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On Some Operators Generated by G-Method and Stack

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Abstract

In this paper, two operators $\varphi_{\mathscr{S}}^{G}, \Psi_{\mathscr{S}}^{G}$ are introduced by using G-method and stack, and a strong generalized topology $\tau_{\mathscr{S}}^{G}$ which is finer than G- generalized topology is obtained via the operator $\Psi_{\mathscr{S}}^{G}$. In addition, two new operators $\Gamma_{\mathscr{S}}^{G}, \Psi_{\Gamma}^{G}$ are defined and a new strong generalized topology $\sigma_{\mathscr{S}}^{G}$ is attained by Ψ_{Γ}^{G}. Basic properties of these operators are investigated and the relationships between these two strong topologies obtained and G-generalized topology are examined.

Keywords: *stack, G-method, G-closure, strong generalized topology*
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1. Introduction

The concept of sequential convergence is an substantial research subject in many branches of mathematics. Many types of convergence as a generalization of convergence have been studied by many researchers in different spaces (see [1, 7, 17, 18, 21]). Connor and Grosse-Erdmann [2] introduced G-methods defined on a linear subspace of the vector space of all real sequences by using linear functional G. Also, they defined G-convergence and G-continuity with the help of G-method. Çakallı [5, 6] defined G-sequential compactness, G-sequential continuity and G-sequential connectedness by extending the concepts to topological groups which satisfy the first axiom of countability. Mucuk and Şahan[12] gave a further investigation of G-sequential continuity these groups. Recently, Lin and Liu [14] introduced G-method and G-convergence in arbitrary sets and investigated the operators of G-hull, G-closure, G-kernel and G-interior. They defined G-generalized topology induced by G-methods. Thus, they expanded and enhanced some results for G-method on first countable topological groups. After than many studies on the G-method have done by Liu and the others in the light of the given paper(see [8, 15, 16, 20]).
The concept of stack like the concept of sequential convergence play an important role in many branches of mathematics such as topology, logic, measure theory. Grimeisen [13] and Thron [19] introduced the stack on a set. Later, many researchers as Hosny[9], Hosny and Al-Kadi[10] and Min and Kim[11] have studied on stacks.
In this paper, we introduce two operators $\varphi_{\mathscr{S}}^{G}, \Psi_{\mathscr{S}}^{G}$ by using G-method and stack. Then, we investigate basic properties of them and generated a strong generalized topology which is finer than G-generalized topology via the operator $\Psi_{\mathscr{S}}^{G}$. Also, we define new two operators $\Gamma_{\mathscr{S}}^{G}, \Psi_{\Gamma}^{G}$ and characterize them. Then, we obtain another strong generalized topology with the help of Ψ_{Γ}^{G}. In particular, we investigated when these two strong generalized topologies could be equal to G-generalized topology. We also gave various counter examples for supporting the study.

2. Preliminaries

Let X be a set and $s(X)$ denotes the set of all X-valued sequences, i.e. $\mathbf{x} \in S(X)$ if and only if $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ is a sequence with each $x_n \in X$. If $f : X \to Y$ is a mapping, then $f(\mathbf{x}) = \{f(x)\}_{n\in\mathbb{N}}$ for each $\mathbf{x} \in s(X)$. If X is a topological space, $c(X)$ denotes the set of all X-valued convergent sequences. Throughout the paper, all topological spaces are assumed to satisfy the T_2-separation property.

Definition 2.1. *[14] Let X be a set .*

1. *A method on X is a function $G : c_G(X) \to X$ defined on a subset $c_G(X)$ of $s(X)$.*
2. *A sequence $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}}$ in X is said to be G-convergent to $l \in X$ if $\boldsymbol{x} \in c_G(X)$ and $G(\boldsymbol{x}) = l$.*

Definition 2.2. *[14] Let X be a topological space.*

1. *A method $G : c_G(X) \to X$ is called regular if $c(X) \subseteq c_G(X)$ and $G(\boldsymbol{x}) = \lim \boldsymbol{x}$ for each $\boldsymbol{x} \in c(X)$.*
2. *A method $G : c_G(X) \to X$ is called subsequential if, whenever $\boldsymbol{x} \in c_G(X)$ is G-convergent to $l \in X$, then there exists a subsequence $\boldsymbol{x}' \in c(X)$ of $\boldsymbol{x}$ with $\lim \boldsymbol{x}' = l$.*

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Definition 2.3. *[14] Let X be a set, G be a method on X and $A \subseteq X$.*

