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Uniqueness theorems for some classes of nonlinear fractional differential equations in the Riemann-Liouville sense

Müfit ŞAN*

Çankırı Karatekin University, Faculty of Science, Department of Mathematics, Uluyazı Campus, Çankırı.

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Abstract

In this study, some classes of Riemann-Liouville fractional differential equations with right-hand side functions having a singularity with respect to their first variable and with a nonhomogeneous initial condition are considered. First, it is briefly stated that under which conditions the existence of a local continuous solution of this initial value problem occurs. Later, uniqueness theorems were developed in types of Krasnosel'skii-Krein, Kooi, Roger and Banaś-Rivero, respectively. These theorems improve the previously obtained results, and for their proofs pre-existing techniques are enriched by the tools of Lebesgue spaces.

Keywords: *Fractional differential equations, Riemann-Liouville derivative, existence and uniqueness, Lebesgue spaces.*

Riemann-Liouville tip kesirli türevli lineer olmayan denklemlerin bazı sınıfları için teklik teoremleri

Öz

Bu çalışmada. sağ yan fonksiyonları birinci değişkenlerine göre singülerliğe sahip ve başlangıç koşulu homojen olmayan Riemann-Liouville kesirli diferansiyel denklemlerinin bazı sınıfları göz önüne alınmıştır. İlk önce, bu başlangıç-değer probleminin bir lokal sürekli çözümünün varlığını hangi koşular altında gerçekleştiği kısaca ifade edilmiştir. Daha sonra ise, sırasıyla Krasnosel'skii-Krein, Kooi, Roger ve Banaś-Rivero tiplerinde teklik teoremleri ortaya çıkarılmıştır. Bu teoremler daha önceden elde edilen sonuçları geliştirken, bu teoremlerin ispatları için, daha önceden var olan teknikler Lebesgue uzaylarının araçları ile zenginleştirilmiştir.

* Müfit ŞAN, mufitsan@karatekin.edu.tr, http://orcid.org/0000-0001-6852-1919

Anahtar kelimeler: *Kesirli mertebeden türevli denklemler, Riemann-Liouville türevi, varlık ve teklik, Lebesgue uzayları.*

1. Introduction

Fractional Calculus (FC) is nowadays one of the most discussed and developing areas of mathematics (see, [1-6]). Perhaps the most important reason for this is that integral and derivative operators in FC, when compared with their integer order counterpart in classical calculus, are not uniquely defined, but rather one can witness that they are defined in many different ways in the literature. These so-called fractional derivatives are classified by some of their features, for example, by their kernels, singular or non-singular and by the time they were defined: newly or old-time defined (see, for example [1,4]). Riemann-Liouville (R-L), Grünwald-Letnikov, Hadamard fractional derivatives are counted as longstanding definitions, while Caputo-Fabrizio, Atangana-Baleanu, conformable are examples for newly defined operators. Recently, besides investigation of properties for each one of the newly defined operators, qualitative and quantitative properties of solutions to differential equations associated with them and their applications to physical and engineering problems are currently available in the literature and are still extensively studied. Although newly defined operators and related studies are recently popular in the theory and applications of FC, qualitative and quantitative properties such as existence, uniqueness and continuation, of solutions to the fractional differential equations in sense of Riemann-Liouville and Caputo fractional derivatives continue to be studied as before (see the latest work in [7-13]).

In this work, we use the R-L derivative and reveal some uniqueness criteria to a nonlinear fractional differential equation (FDE) involving the R-L derivative, given as follows:

$$\begin{aligned} {}_{RL}D^{\gamma}\upsilon(x) &= f(x,\upsilon(x)), 0<x<T \\ \upsilon(0) &= \upsilon_0, \end{aligned} \tag{1}$$

where $0<\gamma<1$ (is valid throughout the paper), $T>0$ is an arbitrary real number, the right hand side function $f:(0,T]\times\mathbb{R}\to\mathbb{R}$ will be given later, $\upsilon_0\neq 0$ is a real number and ${}_{RL}D^{\gamma}$ stands for the R-L derivative of order γ, which is formularized by applying the ordinary derivative to R-L integral I^{γ} as below:

$${}_{RL}D^{\gamma}\upsilon(x) := \frac{d}{dx}\left[I^{1-\gamma}\upsilon(x)\right] \text{ and } {}_{RL}I^{\gamma}\upsilon(x) := \frac{1}{\Gamma(\gamma)}\int_0^x \frac{\upsilon(\eta)}{(x-\eta)^{1-\gamma}}d\eta.$$

Here, $\Gamma(.)$ is the well-known Gamma function.

