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On the decomposition formulae for the solutions of a class of partial pifferential equations of even order

by

S. SÜRAY AND A. O. ÇELEBİ

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On the decomposition formulae for the solutions of a class of partial differential equations of even order

S. SÜRAY AND A. O. ÇELEBİ

SUMMARY

A Weinstein and L. E. Payne had obtained decomposition formulae for the solutions of the equation

$$L_k^m \; u = O,$$

where

$$L_k = \sum_{i=1}^{n} \frac{\partial^2}{\partial \; x_i^2} + \frac{\partial^2}{\partial y^2} + \frac{k}{y} \frac{\partial}{\partial y}$$

In this paper, we will extend the decomposition formulae for the solutions of the equation

$$\left(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{v} \frac{\alpha_i}{x_i} \frac{\partial}{\partial x_i} \right)^m u = 0.$$

1. Let us consider the differential operator

$$L\alpha_k = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\alpha_k}{x_k} \frac{\partial}{\partial x_k} \quad (1 \leq k \leq n)$$

where $\alpha_k \in R$ is an arbitrary constant. Let the function $u = u(x_1, x_2, \dots x_n) \in C^2$ be the solution of the equation

$$L\alpha_k \; u = 0.$$

Such a solution will be denoted by $u\{\alpha_k\}$. The decomposition

$$w = \sum_{j=0}^{p-1} x_h^{\,j} \, u_j \; \{\alpha_k\} \tag{1}$$

with $h \neq k$, for the solution $w\{\alpha_k\}$ of the equation

$$L^p\alpha_k \; w = 0$$

which is obtained by the iterated applications of the operator L_{α_k} to the function $w = w(x_1, x_2, \ldots, x_n) \in C^{2p}$ p times is known [1].

2. The first thing we will do is to extend formula (1), to the case of more than one coefficient which are called the parameters. Let us assume, for example, that there are ν parameters $(1 \leq \nu \leq n-1)$, and let us denote the solution of the equation

$$\mathcal{Q}^p w = 0$$

by $w = w\{\alpha_1, \alpha_2, \ldots, \alpha_\nu\}$, where

$$\mathcal{Q} \equiv L_{\alpha_1, \alpha_2, \ldots, \alpha_\nu} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{\nu} \frac{\alpha_i}{x_i} \frac{\partial}{\partial x_i} .$$

We shall show that the function w admits the decomposition

$$w = \sum_{j=0}^{p-1} x_h^j \, u_j \, \{\alpha_1, \alpha_2, \ldots, \alpha_\nu\} \tag{2}$$

provided that $h \neq i$ $(i = 1, 2, \ldots, \nu)$. It is assumed that each function u_j in (2) is a solution of the equation

$$\mathcal{Q}\, u = 0.$$

We already know that formula (2) is valid in the case of only one parameter [formula (1)]. In order to show that it is also valid in the case of ν parameters we must verify

$$\mathcal{Q}^p x_h^q u_q = 0$$

for $o \leq q \leq p-1$.

$$\mathcal{Q}^p u_o = 0$$

$$\mathcal{Q}^p x_h u_1 = \mathcal{Q}^{p-1} \mathcal{Q}\, x_h u_1 = \mathcal{Q}^{p-1} \Big[x_h \mathcal{Q}\, u_1 - x_h \frac{\partial^2 u_1}{\partial x_h^2}$$

$$+ \frac{\partial^2}{\partial x_h^2} (x_h u_1) \Big] = 2\, \mathcal{Q}^{p-1} \frac{\partial u_1}{\partial x_h} = 2 \frac{\partial}{\partial x_h} \mathcal{Q}\, u_1 = 0$$

and in a similar way, we can see that

$$\mathcal{Q}^p x_h^q u_q = K\, \mathcal{Q}^{p-q} \frac{\partial^q u_2}{\partial x_h^q} = K \frac{\partial^q}{\partial x_h^q} \mathcal{Q}^{p-q} u_q = 0$$

where K ($\neq$ 0) is a fixed but arbitrary constant. That is, if we write a decomposition in the form (2)

$$\mathcal{Q}^p\ w = \mathcal{Q}^p \sum_{J=0}^{p-1} x_h{}^j\ u_j\ \{\alpha_1,\ \alpha_2, \ldots, \alpha_\nu\}$$

$$= \sum_{J=0}^{p-1} \mathcal{Q}^p\ x_h{}^j\ u_j\ \{\alpha_1,\ \alpha_2, \ldots, \alpha_\nu\} = 0.$$