1. *A is called a G-closed set of X if whenever $\mathbf{x} \in s(A) \cap c_G(X)$, then $G(\mathbf{x}) \in A$. $X \backslash A$ is a G-open set if A is a G-closed set.*
2. *The G-closure of A is defined as the intersection of all G-closed sets containing A, and the G-closure of A is denoted by $\overline{A}^G$. The G-interior of A is defined as the union of all G-open sets contained in A, and the G-interior of A is denoted by $A^{\circ G}$.*

Lemma 2.4. *[14] Let X be a topological space.*

1. *If G is a regular method on X, then every G-closed set of X is sequentially closed.*
2. *If G is a subsequential method on X, then every sequentially closed set of X is G-closed.*

Definition 2.5. *[14] Let G be a method on a topological space X. X is said to be a G-sequential space if every G-closed set in X is closed.*

Proposition 2.6. *[14] Let G be a method on a set X and $A \subseteq X$. Then $x \in \overline{A}^G$ if and only if every subset U of X with $x \in U^{\circ G}$ intersects A.*

Definition 2.7. *[14] Let X be a set, G be a method on X and $Y \subseteq X$. Put $c_{G|Y}(Y) = \{\mathbf{x} \in s(Y) \cap c_G(X) : G(\mathbf{x}) \in Y\}$. The function $G|_Y : c_{G|Y}(Y) \to Y$ is called the submethod of G on Y.*

Definition 2.8. *[14] Let G be a method on a set X. The family $\tau_G = \{U \subseteq X : U \text{ is } G\text{-open in } X\}$ is a generalized topology on X and it is called the G-generalized topology on X.*

Definition 2.9. *[13, 19] A collection $\mathscr{S}$ of the subsets of a set X is called a stack if $A \in \mathscr{S}$ whenever $B \in \mathscr{S}$ and $B \subseteq A$. A stack is proper if $\emptyset \notin \mathscr{S}$.*

Definition 2.10. *[3, 4] Let X be a nonempty set and μ be a collection of subsets of X. μ is called a generalized topology on X if and only if $\emptyset \in \mu$ and μ is closed under arbitrary unions. Moreover, if $X \in \mu$ then μ is called a strong generalized topology.*

3. The Operator $\varphi_{\mathscr{S}}^G$

Definition 3.1. *Let G be a method and $\mathscr{S}$ be a stack on a set X. For $A \subseteq X$, we define the following operator:*

$$\varphi_{\mathscr{S}}^G(A) = \{x \in X : A \cap U \in \mathscr{S} \text{ for every } G\text{-open set } U \text{ containing } x\}.$$

Proposition 3.2. *Let G be a method and $\mathscr{S}, \mathscr{S}_1, \mathscr{S}_2$ be three stacks on a set X. For $A, B \subseteq X$, the following statements hold:*

1. *$A \subseteq B$ implies $\varphi_{\mathscr{S}}^G(A) \subseteq \varphi_{\mathscr{S}}^G(B)$.*
2. *$\mathscr{S}_1 \subseteq \mathscr{S}_2$ implies $\varphi_{\mathscr{S}_1}^G(A) \subseteq \varphi_{\mathscr{S}_2}^G(A)$.*
3. *If $T \notin \mathscr{S}$, then $\varphi_{\mathscr{S}}^G(T) = \emptyset$.*

Proof.

1. Let $x \notin \varphi_{\mathscr{S}}^G(B)$. Then, there exists a G-open set U containing x such that $B \cap U \notin \mathscr{S}$. Since $A \subseteq B$, we have $A \cap U \notin \mathscr{S}$. Thus, $x \notin \varphi_{\mathscr{S}}^G(A)$.
2. Assume that $\mathscr{S}_1 \subseteq \mathscr{S}_2$. Let $x \notin \varphi_{\mathscr{S}_2}^G(A)$. Then, there is a G-open set U containing x such that $A \cap U \notin \mathscr{S}_2$. By hypothesis, we get $A \cap U \notin \mathscr{S}_1$. Hence $x \notin \varphi_{\mathscr{S}_1}^G(A)$.
3. Since $T \cap U \subseteq T$ and $T \notin \mathscr{S}$ for each G-open set U containing $x \in X$, we obtain $T \cap U \notin \mathscr{S}$. Thus, $\varphi_{\mathscr{S}}^G(T) = \emptyset$.

□

Proposition 3.3. *Let G be a method and $\mathscr{S}$ be a proper stack on a set X. For $A \subseteq X$, the following statements hold:*

1. *$\varphi_{\mathscr{S}}^G(A) \subseteq \overline{A}^G$.*
2. *$\varphi_{\mathscr{S}}^G(\varphi_{\mathscr{S}}^G(A)) \subseteq \varphi_{\mathscr{S}}^G(A)$.*

Proof.

