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Modal behaviour investigation of spur gears with lumped parameter and finite element methods

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Abstract

Computation of gear dynamic forces accurately and correct modal behaviour estimation require highly validated modelling techniques in gear dynamics. In this paper, linear modal behaviour of a spur gear pair under prescribed conditions is investigated with lumped parameter and finite element methods. The main aim of this study is to compare the modal analysis results of the spur gear pair with the lumped parameter and finite element methods. For this purpose, a six degrees of freedom dynamic model of a spur gear pair is created using the lumped parameter method. In this method, the gears are assumed to be rigid disks whereas the gear teeth contacts and bearings are considered as flexible, which are modelled with spring elements. Then, a 3D solid model of the spur gear pair is created using the finite element method for high fidelity numerical analyses. In the finite element method, the gears are modelled with flexible three-dimensional solid elements, which is one of the main differences between the two methods. To remove the nonlinearity in the gear pair system, the contact is simplified with a rigid bonding of nodes in the finite element model. The natural frequencies and mode shapes are calculated by linear modal analysis for both methods. The obtained results from the individual methods show that flexible gear body modes, which are seen at higher frequencies, can only be detected with the finite element method. The tooth modes in which the gear bodies acting as a rigid body can be detected successfully with the two methods.

Keywords: *Gear dynamics, spur gears, finite element method, lumped parameter method, modal analysis.*

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Alın dişlilerin yığılı parametre ve sonlu elemenlar yöntemiyle modal davranışının incelenmesi

Öz

Dişli dinamiğinde, dişli dinamik kuvvetlerinin kesin olarak hesaplanması ve doğru modal davranış tahmini yüksek doğruluklu modelleme yöntemlerini gerektirmektedir. Bu çalışmada, bir alın dişli çiftinin doğrusal modal davranışı, belirlenmiş şartlar altında yığılı parametre ve sonlu elemanlar yöntemiyle incelenmektedir. Bu çalışmanın ana amacı yığılı parametre ve sonlu elemanlar yöntemiyle dişli çiftinin modal analiz sonuçlarını karşılaştırmaktır. Bunun için alın dişli çiftinin altı serbestlik dereceli dinamik modeli yığılı parametre yöntemiyle oluşturulmuştur. Bu yöntemde, dişli gövdeleri rijit kabul edilirken dişli temasları ve yatakları esnek kabul edilmiştir. Sonrasında, alın dişli çiftinin üç boyutlu katı modeli yüksek doğruluklu sayısal hesaplamalar için sonlu elemanlar yöntemiyle oluşturulmuştur. Sonlu elemanlar yönteminde, dişliler esnek üç boyutlu katı elemanlarla modellenmiştir. Bu modelde, dişli çifti temasında doğrusal olmama durumunun kaldırılması için temas rijit olarak basitleştirilmiştir. Doğal frekanslar ve mod şekilleri her iki yöntemde doğrusal modal analiz ile hesaplanmıştır. Her iki yöntemle elde edilen sonuçlar, yüksek frekanslarda görülen esnek dişli gövde modlarının sadece sonlu elemanlar yöntemiyle hesaplanabileceğini göstermektedir. Gövdenin hareket etmediği diş modları ise her iki yöntemle başarılı bir şekilde tespit edilebilmektedir.

Anahtar kelimeler: *Dişli dinamiği, alın dişliler, sonlu elemanlar yöntemi, yığılı parametre yöntemi, modal analiz.*

1. Introduction

Gears are commonly used machine elements in rotating systems for power and motion transmission, as well as speed and torque conversion. Particularly, spur gears can be seen in many industrial applications such as electrical motors, internal combustion engines, washing machines and geared pumps because of their advantages of higher efficiency, easy manufacturing process and lower production cost [1]. Contacts between the gear teeth under operating conditions results in a dynamic problem in gear systems [2, 3]. Therefore, dynamic modelling of gear systems is a significant issue to compute gear dynamic forces accurately and predict their modal behaviour properly.

In the literature, there are various analytical models using the lumped parameter method for the dynamic analysis of spur gear pairs [4–6]. The dynamic models are divided into three main groups as purely torsional model, torsional-transverse model and three-dimensional model [7]. The purely torsional model has one degree of freedom, which is initially proposed to investigate torsional vibrations of spur gear pairs [4]. Due to the rotating mass imbalances and significant gyroscopic moments at higher operating speeds, the vibrational motion also occurs in radial directions (transverse) of gears, known also as lateral vibrations [8, 9]. Therefore, the torsional model is expanded to the torsional-transverse model with the consideration of gear vibrations in transverse directions. The so-called torsional-transverse mode has three degrees of freedom (DOF) [10]. On the other hand, the torsional-transverse model could not be sufficient for the dynamic analysis of helical gear pairs because of the occurrence of axial forces in them. For this reason, the three-dimensional model, which has six degrees of freedom, has recently been developed by adding axial and tilting motions to the torsional – transverse model [6, 8, 11]. Furthermore, computed vibrational responses from the linear lumped parameter models of gear systems were validated with the experimental studies, whereas these studies does not include the validation of the modal behaviour of gear pairs [10, 11]. Nonlinear vibration behaviour of spur gears is also shown by numerous numerical [12–25] and experimental studies [26–28]. Throughout these studies, the main reason for the nonlinearity is identified as losing the gear contact within backlash (clearance nonlinearity). In addition to the analytical studies, there are some numerical studies in which dynamic analyses are carried out with different modelling approaches such as dynamic finite element/contact mechanics model [29], finite element/lumped parameter model [30], frequency-domain finite element model [31]. Although there are numerous linear and nonlinear vibration analyses conducted with the analytical gear dynamic models, there are limited studies comparing the analytical methods with the numerical methods such as finite element method [32, 33]. From these two studies, Vinayak and Singh [32] numerically computed natural frequencies with multibody dynamics, including rigid and flexible gear body modelling, and finite element models; however they did not identify the vibration modes systematically. Ambarisha and Parker [33] analysed the nonlinear dynamic response of spur planetary gears using two-dimensional lumped parameter and finite element models, while they did not conduct a linear modal analysis for a spur gear pair using the lumped parameter method. When these studies are taken into consideration, modal behaviour comparison of spur gear pairs using lumped parameter and finite element methods is still not well understood in the gear dynamics field. Therefore, a detailed modal analysis of spur gear pairs with three-dimensional lumped parameter and finite element models is needed.