On the other hand, it is evident that if

$$\mathcal{Q}^p\ w\ \{\alpha_1,\ \alpha_2, \ldots, \alpha_\nu\} = 0$$

then the solution w of this equation admits a decomposition of the form

$$w\ \{\alpha_1,\ \alpha_2, \ldots, \alpha_\nu\} = \sum_{J=0}^{p-1} x_h{}^j\ u_j\ \{\alpha_1\ \alpha_2, \ldots, \alpha_\nu\}.$$

3. In this paragraph we shall try to extend the decomposition

$$w = \sum_{i=1}^{p} u\ \{\alpha_i - 2\ (p - i)\}$$

which A. Weinstein [2] has given for the solution of

$$L\alpha_1\ L\alpha_2\ \ldots\ L\alpha_p\ w = 0$$

to the case where the operators contain equal number and more than one parameter.

First we will prove the following theorem for two operators and ν ($1 \leqq \nu \leqq n$) parameters where only the parameter in the h th place differs from each other and the rest is the same.

THEOREM I Under the hypothesis

$$\alpha_h{}^2 \leqq \alpha_h{}^1 - 2,$$

the solution w of the equation

$$L\alpha_1, \alpha_2, \ldots, \alpha_h{}^1, \ldots, \alpha_\nu\ L\ \alpha_1, \alpha_2, \ldots, \alpha_h{}^2, \ldots, \alpha_\nu\ w = 0 \qquad (3)$$

admits the decomposition

$$w = u\ \{\alpha_1,\ \alpha_2, \ldots, \alpha_h{}^2, \ldots, \alpha_\nu\}$$
$$+ u\ \{\alpha_1,\ \alpha_2\ \ldots, \alpha_h{}^1 - 2, \ldots, \alpha_\nu\}. \qquad (4)$$

Proof: For the sake of simplicity, let us denote the operators in (3), from left to right, by $\mathcal{Q}_1$, and $\mathcal{Q}_2$. Equation (3) is equivalent to the equation

$$\mathcal{Q}_2\ w = u\ \{\alpha_1, \alpha_2, \ldots, \alpha_h^1, \ldots \alpha_\nu\}. \tag{5}$$

Under the assumption $\alpha_h^2 \neq \alpha_h^1 - 2$ equation (5) has a solution of the form $v\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2, \ldots, \alpha_\nu\}$. In fact, considering the equality

$$\mathcal{Q}_2 = \mathcal{Q}\ \alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2, \ldots, \alpha_\nu$$
$$+ \alpha_h^2 - \alpha_h^1 + 2)\ x_h^{-1} \frac{\partial}{\partial x_h} \tag{6}$$

which can easily be established, and the relation

$$x_h^{-1} \frac{\partial}{\partial x_h}\ u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2, \ldots, \alpha_\nu\}$$
$$= u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1, \ldots, \alpha_\nu\} \tag{7}$$

which is given by A. O. Çelebi [3], equation (5) can be written as

$$\mathcal{Q}_2\ v\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2, \ldots, \alpha_\nu\}$$
$$= u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1, \ldots, \alpha_\nu\}$$

or

$$(\alpha_h^2 - \alpha_h^1 + 2)\ v\{\alpha_1, \alpha_2, \ldots, \alpha_h^1, \ldots, \alpha_\nu\}$$
$$= u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1, \ldots, \alpha_\nu\}. \tag{8}$$

It is clear that, if $\alpha_h^2 - \alpha_h^1 + 2 \neq 0$ or $\alpha_h^2 \neq \alpha_h^1 -2$ then,

$$v = \frac{u}{\alpha_h^2 - \alpha_h^1 + 2};$$

since the operator $\mathcal{Q}_1$ is linear, c u is also a solution of the equation $\mathcal{Q}_1\ u = 0$ where c is a constant; that is, neglecting $(\alpha_h^2 - \alpha_h^1 + 2)^{-1}$ we can take $v = u$, so,

$$\mathcal{Q}_2\ u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2, \ldots, \alpha_\nu\} = u$$
$$\{\alpha_1, \alpha_2, \ldots \alpha_h^1, \ldots, \alpha_\nu\} \tag{9}$$

Subtracting equation (9) from (5), we get

$$\mathcal{Q}_2\ [w - u\ \{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2, \ldots, \alpha_\nu\}] = 0 \tag{10}$$

The meaning of equation (10) is

$$w - u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2, \ldots, \alpha_\nu\} = u\{\alpha_1, \alpha_2, \ldots, \alpha_h^2, \ldots, \alpha_\nu\}$$

or

$$w = u\{\alpha_1, \alpha_2, \ldots, \alpha_h^2, \ldots, \alpha_\nu\} + u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2, \ldots, \alpha_\nu\}.$$

Thus the theorem is proved.