1. Let $x \notin \overline{A}^G$. Then, there exists a subset U of X with $x \in U^{\circ G}$ such that $U \cap A = \emptyset \notin \mathscr{S}$. Hence, we get $U^{\circ G} \cap A \notin \mathscr{S}$. Since $U^{\circ G}$ is G-open and $x \in U^{\circ G}$, we say $x \notin \varphi_{\mathscr{S}}^G(A)$.
2. Suppose that $x \in \varphi_{\mathscr{S}}^G(\varphi_{\mathscr{S}}^G(A))$. Then, we have $\varphi_{\mathscr{S}}^G(A) \cap U \in \mathscr{S}$ for every G-open set U containing x. That is, $\varphi_{\mathscr{S}}^G(A) \cap U \neq \emptyset$. Thus, there exists an element $y \in \varphi_{\mathscr{S}}^G(A) \cap U$. Since U is G-open set containing y and $y \in \varphi_{\mathscr{S}}^G(A)$, we get $A \cap U \in \mathscr{S}$. Hence, $x \in \varphi_{\mathscr{S}}^G(A)$.

□

The following examples show that the converse implications of Proposition 3.2(1) and the equalities in Proposition 3.3(1) and (2) are not true in general.

Example 3.4. *Let's take the example of the G-method in [[14], Example 2.13(1)]. Let X be the set of all integers. Put $c_G(X) = s(X)$ and $G : c_G(X) \to X$ is defined by $G(\mathbf{x}) = 0$ for each $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}} \in c_G(X)$. Let $\mathscr{S} = \{S \subseteq X : 1 \in S\}$ be a stack on X. For $A = \{1,2\}$ and $B = \{3\}$, we have $\varphi_{\mathscr{S}}^G(A) = \{0,1\}$ and $\varphi_{\mathscr{S}}^G(B) = \emptyset$. Also, we get $\overline{A}^G = \{0,1,2\}$.*

Example 3.5. *Let's take the example of G-method in [[14], Example 7.4(1)]. Let X be the set of all integers with the discrete topology. Put $c_{G_1}(X) = \{\{x_n\}_{n\in\mathbb{N}} \in S(X)$: there exists $m \in \mathbb{N}$ such that $\{x_n - x_{n-1}\}_{n>m}$ is a constant sequence $\}$. $G_1 : c_{G_1}(X) \to X$ is defined by $G_1(\mathbf{x}) = \lim_{n\to\infty}(x_{n+1} - x_n)$ for each $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}} \in c_{G_1}(X)$. Let $Y = 2\mathbb{N}$ be a space of X with $G_2 = G_1|_Y$ and $\mathscr{S} = \{S \subseteq Y : \{2,4\} \subseteq S\}$ be a stack on Y. For $A = \{0,2,4\}$, we have $\varphi_{\mathscr{S}}^G(\varphi_{\mathscr{S}}^G(A)) = \emptyset \neq \varphi_{\mathscr{S}}^G(A) = \{0\}$.*

Remark 3.6. *Note that $\varphi^G_{\mathscr{S}}(X) \subseteq X$ but the equality is not true in general. Moreover, there is no relationship between $\varphi^G_{\mathscr{S}}(A)$ and A for $A \subseteq X$. We can easily see that in Example 3.4 because $\varphi^G_{\mathscr{S}}(X) = \{0,1\} \neq X$ and $\varphi^G_{\mathscr{S}}(\{1,2\}) = \{0,1\}$. Also, $\varphi^G_{\mathscr{S}}(\emptyset) = \emptyset$ for a proper stack $\mathscr{S}$.*

Proposition 3.7. *Let G be a method and $\mathscr{S}$ be a stack on a set X. $\mathscr{S}$ is the superset of all G-open sets other than empty set if and only if $\varphi^G_{\mathscr{S}}(X) = X$.*

Proof. It is clear. □

Lemma 3.8. *Let G be a method, $\mathscr{S}$ be a proper stack on a set X and $A \subseteq X$. If $A \cap U \notin \mathscr{S}$ for some G-open set U containing x, then $\varphi^G_{\mathscr{S}}(A) \cap U \notin \mathscr{S}$. Moreover, $\varphi^G_{\mathscr{S}}(A) \cap U = \emptyset$.*

Proof. Let U be a G-open set containing x such that $A \cap U \notin \mathscr{S}$. Assume that $\varphi^G_{\mathscr{S}}(A) \cap U \in \mathscr{S}$. From here, $\varphi^G_{\mathscr{S}}(A) \cap U \neq \emptyset$. Then, there exists an element $y \in \varphi^G_{\mathscr{S}}(A) \cap U$. Since $y \in \varphi^G_{\mathscr{S}}(A)$ and U is G-open containing y, we have $A \cap U \in \mathscr{S}$. This contradicts our hypothesis. Thus, $\varphi^G_{\mathscr{S}}(A) \cap U \notin \mathscr{S}$. Similarly, it is shown that $\varphi^G_{\mathscr{S}}(A) \cap U = \emptyset$. □

Proposition 3.9. *Let G be a method and $\mathscr{S}$ be a proper stack on a set X. For $A \subseteq X$, the following statements hold:*

1. *$\varphi^G_{\mathscr{S}}(A)$ is G-closed.*
2. *If A is G-closed then $\varphi^G_{\mathscr{S}}(A) \subseteq A$.*

Proof.