In this study, free vibration of a spur gear pair is investigated with the lumped parameter method (LPM), analytically, and with a linearized finite element method (FEM), numerically. In the lumped parameter model, the gear teeth contacts and supporting elements are assumed to be flexible while gear bodies are assumed to be rigid, whereas the finite element model considers gear teeth and disks as flexible bodies. In terms of linearization, the contact at the gear teeth is assumed to be rigid bonding. Basically, this removes the nonlinearity and simplifies the model significantly. Modal analyses are performed for both models, and modal parameters such as natural frequencies and mode shapes are investigated in detail. Finally, a comparison of these methods is done, and the advantages of each method are discussed.

2. Dynamic modelling of a spur gear pair

In this section, dynamic modelling of a spur gear pair with lumped parameter and finite element methods is demonstrated.

2.1. Lumped parameter model

A spur gear pair consists of a driving gear (pinion) and a driven gear with their supports. To create a dynamic model of a spur gear pair, six degrees of freedom linear lumped parameter models are generally used [8, 11]. In the lumped parameter dynamic model, pinion and driven gear bodies are assumed to be rigid, whereas gear teeth contacts and supporting elements are assumed to be flexible. Moreover, time-varying mesh stiffness, clearance nonlinearity due to losing the gear contact within backlash, damping at the gear meshes and supports, and gyroscopic effects are neglected. Gear teeth contact (mesh) stiffnesses and gear support (bearing) stiffnesses are modelled with linear spring elements as shown in Figure 1. Based on the linear spring element modelling approach, gear mesh stiffness does not change with respect to the time and position [8, 11].

In this dynamic model, generalized coordinates for each gear can be written as

$$\boldsymbol{q}_i = [x_i, y_i, z_i, \theta_{xi}, \theta_{yi}, \theta_{zi}], \tag{1}$$

where $i = 1, 2$. The first and second indices represent the pinion and driven gear, respectively. Generalized coordinates of this system are then written as

$$\boldsymbol{q} = [x_1, y_1, z_1, \theta_{x1}, \theta_{y1}, \theta_{z1}, x_2, y_2, z_2, \theta_{x2}, \theta_{y2}, \theta_{z2}]. \tag{2}$$

Based on the x-y reference plane in Figure 1, x_i and y_i represent in-plane translation, z_i represent out of plane translation, θ_{xi} and θ_{yi} represent out of plane rotation, θ_{zi} represent in-plane rotation. Both pinion and driven gears have six degrees of freedom, which leads twelve degrees of freedom in total for the gear pair system.

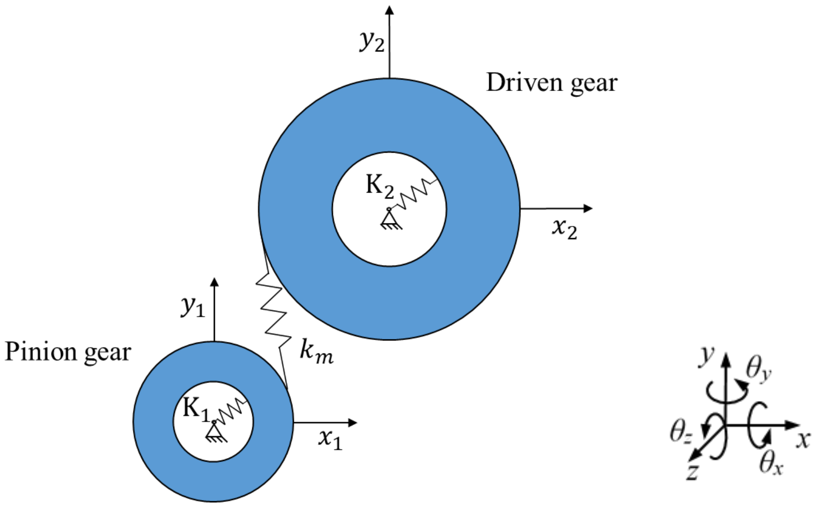


Figure 1. Dynamic model of a spur gear pair.