In [11], Şan and Sert investigated problem (1) under the following conditions:

(C1) $f(x,\upsilon)$ is continuous on $(0,T]\times\mathbb{R}$ so that $x^{\gamma}f(x,\upsilon)$ is continuous on $[0,T]\times\mathbb{R}$.

(C2) There exists a real number $M > 0$ so that

$$\left| x^{\gamma} f(x,\upsilon) - \frac{\upsilon_0}{\Gamma(1-\gamma)} \right| \le M \max\left(x, \frac{|\upsilon - \upsilon_0|}{r} \right)$$

holds for all $x \in [0,T]$ and for all $t \in [\upsilon_0 - r, \upsilon_0 + r]$, where r is a fixed positive real number.

They proved that problem (1) has at least one solution in $C([0,T_0])$, the space of continuous functions defined on $[0,T_0]$, where

$$T_0 := \begin{cases} \dfrac{r}{M\Gamma(1-a)}, & \text{if } \quad M\Gamma(1-a) \ge r, \\ T & \text{if } \quad M\Gamma(1-a) \le r. \end{cases}$$

It is worth noting that the inequality in (C2) implies $x^{\gamma} f(x,\upsilon_0)\big|_{x=0} = \upsilon_0 / \Gamma(1-\gamma)$, which is a necessary condition for the existence of a continuous solution to the problem as shown in [11]. Furthermore, since condition (C1) is not sufficient to give a Peano-type existence theorem by the aid of well-known theorems or methods (fixed point theorems, successive approximation etc.), they had to set another condition in (C2). For the uniqueness of solution to the problem, they gave a Nagumo-type criterion with the right hand side function f satisfying the inequality

$$x^{\gamma} \left| f(x,\upsilon_1) - f(x,\upsilon_2) \right| \le L |\upsilon_1 - \upsilon_2| \tag{2}$$

where L is the Nagumo constant by $L = \left[\Gamma(1-\gamma)\right]^{-1} < 1$.

In this paper, our first aim is to obtain a better estimation for L. For this, in addition to the Nagumo-type condition, we will put forward a hypothesis which is a generalization of Krasnosel'skii-Krein-type condition given by [14-16]. The estimation we obtain for L will be greater than its previous value $\left[\Gamma(1-\gamma)\right]^{-1}$ but still too limited (see, Remark 2.2). On the other hand, the continuity of the right-hand side function f of the FDE in (1) is the only criteria for determining the existence interval of solutions to the problem and it is sufficient for proving the existence of solutions. However, under each different condition such as Nagumo-type, Krasnosel'skii-Krein-type condition (see, [17-18] and other existence results of authors in references therein) existence theorems were proved by the method of successive approximations, which yield same existence interval for solutions. It is probably due to the desire for making contribution to the theory of FDE. Contrary to this we will not give an extra existence theorem and use the Peano-type existence theorem given in [11]. Moreover, the generalization of Kooi-type theorem in [16,19] will be given. When x^{γ} is replaced by $x^{1+\gamma}$ in the inequality (2), we will establish the Roger-type uniqueness criterion which has a growth condition on f as in [16,20]. Finally, we generalize the lemma and the uniqueness theorem for ordinary differential equations of first order given by Banaś and Rivero in [21] to those for the FDE in (1). For proofs of all aforementioned uniqueness results we use techniques used before with the tools of Lebesgue spaces such as Hölder inequality. For this reason, our results are convenient to be generalized for FDE in the weak sense.