1. Let $x \notin \varphi^G_{\mathscr{S}}(A)$. Then, there exists a G-open set U containing x such that $A \cap U \notin \mathscr{S}$. From Lemma 3.8, we have $\varphi^G_{\mathscr{S}}(A) \cap U = \emptyset$. Since U is G-open, we get $x \notin \overline{\varphi^G_{\mathscr{S}}(A)}^G$. This implies that $\overline{\varphi^G_{\mathscr{S}}(A)}^G \subseteq \varphi^G_{\mathscr{S}}(A)$.Thus, we get $\varphi^G_{\mathscr{S}}(A) = \overline{\varphi^G_{\mathscr{S}}(A)}^G$. That is, $\varphi^G_{\mathscr{S}}(A)$ is G-closed.
2. Assume that A is G-closed and $x \notin A$. Then, $X \backslash A$ is G-open set containing x. Since $(X \backslash A) \cap A = \emptyset \notin \mathscr{S}$, we have $x \notin \varphi^G_{\mathscr{S}}(A)$.

□

Proposition 3.10. *Let G be a method and $\mathscr{S}$ be a proper stack on a set X. For $A \subseteq X$, $\varphi^G_{\mathscr{S}}(A \cup \varphi^G_{\mathscr{S}}(A)) = \varphi^G_{\mathscr{S}}(A)$.*

Proof. By Proposition 3.2(1), we have $\varphi^G_{\mathscr{S}}(A) \subseteq \varphi^G_{\mathscr{S}}(A \cup \varphi^G_{\mathscr{S}}(A))$. Let's show that the converse inclusion and $x \notin \varphi^G_{\mathscr{S}}(A)$. Then, there exists a G-open set U containing x such that $A \cap U \notin \mathscr{S}$. By Lemma 3.8, we get $\varphi^G_{\mathscr{S}}(A) \cap U = \emptyset$. Then, $(A \cup \varphi^G_{\mathscr{S}}(A)) \cap U = A \cap U \notin \mathscr{S}$. Thus, $x \notin \varphi^G_{\mathscr{S}}(A \cup \varphi^G_{\mathscr{S}}(A))$. □

Theorem 3.11. *Let G be a method and $\mathscr{S}$ be a proper stack on a set X.*

$$A \subseteq \varphi^G_{\mathscr{S}}(A) \text{ if and only if } \varphi^G_{\mathscr{S}}(A) = \overline{A}^G.$$

Proof. Assume that $A \subseteq \varphi^G_{\mathscr{S}}(A)$. Since $\varphi^G_{\mathscr{S}}(A)$ is G-closed, we have $\overline{A}^G \subseteq \varphi^G_{\mathscr{S}}(A)$. Also, $\varphi^G_{\mathscr{S}}(A) \subseteq \overline{A}^G$ from Proposition 3.3(1). Thus, $\overline{A}^G = \varphi^G_{\mathscr{S}}(A)$. The converse implication is clear from the definition of $\overline{A}^G$. □

Definition 3.12. *Let G be a method and $\mathscr{S}$ be a stack on a set X. For $A \subseteq X$, we define the following operator:*

$$\Psi^G_{\mathscr{S}}(A) = A \cup \varphi^G_{\mathscr{S}}(A).$$

Proposition 3.13. *Let G be a method and $\mathscr{S}$ be a stack on a set X. For $A, B \subseteq X$, the following statements hold:*

1. *$A \subseteq B$ implies $\Psi^G_{\mathscr{S}}(A) \subseteq \Psi^G_{\mathscr{S}}(B)$.*
2. *$A \subseteq \Psi^G_{\mathscr{S}}(A)$.*
3. *$\Psi^G_{\mathscr{S}}(A \cap B) \subseteq \Psi^G_{\mathscr{S}}(A) \cap \Psi^G_{\mathscr{S}}(B)$ and $\Psi^G_{\mathscr{S}}(A) \cup \Psi^G_{\mathscr{S}}(B) \subseteq \Psi^G_{\mathscr{S}}(A \cup B)$.*

Proof.