Support stiffnesses of pinion and driven gears, $\mathbf{K}_i$ are written as

$$\mathbf{K}_i = \begin{bmatrix} k_{xi} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{yi} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{zi} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{Q_{xi}} & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{Q_{yi}} & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{Q_{zi}} \end{bmatrix} \tag{3}$$

where the support stiffnesses consist of six degrees of freedom spring elements, which can do both translational and rotational motion [8]. Based on this equation, support elements have stiffness terms in all directions except the torsional (in-plane rotation) direction. Torsional stiffnesses of the supports are defined as zero ($k_{Q_{zi}} = 0$) since the gears can rotate freely on their bearings during the power transmission without any resistance. As can be clearly seen in Eq. (3), support stiffness is in diagonal form, as a result, there is no coupling term in the stiffness matrix, making the stiffness matrix uncoupled.

The equation of motion of the spur gear pair can be easily obtained using the Lagrange`s equations. For this purpose, kinetic and potential energies of the system should be computed. The kinetic energy of the spur gear pair is written as

$$T=\frac{1}{2}\sum_{i=1}^{2} m_i\left(\dot{x}_i{}^2+\dot{y}_i{}^2+\dot{z}_i{}^2\right)+I_{di}\left(\dot{\theta}_{xi}{}^2+\dot{\theta}_{yi}{}^2\right)+I_{pi}\dot{\theta}_{zi}{}^2 \tag{4}$$

where m_i, I_{di} and I_{pi} are the mass, diametral and polar mass moment of inertia, respectively. The strain energies of the support elements for the pinion and driven gear are written as

$$V_s=\frac{1}{2}\sum_{i=1}^{2} k_{xi}x_i{}^2+k_{yi}y_i{}^2+k_{zi}z_i{}^2+k_{Q_{xi}}\theta_{xi}{}^2+k_{Q_{yi}}\theta_{yi}{}^2+k_{Q_{zi}}\theta_{zi}{}^2 \tag{5}$$

The gear mesh strain energy due to the gear teeth deflection is also written as

$$V_m=\frac{1}{2}k_m\delta_m{}^2 \tag{6}$$

where k_m represents the mesh stiffness between the pinion and driven gear, δ_m represents the relative displacement between the pinion and driven gears. The relative displacement is expressed as [8, 11]

$$\begin{aligned}\delta_m\ =&\left[\left(x_p-x_g\right)\sin\emptyset+\left(y_p-y_g\right)\cos\emptyset+\left(r_p\theta_{zp}+r_g\theta_{zg}\right)\right]\cos\beta\\&+\left[\left(r_p\theta_{xp}+r_g\theta_{xg}\right)\sin\emptyset+\left(r_p\theta_{yp}+r_g\theta_{yg}\right)\cos\emptyset+\left(z_g-z_p\right)\right]\sin\beta\end{aligned} \tag{7}$$

In Eq. (7), β and $\emptyset$ represent the gear helix angle and pressure angle, respectively. It should be noted that the gear helix angle becomes zero for the spur gears. Following, the total potential energy of the system is computed as

$$V=V_s+V_m. \tag{8}$$

Finally, the equation of motion of the system for the free vibrations can be obtained using Lagrange`s equation as

$$\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial\dot{q}_k}\right)-\frac{\partial(T-V)}{\partial q_k}=0 \qquad (k=1,2,\dots,n). \tag{9}$$

Here, n represents the total degree of freedom. After solving the Eq. (9) for the free vibration of the system. The equation of motion of the system is derived as

$$\mathbf{M}\ddot{\boldsymbol{q}}(t)+\mathbf{K}\boldsymbol{q}(t)=0. \tag{10}$$

In Eq. (10), $\mathbf{M}$ and $\mathbf{K}$ represent the mass and stiffness matrices of the gear system, respectively. Here, $\boldsymbol{q}(t)$ also represents the generalized coordinates of the gear system. The mass and stiffness matrices of the gear system can be seen explicitly in reference [8].

The internal structure of the mass and stiffness matrices are seen in Figure 2, where all the elements are zero except the plotted points.

