q' - Q)\operatorname{cotg}\alpha =$$

$$\frac{P}{\cos\alpha}(\operatorname{cotg}\vartheta' - \operatorname{cotg}\vartheta) - (Q' - Q)\operatorname{tg}\alpha$$

and using (18) the Euler-Savary formula :

$$\frac{P}{\sin\alpha}(\operatorname{cotg}\varphi - \operatorname{cotg}\varphi') = 0$$

can be written, which means $\varphi = \varphi'$. This shows that the great circles $\widehat{MX}$ and $\widehat{M'X'}$ intersect at a point $\vec{Q}$ of the circle f_2. Thus the correctness of the construction is proved. It means that by construction the corresponding point of X, X′, can be determined if the centers of spherical curvatures M and M′ of the polar curves are known.

On the contrary if X′ is known, with the same method the point X can be determined by construction.

If the point X is situated on the mutual spherical normal of the polar curves, this construction does not hold. Then to find a method of construction we have to apply the theorem called the Projection theorem.

Theorem : The ortogonal spherical projections of the two corresponding points X and X′ on any spherical polar ray f_0, (these points being situated on the mutual spherical normal of the polar curves) are also point-pairs X_0 and X_0' which correspond one to another. Fig (3)

If the orthogonal spherical projections of the points X and X′ on any spherical polar ray f_0 (different from the spherical normal of the polar curve) be X_0 and X_0', then :

$$\widehat{PX} = \vartheta, \quad \widehat{PX'} = \vartheta', \quad \widehat{PX_0} = \vartheta_0, \quad \widehat{PX_0'} = \vartheta_0'$$

and from the corresponding spherical triangles we have :

$$\operatorname{cotg}\vartheta = \operatorname{cotg}\vartheta_0 \sin\alpha\,, \qquad \operatorname{cotg}\vartheta' = \operatorname{cotg}\vartheta_0' \sin\alpha \ (*)$$

(*) Let ABC be a spherical triangle with its four elements in a row B, a C, b the relation :

$$\cos a \cos C = \sin a \operatorname{cotg} b - \sin C \operatorname{cotg} B$$

holds. If B = 90°, then it becomes :

$$\operatorname{cotg} b = \operatorname{cotg} a \cos C.$$

From these relations by (18) we hawe :

$$\text{cotg}\, \vartheta' - \text{cotg}\, \vartheta = (\text{cotg}\, \vartheta_0' - \text{cotg}\, \vartheta_0) \sin \alpha = \frac{Q' - Q}{P} = \frac{\psi}{P}$$

This shows that the points X and X′ are connected. Then we have this construction :

Firs we find the orthogonal spherical projection X_0 of X on f_0, then according to the general way of construction we find X_0'. Then this point is projected again spherically on the spherical normal of the polar curve, fig (3).

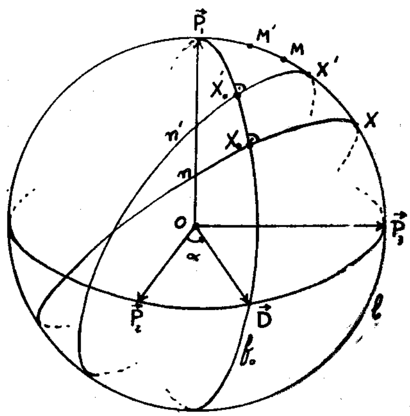

Fig. 3

Now it can easily be deduced that the centers of spherical curvatures of the orbits of all the points of the circle with the spherical diameter $\widehat{PX}$ are situated on the circle with the spherical diameter $\widehat{PX}'$.

§ 5. The extension of the Savary construction to the Line space.

Let us consider that the points of the spheres K and K′ are extended to dual points. Then to the curves (P) and (P′) on the spheres, the ruled surfaces (P) and (P′) of the line space correspond. Then P is the line of intersection of these two ruled surfaces. Because of $dS = dS'$, the corresponding central normals of these surfaces coincide at every instant. This gives that at every instant these two surfaces have the same principal trihedral $B = (P_1, P_2, P_3)$. The axes of instantaneous rotation of the motions B/K and B/K' are the lines M and M′. Thus we get to the extension of the Euler - Savary construction to the line space. The problem is at a definite instant of the motion if the lines M and M′ are known then by construction to find the axe of instantaneous rotation of the principal trihedral (X_1, X_2, X_3) of the ruled surface generated by a fixed line X of the K line space.