1. It is clear from the Proposition 3.2(1).
2. It is obvious.
3. The proofs are obvious from Proposition 3.2(1).

□

Proposition 3.14. *Let G be a method and $\mathscr{S}$ be a proper stack on a set X. For $A \subseteq X$, $\Psi^G_{\mathscr{S}}(\Psi^G_{\mathscr{S}}(A)) = \Psi^G_{\mathscr{S}}(A)$.*

Proof. $\Psi^G_{\mathscr{S}}(\Psi^G_{\mathscr{S}}(A)) = \Psi^G_{\mathscr{S}}(A \cup \varphi^G_{\mathscr{S}}(A)) = (A \cup \varphi^G_{\mathscr{S}}(A)) \cup \varphi^G_{\mathscr{S}}(A \cup \varphi^G_{\mathscr{S}}(A))$. From Proposition 3.10, we have $\Psi^G_{\mathscr{S}}(\Psi^G_{\mathscr{S}}(A)) = A \cup \varphi^G_{\mathscr{S}}(A) = \Psi^G_{\mathscr{S}}(A)$. □

The following examples show that the equalities in Proposition 3.13 are not true in general.

Example 3.15. *Consider Example 3.5. For $A = \{2\}$ and $B = \{4\}$, we have $\varphi^G_{\mathscr{S}}(A) = \emptyset = \varphi^G_{\mathscr{S}}(B)$ and $\varphi^G_{\mathscr{S}}(A \cup B) = \{0\}$. Thus, $\Psi^G_{\mathscr{S}}(A) \cup \Psi^G_{\mathscr{S}}(B) = \{2,4\} \neq \Psi^G_{\mathscr{S}}(A \cup B) = \{0,2,4\}$.*

Example 3.16. *Let $X=\{a,b,c,d\}$ and $\mathscr{S}=\{\{a,c\},\{a,b,c\},\{a,c,d\},\{a,b,d\},\{c,d\},\{b,c,d\},X\}$ be a stack on X. Put $c_G(X)=s(X)$ and $G: c_G(X)\to X$ is defined by $G(\mathbf{x})=d$ for each $\mathbf{x}\in c_G(X)$. For $A=\{a,c\}$ and $B=\{c,d\}$, we have $\varphi^G_{\mathscr{S}}(A)=\{d\}=\varphi^G_{\mathscr{S}}(B)$. Hence, $\Psi^G_{\mathscr{S}}(A)\cap\Psi^G_{\mathscr{S}}(B)=\{c,d\}\neq\Psi^G_{\mathscr{S}}(A\cap B)=\{c\}$.*

Remark 3.17. *Note that $\Psi^G_{\mathscr{S}}(X)=X$. Also, for a proper stack, $\Psi^G_{\mathscr{S}}(\emptyset)=\emptyset$ since $\varphi^G_{\mathscr{S}}(\emptyset)=\emptyset$.*

Theorem 3.18. *Let G be a method and $\mathscr{S}$ be a proper stack on a set X.*

$$\tau^G_{\mathscr{S}}=\{H\subseteq X:\Psi^G_{\mathscr{S}}(X\backslash H)=X\backslash H\}$$

is a strong generalized topology on X with $\tau_G\subseteq\tau^G_{\mathscr{S}}$.

Proof. From Remark 3.17, we have $\emptyset,X\in\tau^G_{\mathscr{S}}$. Let $H_i\in\tau^G_{\mathscr{S}}$ for $i\in I$. $\Psi^G_{\mathscr{S}}(X\backslash\cup_{i\in I}H_i)=(X\backslash\cup_{i\in I}H_i)\cup\varphi^G_{\mathscr{S}}(X\backslash\cup_{i\in I}H_i)=X\backslash\cup_{i\in I}H_i$ since $\varphi^G_{\mathscr{S}}(X\backslash\cup_{i\in I}H_i)\subseteq\varphi^G_{\mathscr{S}}(X\backslash H_i)\subseteq X\backslash H_i$. From here, $\cup_{i\in I}H_i\in\tau^G_{\mathscr{S}}$. Thus, $\tau^G_{\mathscr{S}}$ is a strong generalized topology on X. Also, let $U\in\tau_G$. Then, $X\backslash U$ is G-closed. From Proposition 3.9(2), $\varphi^G_{\mathscr{S}}(X\backslash U)\subseteq X\backslash U$. Thus, $\Psi^G_{\mathscr{S}}(X\backslash U)=X\backslash U$. That is, $U\in\tau^G_{\mathscr{S}}$ and $\tau_G\subseteq\tau^G_{\mathscr{S}}$. □

Example 3.19. *Consider Example 3.4, we obtain $\tau^G_{\mathscr{S}}=\{X\}\cup\{U\subseteq X:0\notin U\}\cup\{H\subseteq X:1\in H\}$ such that $\tau_G\subseteq\tau^G_{\mathscr{S}}$.*

The following example shows that $\tau^G_{\mathscr{S}}$ is not a topology in general.