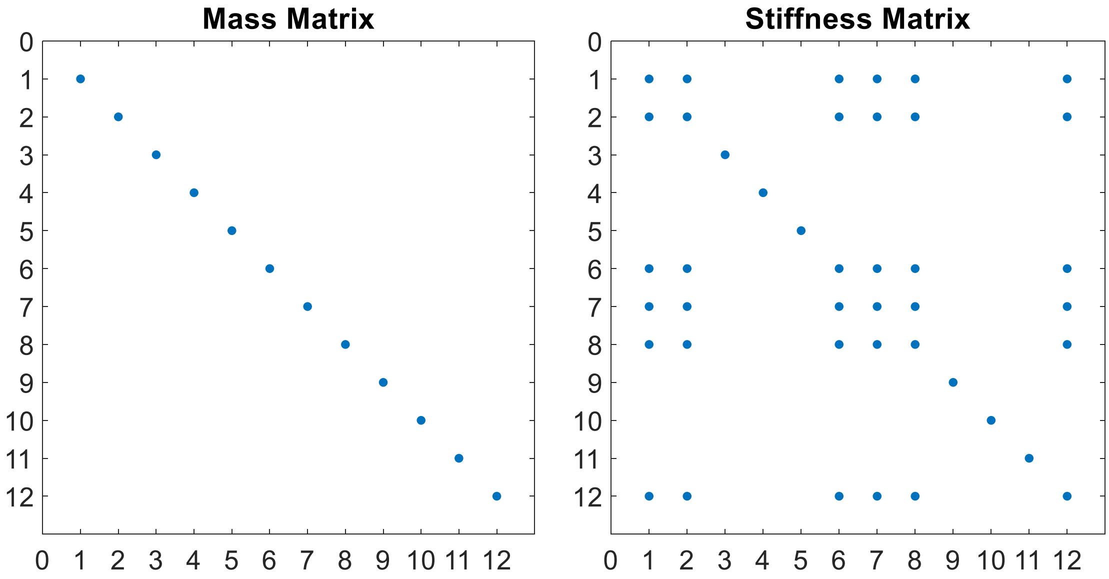


Figure 2. Mass and stiffness matrices of a spur gear pair.

For the free-free vibration problem, Eq. (10) can be converted into an eigenvalue problem by assuming displacement as harmonic, $\boldsymbol{q}(t) = \{\psi\}e^{i\omega t}$.

$$([\boldsymbol{K}] - \omega_r{}^2[\boldsymbol{M}])\{\psi\}_r = \{0\} \qquad r = 1, 2, \dots, k \tag{11}$$

Here, $\omega_r{}^2$ and $\{\psi\}_r$ represent eigenvalues and eigenvectors, from which natural frequencies and mode shapes can be obtained, respectively. For a non-singular solution of the Eq. (11), the determinant below

$$det([\boldsymbol{K}] - \omega_r{}^2[\boldsymbol{M}]) = 0 \tag{12}$$

should be equal to zero. After solving the Eq. (12), $\omega_r{}^2$ and $\{\psi\}_r$ are obtained. Modal matrix of the system can then be expressed using the eigenvectors as below

$$\mathbf{U} = [\psi_1, \psi_2, \dots, \psi_n]. \tag{13}$$

2.2. Finite element model

A three-dimensional solid model of the spur gear pair is created in accordance with DIN 867 standards by using a CAD (Computer Aided Design) software. The prepared geometry of the spur gear pair is then transferred to a CAE (Computer Aided Engineering) software for the finite element analysis. The second-order tetrahedral elements are used when forming the mesh structure, because its ability to represent such geometries is relatively better and of course for better accuracy. The mesh structure of the gear pair consists of 7020 quadratic tetrahedral elements with 9774 nodes and 8421 quadratic hexahedral elements with 40559 nodes alongside with two spring elements with six degrees of freedom (fully compliant in θ_z direction) per each, which is shown in Figure 3. The gear teeth have a more complex geometry. To represent the geometry in a more accurate way, tetrahedral elements are preferred over the hexahedral ones.

The reason of using two different types of elements can be explained with the balance of accuracy and computational cost. To avoid any convergence issues around the critical regions, such as contact area, fillet area and their neighbourhoods, the mesh is kept finer than the rest of the model. For the other parts, simpler geometries appear and therefore straightforward meshing could be applied. This leads to the usage of quadratic hexahedral elements and relatively coarser mesh structure. This also reduces the computational cost of the model.

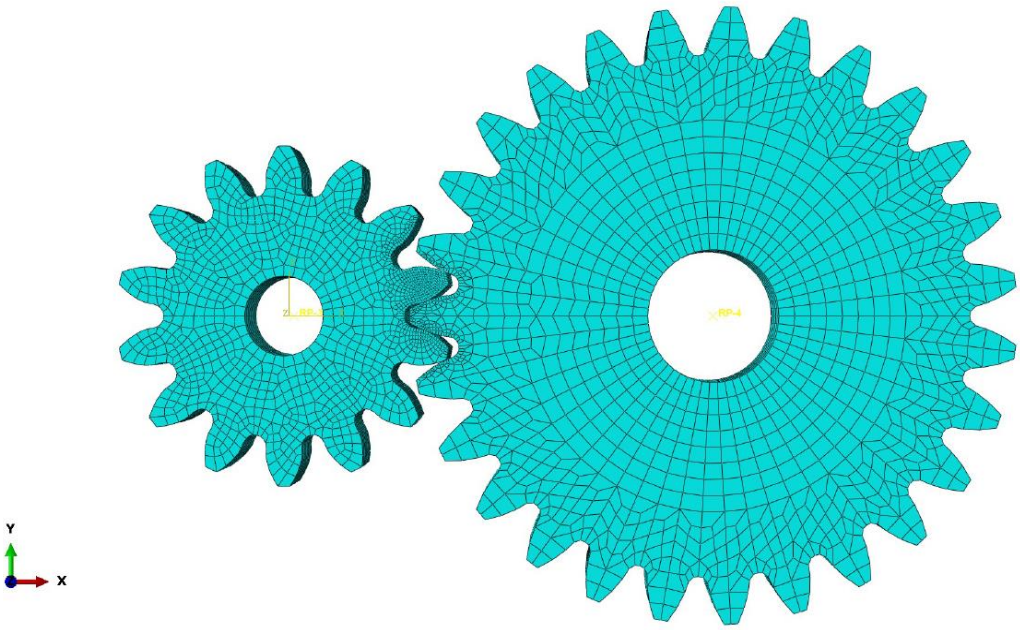


Figure 3. Geometry and mesh structure of the spur gear pair.