The following lemma given in [11] will be main tool to reveal uniqueness results for problem (1):

Lemma 1.1 Let condition (C1) and $x^{\gamma} f(x,\upsilon_0)\big|_{x=0} = \upsilon_0 / \Gamma(1-\gamma)$ be fulfilled. Then $\upsilon \in C\left(\left[0,T\right]\right)$ is a solution of problem (1) if and only if it is a solution of the Volterra-type integral equation

$$\upsilon(x) = \frac{1}{\Gamma(\gamma)} \int_0^x \frac{f(\eta,\upsilon(\eta))}{(x-\eta)^{1-\gamma}} d\eta. \tag{3}$$

Lastly, we point out that the continuity condition of $x^{\gamma} f(x,\upsilon)$ on $[0,T]\times\mathbb{R}$ is superfluous and thus can be dropped since condition (C2) with continuity of $f\left(x,\upsilon\right)$ on $(0,T]\times\mathbb{R}$ already implies it. Let us see this in the following remark:

Remark 1.2 It is obvious that $x^{\gamma} f(x,\upsilon) \in C\left((0,T]\times\mathbb{R}\right)$. Now, let $\left\{\left(x_n,\upsilon_n\right)\right\}_{n=1}^{\infty}$ be a sequence in $(0,T]\times\mathbb{R}$ so that $\left(x_n,\upsilon_n\right)\to\left(0,\upsilon\right)$ for an arbitrary $\upsilon\in\mathbb{R}$. By using this in the inequality in condition (C2), one obtains

$$\begin{aligned}\left| x_n^{\gamma} f(x_n,\upsilon_n-\upsilon+\upsilon_0) - \frac{\upsilon_0}{\Gamma(1-\gamma)} \right| &\le M \max\left(x_n, \frac{\left|\upsilon_n-\upsilon+\upsilon_0-\upsilon_0\right|}{r} \right) \\ &= M \max\left(x_n, \frac{\left|\upsilon_n-\upsilon\right|}{r} \right) \to 0,\end{aligned}$$

which implies that $x_n^{\gamma} f(x_n,\upsilon_n-\upsilon+\upsilon_0) \to \dfrac{\upsilon_0}{\Gamma(1-\gamma)} = x^{\gamma} f(x,\upsilon_0)\big|_{x=0}$. From here, one can conclude $x_n^{\gamma} f(x_n,\upsilon_n) \to x^{\gamma} f(x,\upsilon)\big|_{x=0}$ showing that $x^{\gamma} f(x,\upsilon)$ is sequentially continuous. Therefore it is continuous on $\left\{0\right\}\times\mathbb{R}$ as well. Consequently, $x^{\gamma} f(x,\upsilon) \in C\left([0,T]\times\mathbb{R}\right)$.

2. Main results

We will begin with a Krasnosel'skii-Krein-type uniqueness theorem given as follows:

Theorem 2.1 Let $0<\gamma<1$ and conditions (C1)- (C2) be fulfilled. Furthermore, suppose that there exists a $L>0$ fulfilling the inequality $\Gamma(L-\gamma+1)<\Gamma(L)$ so that

$$x^{\gamma}\left| f(x,\upsilon_1) - f(x,\upsilon_2)\right| \le L\left|\upsilon_1-\upsilon_2\right| \tag{4}$$

holds for all $(x,\upsilon_1),(x,\upsilon_1)\in[0,T]\times\mathbb{R}$, and that there exist $C>0$ and $\alpha\in(0,1)$ satisfying $\gamma-L(1-\alpha)>0$ such that

$$\left|f(x,\upsilon_1)-f(x,\upsilon_2)\right|\le C\left|\upsilon_1-\upsilon_2\right|^{\alpha} \tag{5}$$

is fulfilled for all $(x,\upsilon_1),(x,\upsilon_2)\in[0,T]\times\mathbb{R}$.
Then, problem (1) has a unique solution in $C\left(\left[0,T_0\right]\right)$.