Example 3.20. *Let $X=\{a,b,c,d\}$ and $\mathscr{S}=\{\{a,c\},\{a,b,c\},\{a,c,d\},X\}$ be a stack on X. Consider the G-method in Example 3.16. For $A=\{a,d\}$ and $B=\{c,d\}$, we have $A,B\in\tau^G_{\mathscr{S}}$ but $A\cap B\notin\tau^G_{\mathscr{S}}$.*

Theorem 3.21. *Let G be a method and $\mathscr{S}$ be a proper stack on a set X. For $H\in\tau^G_{\mathscr{S}}$, $H=\cup(V\backslash T)$ for G-open set V and $T\notin\mathscr{S}$.*

Proof. Firstly, we must show that $V\backslash T\in\tau^G_{\mathscr{S}}$ for G-open set V and $T\notin\mathscr{S}$. For this, let's prove $\varphi^G_{\mathscr{S}}(X\backslash(V\backslash T))\subseteq(X\backslash(V\backslash T))$. Let $x\in\varphi^G_{\mathscr{S}}((X\backslash V)\cup T)$. Then, for each G-open set U containing x such that $((X\backslash V)\cup T)\cap U=((X\backslash V)\cap U)\cup(T\cap U)\in\mathscr{S}$. If $(X\backslash V)\cap U=\emptyset$ then $T\cap U\in\mathscr{S}$. That is, $T\in\mathscr{S}$. This is a contradiction. In that case, $(X\backslash V)\cap U\neq\emptyset$. Since U is G-open, $x\in U=U^{\circ G}$. From here, $x\in\overline{(X\backslash V)}^G=(X\backslash V)\subseteq(X\backslash V)\cup T$. Thus, $V\backslash T\in\tau^G_{\mathscr{S}}$ for G-open set V and $T\notin\mathscr{S}$. Let $x\in H\in\tau^G_{\mathscr{S}}$. Then, $x\notin\varphi^G_{\mathscr{S}}(X\backslash H)$ and there exists G-open set V containing x such that $V\cap(X\backslash H)\notin\mathscr{S}$. Say $T=V\cap(X\backslash H)$. Then, $x\in V\backslash T\subseteq H$. □

Corollary 3.22. *Let G be a method and $\mathscr{S}=\mathscr{P}(X)\backslash\{\emptyset\}$ be a stack on a set X. Then, $\tau_G=\tau^G_{\mathscr{S}}$.*

Corollary 3.23. *Let G be a method and $\mathscr{S}$ be a proper stack on a topological space (X,τ).*

1. *If G is a subsequential method, then every sequentially closed set is $\tau^G_{\mathscr{S}}$-closed. In addition, if X is first countable space, then every closed set is $\tau^G_{\mathscr{S}}$-closed.*
2. *If G is regular method and $\mathscr{S}=\mathscr{P}(X)\backslash\{\emptyset\}$, then every $\tau^G_{\mathscr{S}}$-closed is sequentially closed. In addition, if X is first countable space, then every $\tau^G_{\mathscr{S}}$-closed set is closed .*
3. *If G is regular subsequential method and $\mathscr{S}=\mathscr{P}(X)\backslash\{\emptyset\}$, then sequentially closed sets coincide with $\tau^G_{\mathscr{S}}$-closed sets. In addition, if X is first countable space then closed sets coincide with $\tau^G_{\mathscr{S}}$-closed sets.*

4. The Operator $\Gamma^G_{\mathscr{S}}$

Definition 4.1. *Let G be a method and $\mathscr{S}$ be a stack on a set X. For $A\subseteq X$, we define the following operator:*

$$\Gamma^G_{\mathscr{S}}(A)=\{x\in X:A\cap\overline{U}^G\in\mathscr{S}\text{ for each }G\text{-open set }U\text{ containing }x\}.$$

Lemma 4.2. *Let G be a method and $\mathscr{S}$ be a stack on a set X. For $A\subseteq X$, $\varphi^G_{\mathscr{S}}(A)\subseteq\Gamma^G_{\mathscr{S}}(A)$.*

Proof. Let $x\in\varphi^G_{\mathscr{S}}(A)$. Then, for each G-open set U containing x, we have $A\cap U\in\mathscr{S}$. Since $A\cap U\subseteq A\cap\overline{U}^G$, we have $A\cap\overline{U}^G\in\mathscr{S}$. Thus, $x\in\Gamma^G_{\mathscr{S}}(A)$. □

The equality in Lemma 4.2 may not be hold in general.