Before solving the problem let us record the following :

In fig (1) the points of intersection of the perpendiculars drawn from O to the planes f_1, f_2, h, h' with sphere be respectively :

$\vec{F}_1 = \vec{X}_2$, $\vec{F}_2$, $\vec{H}$ and $\vec{H}'$ then :

I	II	III	IV
1) $\vec{F}_1 \vec{X} = 0$	1) $\vec{F}_2 \vec{Q} = 0$	1) $\vec{H} \vec{X} = 0$	1) $\vec{H}' \vec{Q} = 0$
2) $\vec{F}_1 \vec{P} = 0$	2) $\vec{F}_3 \vec{P} = 0$	2) $\vec{H} \vec{M} = 0$	2) $\vec{H}' \vec{M}' = 0$
3) $\vec{F}_1 \vec{X}' = 0$	3) $\vec{F}_2 \vec{F}_1 = 0$	3) $\vec{H} \vec{Q} = 0$	3) $\vec{H}' \vec{X}' = 0$

can be written.

Let us consider the dual correspondence of these :

It is known that the mutual perpendicular of two skew lines can be constructed always taking this into consideration :

From	III_1	and	III_2, X	and	with	the help	of	M, H	can be	determined.
»	I_1	»	I_2, X	»	»	»	»	P, $F_1 = X_2$	»	»
»	II_2	»	II_3, P	»	»	»	»	F_1, F_2	»	»
»	II_1	»	III_3, F_2	»	»	»	»	H, Q	»	»
»	VI_1	»	IV_2, Q	»	»	»	»	M′, H′	»	»
»	IV_3	»	I_3, H′	»	»	»	»	F_1, X′	»	»

From (14) and (14′) if we consider $\vec{P_2}\vec{M}=0$ and $\vec{P_2}\vec{M'}=0$, then following the above relations we can realize the desired construction fig. (4).

It is clear that this construction does not hold if the given line X intersects P_2 orthogonaly. To solve this case we have to extend the corresponding case on the surface of the sphere to duals. The following relations can be written from fig (3).

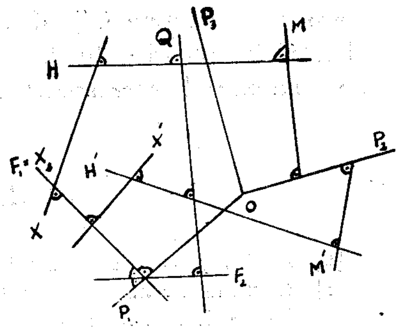

Fig. 4

The great circle passing through $\vec{P_1}$ and $\vec{P_3}$ is l, its pole on the sphere L.

The circle passing through $\vec{X}$ and orthogonal to f_0 is n and its pole N, the great circle passing through $\vec{X'}$ and orthogonal to f_0 is n', its pole N′. The pole of f_0 is F_0. Then we have :

I	II	III	IV
1) $\vec{F_0}\vec{X_0}=0$	1) $\vec{L}\vec{X}=0$	1) $\vec{N}\vec{X}=0$	1) $\vec{N'}\vec{X'}=0$
2) $\vec{F_0}\vec{P}=0$	2) $\vec{L}\vec{X'}=0$	2) $\vec{N}\vec{X_0}=0$	2) $\vec{N'}\vec{X'_0}=0$
	3) $\vec{L}\vec{M}=0$	3) $\vec{N}\vec{F_0}=0$	3) $\vec{N'}\vec{F_0}=0$

These can be extended to duals, so now the problem is, when M, M′ and X is known, to construct X′.

Let us construct a line F_0 which intersects the line P orthogonally and is different from L. Then we can follow the following relations and complete the construction.

From II_1 and II_3, X and with the help of M, L is determined.
» III_1 » III_3, X » » » » » F_0, N » »
» I_1 » III_2, F_0 » » » » » N, X_0 » »

following the general construction X'_0 is determined.

From IV_2 and IV_3, X'_0 and with the help of F_0, N′ is determined.
» IV_1 » II_2, N′ » » » » » L, X′ » »

On the surface of the sphere when the point X is situated on l, then we have mentioned that the centers of spherical curvature of all the points of the circle (c) of spherical diameter $\widehat{PX}$ are on the circle (c') of spherical diameter $\widehat{PX'}$.

If this is extended to line space two hyperboloids with the same generator P correspond to the circles (c) and (c').

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