Proof. Under hypothesis of the theorem, we suppose that problem (1) admits two different solutions represented by $\upsilon_1,\upsilon_2\in C\left(\left[0,T_0\right]\right)$. We will prove by contradiction that they are actually equal. Set $\Upsilon(x)=\left|\upsilon_1(x)-\upsilon_2(x)\right|$ with υ_1,υ_2 satisfying (3). By applying condition (C1) and inequality in (5) we have the following estimation for $\Upsilon(x)$:

$$\begin{aligned}\Upsilon(x)&\le\frac{1}{\Gamma(\gamma)}\int_0^x\frac{\left|f\left(\eta,\upsilon_1(\eta)\right)-f\left(\eta,\upsilon_2(\eta)\right)\right|}{(x-\eta)^{1-\gamma}}d\eta\\&\le\frac{C}{\Gamma(\gamma)}\int_0^x\frac{\Upsilon^{\alpha}(\eta)}{(x-\eta)^{1-\gamma}}d\eta\le\frac{C}{\Gamma(\gamma)}\left(\int_0^x\left(\frac{1}{(x-\eta)^{1-\gamma}}\right)^{q}d\eta\right)^{1/q}\left(\int_0^x\Upsilon^{p\alpha}(\eta)d\eta\right)^{1/p}\\&\le\frac{C}{\Gamma(\gamma)(1+(\gamma-1)q)^{1/q}}x^{\frac{(\gamma-1)q+1}{q}}\Pi^{1/p}(x),\end{aligned}$$

where $q>1$ satisfies $(\gamma-1)q+1>0$ and $p=q/(q-1)$, and $\Pi(x)$ is given by

$$\Pi(x)=\int_0^x\Upsilon^{p\alpha}(t)dt.$$

from here, we have the following inequality

$$\Upsilon^{p}(x)\le C_1x^{p\frac{(\gamma-1)q+1}{q}}\Pi(x). \tag{6}$$

This estimation leads to

$$\Pi'(x)=\Upsilon^{p\alpha}(x)\le C_2x^{p\alpha\frac{(\gamma-1)q+1}{q}}\Pi^{\alpha}(x).$$

Multiplying the both sides of the last inequality with $(1-\alpha)\Pi^{-\alpha}(x)$,

$$(1-\alpha)\Pi^{-\alpha}(x)\Pi'(x)=\frac{d}{dx}\left[\Pi^{1-\alpha}(x)\right]\le(1-\alpha)C_2x^{p\alpha\frac{(\gamma-1)q+1}{q}}$$

is obtained. If both sides of the above inequality are integrated from 0 to x, one gets

$$\Pi^{1-\alpha}(x) \le C_3 x^{p\alpha \frac{(\gamma-1)q+1}{q}+1},$$

where $\Pi(0)=0$ is used. Consequently, this yields the following estimation for $\Pi(x)$

$$\Pi(x) \le C_4 x^{p\alpha \frac{(\gamma-1)q+1}{q(1-\alpha)}+\frac{1}{1-\alpha}}. \tag{7}$$

Determining (7) in (6), one can compute

$$\Upsilon(x) \le C_4 x^{\frac{(\gamma-1)q+1}{q}} x^{\alpha \frac{(\gamma-1)q+1}{q(1-\alpha)}+\frac{1}{p(1-\alpha)}} \le C_4 x^{\frac{\gamma}{1-\alpha}}$$

since $\dfrac{p+q}{pq}=1.$

Now let $\Pi_1(x)=x^{-L}\Pi(x),$ where L fulfils the inequality $\gamma-L(1-\alpha)>0$ as it is assumed. Hence, from the last inequality it is $0 \le \Pi_1(x) \le C_4 x^{\frac{\gamma}{1-\alpha}-L}$.

It can be observed that $\Pi_1(x)$ is continuous on $[0,T_0]$ and $\Pi_1(0)=0.$ Let us now see that $\Pi(x)$ vanishes identically on $[0,T_0]$. For this, suppose otherwise and let $\Pi_1(x)\not\equiv 0$. As a result of its continuity and non-negative valuedness, there is a point $x_0 \in (0,T_0]$ at which $\Pi_1(x)$ takes its maximum value. This enables the following assignation

$$M=\Pi_1(x_0)=\max_{x\in[0,T_0]}\Pi_1(x).$$

By using the inequality in (4),

$$M=\Pi_1(x_0)=x_0^{-L}\Pi(x_0) \le \frac{x_0^{-L}}{\Gamma(\gamma)}\int_0^{x_0} \frac{\eta^{\gamma}\left|f\left(\eta,\upsilon_1(\eta)\right)-f\left(\eta,\upsilon_2(\eta)\right)\right|}{\eta^{\gamma}\left(x_0-\eta\right)^{1-\gamma}}d\eta$$