Example 4.3. *Consider Example 3.16. For $A=\{c,d\}$, we have $\varphi^G_{\mathscr{S}}(A)=\{d\}\neq\Gamma^G_{\mathscr{S}}(A)=\{c,d\}$.*

Proposition 4.4. *Let G be a method and $\mathscr{S},\mathscr{S}_1,\mathscr{S}_2$ be three stacks on a set X. For $A,B\subseteq X$, the following statements hold:*

1. *$A\subseteq B$ implies $\Gamma^G_{\mathscr{S}}(A)\subseteq\Gamma^G_{\mathscr{S}}(B)$.*
2. *$\mathscr{S}_1\subseteq\mathscr{S}_2$ implies $\Gamma^G_{\mathscr{S}_1}(A)\subseteq\Gamma^G_{\mathscr{S}_2}(A)$.*
3. *If $T\notin\mathscr{S}$, then $\Gamma^G_{\mathscr{S}}(T)=\emptyset$.*

Proof.

1. Let $x\notin\Gamma^G_{\mathscr{S}}(B)$. Then, there exists a G-open set U containing x such that $B\cap\overline{U}^G\notin\mathscr{S}$. Since $A\cap\overline{U}^G\subseteq B\cap\overline{U}^G$, we have $A\cap\overline{U}^G\notin\mathscr{S}$. Thsu, $x\notin\Gamma^G_{\mathscr{S}}(A)$.
2. Assume that $\mathscr{S}_1\subseteq\mathscr{S}_2$. Let $x\notin\Gamma^G_{\mathscr{S}_2}(A)$. Then, there is a G-open set U containing x such that $A\cap\overline{U}^G\notin\mathscr{S}_2$. By hypothesis, we get $A\cap\overline{U}^G\notin\mathscr{S}_1$. Hence $x\notin\Gamma^G_{\mathscr{S}_1}(A)$.
3. Since $T\notin\mathscr{S}$ and $T\cap\overline{U}^G\subseteq T$ for each G-open set U containing $x\in X$, we have $T\cap\overline{U}^G\notin\mathscr{S}$. Thus, $x\notin\Gamma^G_{\mathscr{S}}(T)$ i.e. $\Gamma^G_{\mathscr{S}}(T)=\emptyset$.

□

The following example shows that the converse implication of Proposition 4.4(1) is not true in general.

Example 4.5. *Consider Example 3.16. For* $A=\{a,c\}$ *and* $B=\{c,d\}$*, we have* $\Gamma^G_{\mathscr{S}}(A)=\{d\}\subseteq\Gamma^G_{\mathscr{S}}(B)=\{c,d\}$ *but* $A\nsubseteq B$*.*

Remark 4.6. *Note that, there is no relationship between* $\Gamma^G_{\mathscr{S}}(A)$ *and* A*. For instance, if we take the stack* $\mathscr{S}=\{S\subseteq X:0\in S\}$ *in Example 3.4, we have* $\Gamma^G_{\mathscr{S}}(A)=X$ *and* $\Gamma^G_{\mathscr{S}}(B)=\emptyset$ *for* $A=\{0,1\}$ *and* $B=\{2\}$*. Moreover,* $\Gamma^G_{\mathscr{S}}(X)\subseteq X$ *but the equality is not true in general. We can see that in Example 3.4, since* $\Gamma^G_{\mathscr{S}}(X)=\{0,1\}\neq X$*. Also, for a proper stack,* $\Gamma^G_{\mathscr{S}}(\emptyset)=\emptyset$*.*

Lemma 4.7. *Let* G *be a method and* $\mathscr{S}$ *be a proper stack on a set* X *and let* $A\subseteq X$*. If* $A\cap\overline{U}^G\notin\mathscr{S}$ *for some* G*-open set* U *containing* x*, then* $\Gamma^G_{\mathscr{S}}(A)\cap U\notin\mathscr{S}$*. Moreover,* $\Gamma^G_{\mathscr{S}}(A)\cap U=\emptyset$*.*

Proof. Let U be a G-open set containing x such that $A\cap\overline{U}^G\notin\mathscr{S}$. Assume that $\Gamma^G_{\mathscr{S}}(A)\cap U\in\mathscr{S}$. From here, $\Gamma^G_{\mathscr{S}}(A)\cap U\neq\emptyset$. Then, there exists an element $y\in U$ and $y\in\Gamma^G_{\mathscr{S}}(A)$. From here, $A\cap\overline{U}^G\in\mathscr{S}$. This is a contradiction. Hence, $\Gamma^G_{\mathscr{S}}(A)\cap U\notin\mathscr{S}$. Analogously, $\Gamma^G_{\mathscr{S}}(A)\cap U=\emptyset$. □

Proposition 4.8. *Let* G *be a method and* $\mathscr{S}$ *be a proper stack on a set* X*. For* $A\subseteq X$*,* $\Gamma^G_{\mathscr{S}}(A)$ *is* G*-closed.*