One of the most basic conditions for modal analysis when using a typical finite element software is that the system must be linear. The system in question is not substantially linear. The most important factor that disrupts linearity is the contact of gear teeth. In order to linearize the system, the contacts in multi-part systems need to be simplified with some assumptions. In this study, such a simplification is made by applying a rigid bonding condition at the gear teeth contact areas. This is achieved by applying the constraint on the nodes located at the contact patch between the driven and pinion gear. Furthermore, to represent the bearing and the shaft, six degrees of freedom springs are used, and they are connected to the holes at the centres of the gears by applying rigid couplings. The rotation of the gears around the z-axis is set as free. All the other degrees of freedom are restricted as described in Table 1. The stiffness values of these springs are chosen as the same as the analytical lumped parameter method. Similarly, the elastic properties of the gear material are taken as the same as in the lumped parameter model. More on the finite element model, no load is applied to the gears. It should be taken into consideration that the decrease in stiffness due to stresses in load applied gears and thus decrease in natural frequencies is to occur.

It is also to be noted that the basic assumptions in the analytical model are different from those in the finite element model. Most importantly, the lumped parameter model assumes gears as rigid disks, while the finite element model considers them flexible structures. In the lumped parameter model, gear teeth are assumed within the disk mass and their compliance is not considered anywhere else except for the contact. In finite element analysis, other gear teeth are modelled as flexible bodies as well unified with the flexible disk, in their original position and geometry. Therefore, the calculations include the tooth modes and the elastic body modes of the disks that the analytical model cannot capture. Finally, in the analytical model, the gear teeth in contact are only connected by a translational spring. In the finite element analysis, the contact of the gear teeth is considered as rigid, therefore the analysis relies on the flexibility of the gear teeth.

3. Numerical analysis

To compare lumped parameter and finite element methods, a case study was prepared with certain parameters of the spur gear pair. The modal analyses for both lumped parameter and finite element models were performed using the parameters in Table 1. The material of the spur gear pair is assumed to be standard steel. The gear mesh stiffness value was determined based on the initial finite element analyses. Support stiffness values of the gears were selected as reasonable based on the values in the literature [6, 8, 10].

Table 1. Parameters of the spur gear pair.

Parameter	Pinion Gear	Driven Gear
Number of teeth	14	27
Module [mm]	4	4
Inner diameter [mm]	15	25
Thickness [mm]	15	15
Material density [kg/m^3]	7800	7800
Young’s modulus [GPa]	210	210
Shear modulus [GPa]	80	80
Bearing radial stiffness [N/m]	4×10^8	4×10^8
Bearing axial stiffness [N/m]	2×10^8	2×10^8
Bearing tilting stiffness [N.m/rad]	4×10^6	4×10^6
Bearing torsional stiffness [N.m/rad]	0	0
Bearing radial damping [N/(m/s)]	0	0
Bearing axial damping [N/(m/s)]	0	0
Bearing tilting damping [N.m/(rad/s)]	0	0
Bearing torsional damping [N.m/(rad/s)]	0	0
Helix angle β [degree]	0	0
Transverse Pressure angle ø [deg]	22	22
Mesh stiffness [N/m]	4.15×10^8	4.15×10^8
Mesh damping [N/(m/s)]	4.15×10^8	4.15×10^8

After the execution of both methods, the natural frequencies and mode shapes of the test cases were calculated. It should be noted that damping of the bearings, gear contacts and gear bodies were neglected in the spur gear pair because of not affecting the modal behaviour in terms of natural frequency and mode shapes significantly. The presented results for both lumped parameter and finite element methods were computed for the undamped case.

3.1. Lumped parameter method results

For the lumped parameter model of the spur gear pair system, natural frequencies and mode shapes were computed with the linear modal analysis. Natural frequencies are provided with corresponding vibration mode, as seen in Table 2.

Table 2. Computed natural frequencies with lumped parameter method.

Mode #	LPM Natural Frequency [Hz]	Mode Type
1	0	Rotational (Rigid Body)
2	2235	Axial (Gear)
3	2956	Translational – Rotational
4	3160	Translational (Gear)
5	4352	Axial (Pinion)
6	5081	Translational – Rotational
7	6154	Translational (Pinion)
8	11267	Tilting (Gear)
9	11267	Tilting (Gear)
10	12489	Translational – Rotational
11	40687	Tilting (Pinion)
12	40687	Tilting (Pinion)

The modal matrix ($\mathbf{U}$) which shows eigenvectors of the spur gear pair system for each mode is provided in the Eq.14 below. The vibration modes shown in Table 2 were identified based on the eigenvectors in the modal matrix.