$$\le \frac{Lx_0^{-L}}{\Gamma(\gamma)}\int_0^{x_0}\eta^{-\gamma}\left(x_0-\eta\right)^{\gamma-1}\Pi(\eta)d\eta \le \frac{Lx_0^{-L}}{\Gamma(\gamma)}\int_0^{x_0}\eta^{L-\gamma}\left(x_0-\eta\right)^{\gamma-1}\Pi_1(\eta)dt$$

$$\le M\,\frac{\Gamma(L-\gamma+1)}{\Gamma(L)}<M$$

for any $t\in[0,x_0],$ since $\Gamma(L-\gamma+1)\le\Gamma(L).$ However, this is a contradiction. Hence, $\Pi_1(x)\equiv 0$ on $[0,T_0]$, i.e $\upsilon_1=\upsilon_2,$ which is the desired result.

In the following we will give some observations on conditions of the theorem given above and comparison with the Nagumo-type uniqueness theorem given by [20].

Remark 2.2 Consider the inequality $\gamma - L(1-\alpha) > 0$ in the hypothesis of the previous theorem. It is obvious that any value of $\alpha \in \left(0, \gamma\Gamma\left(1-\gamma\right)\delta^{-1}\right)$ with $\delta > 1$ and $L = \beta\left(\Gamma(1-\gamma)\right)^{-1}$ with $1 < \beta < \delta$ satisfy the above inequality. Here, $L = \beta\left(\Gamma(1-\gamma)\right)^{-1} > \left(\Gamma(1-\gamma)\right)^{-1}$, i.e. L is greater than the Nagumo constant. Moreover, $\Gamma(L-\gamma+1) < \Gamma(L)$ means that L cannot be greater than 1. This is the limitation of Theorem 2.1.

A Kooi-type uniqueness result which is known as a generalization of Krasnoselskii-Krein type uniqueness theorem is given as follows. Its proof is here omitted since it can be proved by the similar way used for the above theorem.

Theorem 2.3 Let $0 < \gamma < 1$ and conditions (C1)-(C2) be fulfilled. Furthermore, suppose that there exist a $L > 0$ fulfilling the inequality $\Gamma(L-\gamma+1) \le \Gamma(L)$ so that

$$x^{\gamma}\left|f(x,\upsilon_1) - f(x,\upsilon_2)\right| \le L\left|\upsilon_1 - \upsilon_2\right|$$

holds for all $(x,\upsilon_1),(x,\upsilon_2) \in [0,T]\times\mathbb{R}$, and that there exist $C > 0$, $\beta \in [0,\gamma)$ and $\alpha \in (0,1)$ with $\gamma - \beta - L(1-\alpha) > 0$ such that

$$x^{\beta}\left|f(x,\upsilon_1) - f(x,\upsilon_2)\right| \le C\left|\upsilon_1 - \upsilon_2\right|^{\alpha}$$

is satisfied for all $(x,\upsilon_1),(x,\upsilon_2) \in [0,T]\times\mathbb{R}$.
Then, problem (1) has a unique solution in the space of $C\left(\left[0,T_0\right]\right)$.

Replacing x^{γ} by $x^{1+\gamma}$ in the inequality (4) and by putting forward a growth condition on the right hand side function $f\left(x,\upsilon\right)$, we reveal the Roger-type uniqueness criterion as follows:

Theorem 2.4 Let $0 < \gamma < 1$ and conditions (C1)-(C2) be fulfilled. Furthermore, suppose that there exists c with $c \ge T_0^{\frac{q(1-\gamma)-1}{q}}\Gamma(\gamma)\left[1-q(1-\gamma)\right]^{\frac{1}{q}}$ $\left(q < \left(1-\gamma\right)^{-1}\right)$ so that

$$x^{1+\gamma}\left|f(x,\upsilon_1) - f(x,\upsilon_2)\right| \le c\left|\upsilon_1 - \upsilon_2\right| \tag{7}$$

holds for all $(x,\upsilon_1),(x,\upsilon_1) \in [0,T]\times\mathbb{R}$, and that there exist $p > \dfrac{1}{\gamma}$ so that

$$f(x,\upsilon) = o\left(\exp\left(-\frac{1}{p\left[1-p(1+\gamma)\right]x^{\left[p(1+\gamma)-1\right]}}\right)x^{-(1+\gamma)}\right) \tag{8}$$

is uniformly valid for $0 < \upsilon < \beta$ with an arbitrary $\beta \in \mathbb{R}$.