Proof. Let $x\notin\Gamma^G_{\mathscr{S}}(A)$. Then, there exists a G-open set U containing x such that $A\cap\overline{U}^G\notin\mathscr{S}$. From Lemma 4.7, we have $\Gamma^G_{\mathscr{S}}(A)\cap U=\emptyset$. Since $x\in U=U^{\circ G}$, we get $x\notin\overline{\Gamma^G_{\mathscr{S}}(A)}^G$. Thus, $\overline{\Gamma^G_{\mathscr{S}}(A)}^G\subseteq\Gamma^G_{\mathscr{S}}(A)$. That is, $\Gamma^G_{\mathscr{S}}(A)$ is G-closed. □

Remark 4.9. *Note that* $\Gamma^G_{\mathscr{S}}(A)\cup\Gamma^G_{\mathscr{S}}(B)\subseteq\Gamma^G_{\mathscr{S}}(A\cup B)$ *and* $\Gamma^G_{\mathscr{S}}(A\cap B)\subseteq\Gamma^G_{\mathscr{S}}(A)\cap\Gamma^G_{\mathscr{S}}(B)$ *from Proposition 4.4(1). But the equalities may not be hold in general. Consider Example 3.16. For* $A=\{c\}$ *and* $B=\{d\}$*, we have* $\Gamma^G_{\mathscr{S}}(A)\cup\Gamma^G_{\mathscr{S}}(B)=\emptyset\neq\Gamma^G_{\mathscr{S}}(A\cup B)=\{c,d\}$*. Also, for* $A=\{a,c\}$ *and* $B=\{c,d\}$*, we have* $\Gamma^G_{\mathscr{S}}(A)\cap\Gamma^G_{\mathscr{S}}(B)=\{d\}\neq\Gamma^G_{\mathscr{S}}(A\cap B)=\emptyset$*.*

Definition 4.10. *Let* G *be a method and* $\mathscr{S}$ *be a stack on a set* X*. For* $A\subseteq X$*, we define the following operator:*

$$\Psi^G_\Gamma(A)=X\backslash\Gamma^G_{\mathscr{S}}(X\backslash A).$$

Proposition 4.11. *Let* G *be a method and* $\mathscr{S}$ *be a stack on a set* X*. For* $A,B\subseteq X$*,* $A\subseteq B$ *implies* $\Psi^G_\Gamma(A)\subseteq\Psi^G_\Gamma(B)$*.*

Proof. It is clear from Proposition 4.4(1). □

Proposition 4.12. *Let* G *be a method and* $\mathscr{S}$ *be a proper stack on a set* X*. For* $A\subseteq X$*,* $\Psi^G_\Gamma(A)$ *is* G*-open.*

Proof. By Proposition 4.8, it is obvious. □

Theorem 4.13. *Let* G *be a method and* $\mathscr{S}$ *be a proper stack on a set* X*.*

$$\sigma^G_{\mathscr{S}}=\{H\subseteq X:H\subseteq\Psi^G_\Gamma(H)\}$$

is a strong generalized topology on X *with* $\sigma^G_{\mathscr{S}}\subseteq\tau^G_{\mathscr{S}}$*.*

Proof. Since $\Psi^G_\Gamma(X)=X\backslash\Gamma^G_{\mathscr{S}}(X\backslash X)=X\backslash\Gamma^G_{\mathscr{S}}(\emptyset)=X\backslash\emptyset=X$ and $\emptyset\subseteq\Psi^G_\Gamma(\emptyset)$, we have $X,\emptyset\in\sigma^G_{\mathscr{S}}$. Let $H_i\in\sigma^G_{\mathscr{S}}$ for $i\in I$. From Proposition 4.11 and our hypothesis, $H_i\subseteq\Psi^G_\Gamma(H_i)\subseteq\Psi^G_\Gamma(\cup_{i\in I}H_i)$ for every $i\in I$. Hence, $\cup_{i\in I}H_i\subseteq\Psi^G_\Gamma(\cup_{i\in I}H_i)$. Thus, $\cup_{i\in I}H_i\in\sigma^G_{\mathscr{S}}$. This shows that $\sigma^G_{\mathscr{S}}$ is a strong generalized topology. Also, by Lemma 4.2, we have $\sigma^G_{\mathscr{S}}\subseteq\tau^G_{\mathscr{S}}$. □

Corollary 4.14. *Let* G *be a method and* $\mathscr{S}$ *be a proper stack on a set* X*. If each* G*-open set in* X *is also* G*-closed then* $\sigma^G_{\mathscr{S}}=\tau^G_{\mathscr{S}}$*.*

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