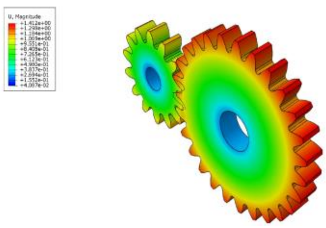
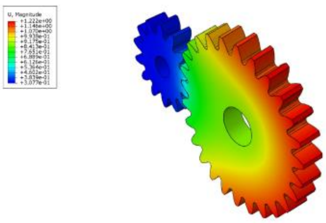
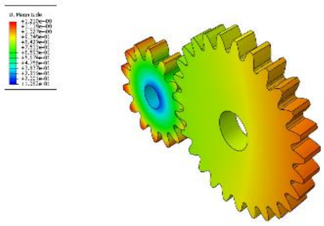
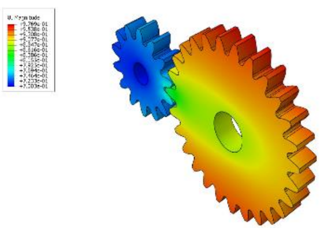
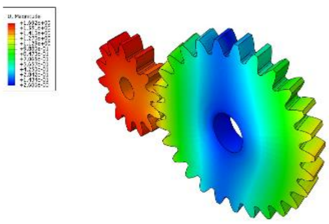
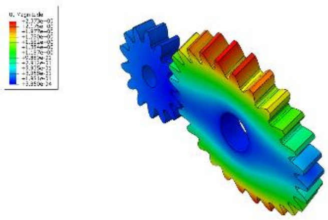
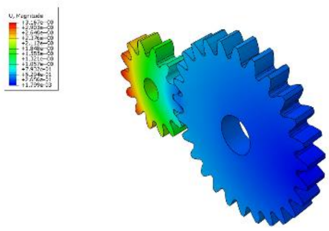
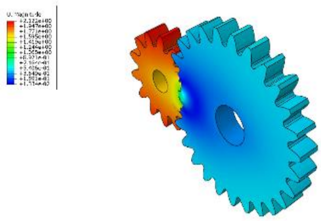
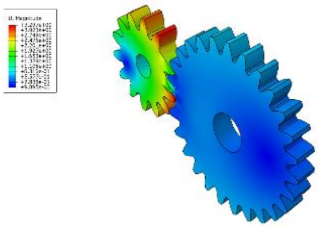
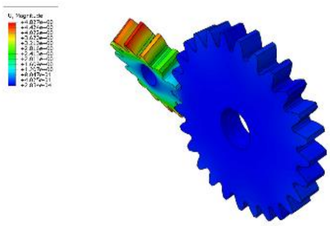
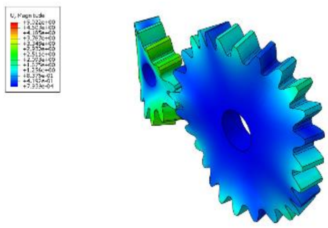
$$\mathbf{U} = \begin{bmatrix} 0 & 0 & -0.06 & 0 & 0 & -0.56 & -1.79 & 0 & 0 & 0.46 & 0 & 0 \\ 0 & 0 & -0.14 & 0 & 0 & -1.39 & 0.72 & 0 & 0 & 1.13 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.93 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 127.82 & -0.01 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.01 & -127.82 \\ -43.38 & 0 & 32.88 & 0 & 0 & 46.60 & 0 & 0 & 0 & 61.35 & 0 & 0 \\ 0 & 0 & 0.34 & -0.92 & 0 & -0.11 & 0 & 0 & 0 & -0.10 & 0 & 0 \\ 0 & 0 & 0.84 & 0.37 & 0 & -0.28 & 0 & 0 & 0 & -0.24 & 0 & 0 \\ 0 & -0.99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 35.37 & 1.44 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.44 & -35.37 & 0 & 0 & 0 \\ 22.49 & 0 & 4.57 & 0 & 0 & 6.48 & 0 & 0 & 0 & 8.53 & 0 & 0 \end{bmatrix} \quad (14)$$

From the Table 2 and computed modal matrix, it is clearly seen that vibration modes consist of one rotational, two axial, two translational, three translational – rotational and four tilting modes. Among these modes, the rigid rotational and translational – rotational modes are also referred to as tooth modes and the remaining modes are referred to as bearing modes. [34]. There is a total of four tooth modes for the gear-pair, while eight bearing modes are available. It is important to point out that only tooth modes are affected by the change in gear teeth contact (mesh) stiffness.