Then, problem (1) possesses at most one solution in the space of $C([0,T_0])$.

Proof. As in the proof of the previous theorem, we again assume that problem (1) has two different solutions such as $\upsilon_1,\upsilon_2 \in C([0,T_0])$. We will prove by contradiction that they are actually equal. By using inequality (7) in the integral representation of $|\upsilon_1(x)-\upsilon_2(x)|$ and by applying Hölder inequality to the obtained inequality, respectively, we have

$$\begin{aligned}
\Upsilon(x) = |\upsilon_1(x)-\upsilon_2(x)| &\le \frac{1}{\Gamma(\gamma)}\int_0^x \frac{\eta^{1+\gamma}\,|f(\eta,\upsilon_1(\eta))-f(\eta,\upsilon_2(\eta))|}{\eta^{1+\gamma}(x-\eta)^{1-\gamma}}dt \\
&\le \frac{c}{\Gamma(\gamma)}\int_0^x \frac{\Upsilon(\eta)}{\eta^{1+\gamma}(x-\eta)^{1-\gamma}}d\eta \\
&\le \frac{c}{\Gamma(\gamma)}\left(\int_0^x \frac{1}{(x-\eta)^{q(1-\gamma)}}\right)^{1/q}\left(\int_0^x \left(\frac{\Upsilon(\eta)}{\eta^{1+\gamma}}\right)^p d\eta\right)^{1/p} \\
&\le \frac{c}{\Gamma(\gamma)}\frac{T_0^{\frac{1-q(1-\gamma)}{q}}}{[1-q(1-\gamma)]^{\frac{1}{q}}}\left(\int_0^x \left(\frac{\Upsilon(\eta)}{\eta^{1+\gamma}}\right)^p d\eta\right)^{1/p} \\
&\le \left(\int_0^x \left(\frac{\Upsilon(\eta)}{\eta^{1+\gamma}}\right)^p d\eta\right)^{1/p} := \Xi^{1/p}(x)
\end{aligned}$$

i.e. $\Upsilon^p(x) \le \Xi(x)$ From here, one can draw the following conclusion

$$\frac{d}{dx}\Xi(x) = \frac{\Upsilon^p(x)}{x^{p(1+\gamma)}} \le \frac{\Xi(x)}{x^{p(1+\gamma)}} \quad \text{and} \quad \frac{d}{dx}\left[e^{H(x)}\Xi(x)\right] \le 0$$

for all $x \in [0,T_0]$, where

$$H(x) = -\frac{1}{[1-p(1+\gamma)]\,x^{[p(1+\gamma)-1]}}.$$

This implies the non-increasing nature of $e^{H(x)}\Xi(x)$. With this fact, if we show that $\lim_{x\to 0^+} e^{H(x)}\Xi(x) = 0$, then it means that $e^{H(x)}\Xi(x) \le 0$ and, therefore, that $\Upsilon(x) \le 0$ for all $x \in [0,T_0]$, i.e. $\Upsilon(x) \equiv 0$. Now, let us show what we need. First, by considering the growth condition for f in (8) one has

$$\Upsilon(x) \le \frac{1}{\Gamma(\gamma)}\int_0^x \frac{|f(\eta,\upsilon_1(\eta))-f(t,\upsilon_2(\eta))|}{(x-\eta)^{1-\gamma}}d\eta$$

$$\le \frac{2\varepsilon}{\Gamma(\gamma)} \int_0^x \frac{e^{-\frac{1}{p[1-p(1+\gamma)]t^{[p(1+\gamma)-1]}}}\eta^{-(1+\gamma)}}{(x-\eta)^{1-\gamma}} d\eta$$

$$\le \frac{2\varepsilon}{\Gamma(\gamma)} \left(\int_0^x \frac{1}{(x-\eta)^{1-\gamma}} d\eta \right)^{1/q} \left(\int_0^x e^{-\frac{1}{[1-p(1+\gamma)]\eta^{[p(1+\gamma)-1]}}} \eta^{-p(1+\gamma)} d\eta \right)^{1/p}$$

$$\le C \left(e^{-\frac{1}{[1-p(1+\gamma)]x^{[p(1+\gamma)-1]}}} \right)^{1/p},$$

which implies that $\Upsilon^p(x) \le C\varepsilon e^{-\frac{1}{[1-p(1+\gamma)]x^{[p(1+\gamma)-1]}}}$. By using this inequality, we obtain

$$e^{-H(x)}\Xi(x) = \frac{1}{\Gamma(\gamma)} e^{-H(x)} \int_0^x \frac{\Upsilon^p(\eta)}{\eta^{p(1+\gamma)}} d\eta$$

$$\le \frac{C\varepsilon}{\Gamma(\gamma)} e^{-H(x)} \int_0^x \frac{e^{-\frac{1}{[1-p(1+\gamma)]\eta^{[p(1+\gamma)-1]}}}}{\eta^{p(1+\gamma)}} d\eta$$

$$\le C\varepsilon e^{-H(x)} e^{H(x)} = C\varepsilon$$

which leads to $\lim_{x\to 0^+} e^{H(x)}\Xi(x) = 0$, as desired. This completes the proof.

To prove Theorem 2.6, we need the following lemma. Its proof can be derived by following the way depicted by Banaś and Rivero in [21].

Lemma 2.5 Let $\Upsilon \in C([0,T])$ be a non-negative function and $g \in C(0,T]$ so that they satisfy the following conditions:

(C3) There exist a function $G \in AC(0,T]$ so that $G'(x) = g^p(x)$ a.e. in $(0,T)$ and $\lim_{x\to 0^+} G(x)$ exists.

(C4) $\Upsilon^p(x) \le \int_0^x g^p(\eta)\Omega^p(\eta)d\eta$ for $x \in [0,T]$ and $p \ge 1$.

(C5) $\Upsilon^p(x) = o(exp(G(x)))$.

Then, $\Upsilon(x) \equiv 0$ on $[0,T]$.

By the aid of this lemma, uniqueness theorem to an initial value problem for the ODE in [22] can be extended to that for problem (1) as follows:

Theorem 2.6 Let conditions (C1) and (C2) be fulfilled. Furthermore, suppose that the function g and G are given as in the previous lemma, and

$c_1 \ge \Gamma(\gamma)[1-q(1-\gamma)]^{\frac{1}{q}} a^{-\frac{1-q(1-\gamma)}{q}}$ $\left(1 < q < \frac{1}{1-\gamma}\right)$ so that the following conditions are satisfied:

$$\left| f(x,\upsilon_1)- f(x,\upsilon_2)\right| \le c_1 g(x)\left|\upsilon_1-\upsilon_2\right| \tag{10}$$

holds for all $(x,\upsilon_1),(x,\upsilon_1)\in[0,T]\times\mathbb{R}$ and,

$$\left| f(x,\upsilon_1)- f(x,\upsilon_2)\right| = o(c_1\left(g(x)\right)^{1/p} exp(\frac{1}{p}G(x))) \qquad (\ p=\frac{q}{q-1}>\frac{1}{\gamma}\ ) \tag{11}$$

is satisfied when $x\to 0$ uniformly with respect to υ_1,υ_2 in each closed interval in the type of $[-b,b]$ for any $b\in\mathbb{R}$. Then, problem (1) has at most one solution in the space of $C\left(\left[0,T_0\right]\right)$.