3.2. Finite element method results

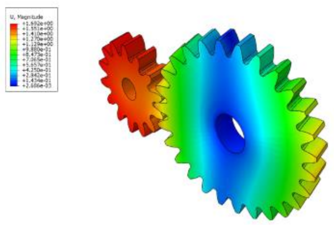
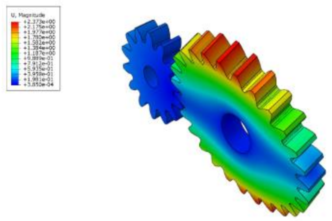
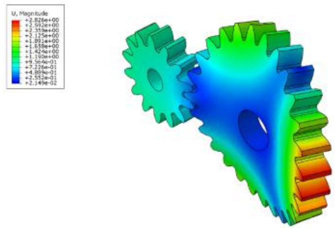
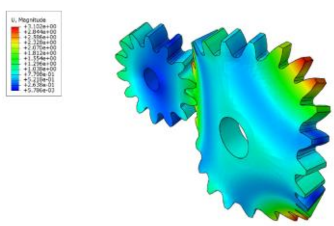
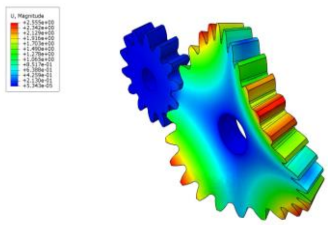
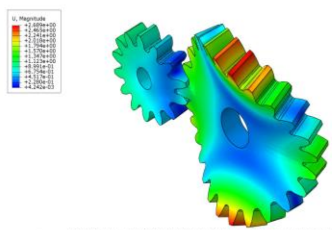
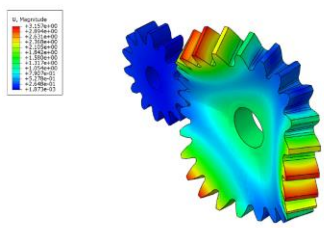
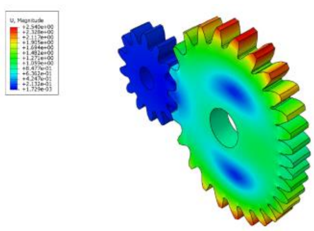
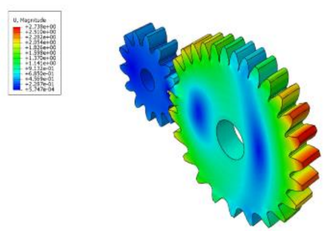
The mode shapes and the respective natural frequencies, calculated with the finite element method, are displayed in Table 3. The first thing to be noted about the results of the finite element method is the rotational mode. This first mode is supposed to be a rigid body motion. However, the simplification of the contact constrains this motion. Therefore, this mode was calculated as a flexible mode as seen in Table 3. The simplification of the contact leads to coupling of some modes as the motions are restrained. An example of coupled modes is the coupling of pinion gear's axial mode and driven gear's tilting mode. Furthermore, the flexible disk modes of the driven gear are coupled with pinion gear's tilting modes. This over constraining of the system can be considered as a drawback of a 3D linearized finite element model.

Table 3. The natural frequencies and the mode shapes of the spur gear pair calculated by the finite element analysis.

1) Rotational – 1011 Hz	2) Axial (Gear) – 2401 Hz	3) Translational-Rotational – 2964 Hz
		
4) Translational (Gear) – 3792 Hz	5,6) Tilting (Gear) / Axial (Pinion) – 4649 Hz	7) Tilting (Gear) – 4966 Hz
		
8) Translational-Rotational – 5098 Hz	9) Translational (Pinion) – 11506 Hz	10) Translational-Rotational - 12436 Hz
		
11) Tilting (Pinion) – 16326 Hz	12) Tilting (Pinion) – 17741 Hz	
		

One of the most significant differences between the lumped parameter model and the finite element model is the flexible body assumption of the finite element method. As mentioned earlier, the lumped parameter method models the gears' main bodies as rigid disks which keep the degrees of freedom at a certain number, to be precise 12. Whereas, the finite element method assumes the whole structure as flexible bodies. Therefore, the degrees of freedom of the finite element model, though it depends on the number of nodes, is much greater than 12. This makes the finite element model capable of catching the natural frequencies and the mode shapes that cannot be calculated by the lumped parameter model. Some of these modes are flexible disk modes. Table 4 displays some examples of flexible disk modes.

Table 4. Some of the flexible disk modes and the respective natural frequencies of the gear pair calculated by the finite element analysis.

1) Gear Nodal Diameter 4649 Hz	2) Gear Nodal Diameter 4966 Hz	3) Gear Nodal Diameter 5013 Hz
		
4) Pinion and Gear Nodal Diameter 13703 Hz	5) Gear Nodal Diameter 7020 Hz	6) Pinion and Gear Nodal Diameter 8642 Hz
		
7) Gear Nodal Diameter 12358 Hz	8) Gear Nodal Circle 24330 Hz	9) Gear Nodal Circle 24856 Hz
		

In flexible disks, nodal diameters and nodal circles determine the mode shapes. In cases with gears, there are essentially two disks connected to each other rigidly which leads to coupling of multiple modes of the two gears/disks. As to be seen from mode 1 in Table 4, the axial mode of the pinion gear, the tilting mode of the driven gear and the first nodal diameter mode of the driven gear are coupled. Mode 2 is also a coupled mode of tilting and the first nodal diameter of the driven gear. In the case of a thin disk, these two nodal diameter modes are supposed to be orthogonal modes and to have the same natural frequencies. Similarly, same comments could be made for the second nodal diameter modes of the driven gear which are shown in modes 5 and 6. The flexible disk modes also exist for pinion gear, but the natural frequencies are much higher than the respective natural frequencies of the driven gear. The reason for this lies in the inertial and stiffness properties of the pinion gear, having a smaller geometry than the driven gear. Therefore, the pinion gear has less mass and is stiffer than driven gear. Nodal circle nodes of the driven gear are given in the last two modes 8 and 9 in Table 4. In fact, the nodal circle modes are supposed to be unique because of the axisymmetric structure of the disk. However, in this case, the connection between the driven gear and pinion gear removes the axial symmetry. Therefore, multiple modes are calculated for nodal circle modes. Another thing to note would be about the modes 3, 4 and 7 have similarly three nodal radii, which is not a case to be observed with disks with perfect symmetry. This also occurs due to the removal of the symmetry by the contact between the gear pair. Similar asymmetry in results can also be obtained by asymmetric loading. Being able to capture these modes, is a major advantage of applying the finite element method in gear modelling.

3.3. Comparison of LPM and FEM results

For the modal behaviour investigation of the spur gear pair, modal analysis results obtained with the finite element and lumped parameter analyses are compared in Table 5.