Proof. Let $\upsilon_1,\upsilon_2\in C\left(\left[0,T_0\right]\right)$ be two different solutions of the problem. By contradiction let us show that they are identical on $\left[0,T_0\right]$. By applying inequality (10) to the integral representation of υ_1,υ_2 in (3) and Hölder inequality with $1<q<\frac{1}{1-\gamma}$ and $p=\frac{q}{q-1}>\frac{1}{\gamma}$ to the resultant inequality, one can deduce

$$\begin{aligned}
\Upsilon(x)=\left|\upsilon_1(x)-\upsilon_2(x)\right| &\le \frac{1}{\Gamma(\gamma)}\int_0^x \frac{\left| f(\eta,\upsilon_1(\eta))- f(\eta,\upsilon_2(\eta))\right|}{(x-\eta)^{1-\gamma}}dt \\
&\le \frac{c_1}{\Gamma(\gamma)}\int_0^x \frac{g(\eta)\Upsilon(\eta)}{(x-\eta)^{1-\gamma}}d\eta \\
&\le \frac{c_1}{\Gamma(\gamma)}\left(\int_0^x \frac{1}{(x-\eta)^{q(1-\gamma)}}\right)^{1/q}\left(\int_0^x g^p(\eta)\Upsilon^p(\eta)d\eta\right)^{1/p} \\
&\le \frac{c_1}{\Gamma(\gamma)}\frac{x^{\frac{1-q(1-\gamma)}{q}}}{\left[1-q(1-\gamma)\right]^{\frac{1}{q}}}\left(\int_0^x g^p(\eta)\Upsilon^p(\eta)dt\right)^{1/p} \\
&\le \left(\int_0^x g^p(\eta)\Upsilon^p(\eta)d\eta\right)^{1/p}
\end{aligned}$$

where $c_1\ge\Gamma(\gamma)\left[1-q(1-\gamma)\right]^{\frac{1}{q}} a^{-\frac{1-q(1-\gamma)}{q}}$. Hence, condition (C4) of Lemma 2.5 holds.

If inequality (11) is applied to the integral representation of υ_1,υ_2 in (3), and if Hölder inequality with same p and q above is used to the obtained inequality, then

$$\begin{aligned}
\Upsilon(x)=\left|\upsilon_1(x)-\upsilon_2(x)\right| &\le \frac{1}{\Gamma(\gamma)}\int_0^x \frac{\left| f(\eta,\upsilon_1(\eta))- f(\eta,\upsilon_2(\eta))\right|}{(x-\eta)^{1-\gamma}}dt \\
&\le \frac{\epsilon c_1}{\Gamma(\gamma)}\int_0^x \frac{\left[g(\eta)exp(G(\eta))\right]^{1/p}}{(x-\eta)^{1-\gamma}}d\eta
\end{aligned}$$

$$\leq \frac{\epsilon c_1}{\Gamma(\gamma)}\left(\int_0^x \frac{1}{(x-\eta)^{1-\gamma}}\right)^{1/q}\left(\int_0^x g(\eta)exp(G(\eta))d\eta\right)^{1/p}$$

$$\leq \frac{\epsilon c_1}{\Gamma(\gamma)}\frac{a^{\frac{1-q(1-\gamma)}{q}}}{\left[1-q(1-\gamma)\right]^{\frac{1}{q}}}\left(\int_0^x g(\eta)exp(G(\eta))d\eta\right)^{1/p}$$

which yields

$$\Upsilon(x) \leq \epsilon\int_0^x g(t)exp(G(t))dt = \epsilon exp(\frac{1}{p}G(x)).$$

This means that condition (C5) is fulfilled. Thus, it has been shown that all conditions of Lemma 2.5 are fulfilled. As a result of the lemma, $\Upsilon(x)$ vanishes identically on $\left[0,T_0\right]$, which indicates that problem (1) cannot have more than one solution.

3. Conclusion

In this work, we studied with Riemann-Liouville fractional differential equations with right-hand side functions having a singularity with respect to their first variable. The existence of local continuous solutions to the problem was briefly addressed. Furthermore, a Krasnosel'skii- Krein type uniqueness theorem for the problem was given for getting a better estimate for the Nagumo constant. For this result, Hölder's inequality, which is one of the most important tools of Lebesgue spaces, has been adapted to the available proof method. By using same inequality with the available methods, Kooi-type, Roger-type and Banaś-Rivero-type theorems were revealed under some appropriate conditions.

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