Table 5. Natural frequency comparison of lumped parameter (LPM) and finite element method (FEM)

Mode #	LPM Natural Frequency [Hz]	FEM Natural Frequency [Hz]	Mode Type
1	0	1011.3	Rotational (Rigid Body)
2	2235	2401	Axial (Gear)
3	2956	2964.2	Translational – Rotational
4	3160	3791.7	Translational (Gear)
5	4352	4648.8	Axial (Pinion)
6	5081	5097.9	Translational – Rotational
7	6154	11506	Translational (Pinion)
8	11267	4648.8	Tilting (Gear)
9	11267	4966.1	Tilting (Gear)
10	12489	12436	Translational – Rotational
11	40687	16326	Tilting (Pinion)
12	40687	17741	Tilting (Pinion)

Although there are significant differences between the assumptions of both analyses, consistent results are obtained. For instance, natural frequencies and mode shapes are obtained using the finite element analysis, which is made by employing the described mesh structure and boundary conditions. The gears are represented with flexible solid elements. For gear teeth contact, the contacting finite element nodes in both gears are rigidly connected. On the other hand, the gears are connected flexibly with translation and rotation springs in the lumped parameter model where the gears can rotate freely. It should be noted that the gear wheels are assumed to be not loaded in both two modelling techniques.

4. Discussion

The two methods which are lumped parameter and finite element methods are based on different assumptions. When comparing the results of these two methods, it should be pointed out that some of the natural frequencies of the respective modes are in good agreement. The natural frequencies of the axial and translational mode of the gear, the axial mode of the pinion gear, three of the translational – rotational modes are obtained with higher accuracy. The natural frequencies of these modes given by both methods are very close and consistent. This is because both methods use the same bearing stiffness values. The fact that the natures of the structures disk and gear teeth, as well as contact, are modelled differently in both methods leads to the calculation of slightly different frequencies.

However, the rest of the natural frequencies that are calculated with these two models have significant differences. The frequency that corresponds to the rotational mode is zero for the lumped parameter model whereas the finite element yields a nonzero frequency for the same mode. The reason behind this difference is, as mentioned earlier, essentially based on the modelling assumptions of the contact between the driven gear and the pinion gear. The rigid connection in the finite element model does not let the driven gear and pinion gear rotate freely while the spring on the lumped parameter model does not restrict that motion. Similarly, for the difference between the calculated axial modes, the same reason can be named. More precisely, in the lumped parameter model, it is assumed that the gear teeth can slide freely on each other, whereas in the finite element model, the contacting nodes in the gear teeth are connected to each other. Moreover, since the mass of the driven gear is larger than the pinion gear and the bending stiffness is small, the frequency values of the axial mode are very different.

Another difference to mention is about tilting modes around x and y axes. The tilting motion around the x-axis is coupled with other forms of motion in finite element method because of the 3D geometry of the gear teeth. Whereas in the lumped parameter method there is nothing that constrains the tilting motion of the gears. Therefore, these modes are not coupled with any other mode. In the results taken from the finite element method, the coupling is observed in tilting modes about y-axis as well. However, the reason behind this coupling is the over-restriction of the contact. Unlike the finite element model, the lumped parameter model displays uncoupled results for those modes. Thus, there is no restriction of this motion. It is seen that the frequencies of the in-plane mode of the gears, which express the vibration motion like a rigid disk, are different. This is due to the difference in gear teeth contact modelling assumptions as well as the flexible disk assumption of the finite element model.

As of the last comments, the finite element generally computes lower natural frequency values than the lumped parameter method. This roots from the difference between basic structural modelling assumptions. The lumped parameter model considers the disks as rigid. The only sources of flexibility in this model are the springs defined as the bearings and the spring that define the mesh stiffness. The same springs are used to define the bearings in the finite element model too. But additionally, the disks are also flexible in that model. This reduces the stiffness of the model. The only exception to this is the pinion gear’s translational mode. The reason for the higher frequency in finite element method lies in the contact again. The contact restricts the motion in this direction, whereas, in the lumped parameter model, there is no restriction.

5. Conclusion

The natural frequencies and mode shapes of a spur gear pair under no-load are analysed with lumped parameter and finite element methods. In the lumped parameter model, gear bodies are assumed to be rigid, and they are grounded by uncoupled flexible bearing elements. Here, the gear teeth contacts are also considered as flexible. In the finite element model, gear bodies are modelled with 3D solid elements. Gear teeth contacts (meshes) are modelled as connected rigidly and bearing stiffnesses are modelled the same way as in the analytical method.

Overall, both methods show consistent modal behaviour despite the different assumptions on which they are built. The lumped parameter models can estimate the tooth modes satisfactorily, which is consistent with the finite element analysis results. On the other hand, lumped parameter model cannot compute the flexible gear body modes, which can only be detected with the finite element model. The analyses result shows that mode shapes predicted by the finite element and lumped parameter methods are mostly consistent, whereas some of the natural frequencies calculated by these two methods have significant differences. The reason for the differences between some natural frequencies lies in the gear contact and body modelling assumptions of the two methods.

Briefly, flexible gear body and gear contact modelling is necessary for an accurate modal parameter prediction of a spur gear pair system. Flexible gear body and contact modelling should be considered for a reliable gear dynamic analysis.

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