Now we will consider the general case with p operators, each having ν parameters and only the parameters in the h th place differ from each other, and the rest of them are the same.

THEOREM II Under the hyphothesis

$$\alpha_h^r \neq \alpha_h^s - 2\,(r - s),\ s < r = 2, 3, \ldots, p$$

the solution w of the equation

$$\prod_{i=1}^{p} L_{\alpha_1, \alpha_2, \ldots, \alpha^i_h, \ldots, \alpha_\nu}\, w = 0 \tag{11}$$

admits the decomposition

$$w = \sum_{i=1}^{p} u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2\,(p - i), \ldots, \alpha_\nu\}$$

where $1 \leqq \nu \leqq n$.

Proof. Denoting the operators by a simplified notation we can write equation (11) as

$$\prod_{1=i}^{p} \mathcal{Q}_i\, w = 0. \tag{12}$$

Equation (12) is equivalent to the following:

$$\prod_{i=2}^{p} \mathcal{Q}_i\, w = u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1, \ldots, \alpha_\nu\}. \tag{13}$$

By Theorem I, under the assumption $\alpha_h^2 \neq \alpha_h^1 - 2$, there is a function $u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2, \ldots, \alpha_\nu\}$ such that

$$\mathcal{Q}_2\, u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2, \ldots, \alpha_\nu\} = u\{\alpha_1, \alpha_2, \ldots\ \alpha_h^1, \ldots, \alpha_\nu\} \tag{14}$$

By the same argument we can write

$$\begin{aligned}\mathcal{Q}_3\, u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2.2, \ldots, \alpha_\nu\} \\ = u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2, \ldots, \alpha_\nu\},\end{aligned}$$

if $\alpha_h^2 \neq \alpha_h^1 - 4$. That is, equation (14) takes the form

$$\begin{aligned}\mathcal{Q}_2\, \mathcal{Q}_3\, u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2.2, \ldots, \alpha_\nu\} \\ = u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2, \ldots, \alpha_\nu\}.\end{aligned}$$

Proceeding in this manner we obtain

$$\begin{aligned}\prod_{i=2}^{p} \mathcal{Q}_i\, u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2\,(p-1), \ldots, \alpha_\nu\} \\ = u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1, \ldots, \alpha_\nu\}.\end{aligned} \tag{15}$$

Subtracting equation (15) from (13) we get

$$\prod_{i=2}^{p} \mathcal{Q}_i\, [w - u\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2\,(p-1), \ldots, \alpha_\nu\}] = 0. \tag{16}$$

Thus,equation (16) takes the form

$$\prod_{i=2}^{p} \mathcal{Q}_i\, w_1 = 0 \tag{17}$$

where

$$w_1 = w - u\,\{\alpha_1, \alpha_2, \ldots, \alpha_h^1 - 2\,(p-1), \ldots, \alpha_\nu\};$$

that is, this procedure reduces the solution of (12) to the solution of (17). Repeating this argument we reduce the solution of (12) to the problem considered in Theorem I. Hence the proof of Theorem II is completed.

4. L. E. Payne has given the decomposition

$$w = \sum_{i=0}^{p-1} y^{2i}\, u\{\alpha + 2\,i\}$$

for the solution of the equation $L_\alpha^p\, w = 0$, [1]. Now we want to extend this formula to the case of more than one parameter; that is we will establish the same decomposition for the solution of the equation

$$\mathcal{Q}^{p} \, w = 0$$

where

$$\mathcal{Q} = L_{\alpha_1, \alpha_2, \ldots, \alpha_\nu}$$

For this purpose we will first prove the following lemma.

Lemma

$$\mathcal{Q}^{p} \, w = x_h^{1-\alpha_h} \, \mathcal{Q}^{*p} \, w^* \tag{18}$$

where $w = x_h^{1-\alpha_h} w^*$, $1 \leqq \nu \leqq n$, $h = i$ $(i = 1, 2, \ldots, \nu)$ and

$$\mathcal{Q}^* = L_{\alpha_1, \alpha_2, \ldots, 2 - \alpha_h, \ldots, \alpha_\nu} .$$

Proof: We will use the method of mathematical induction. First, let us take $p = 1$ and note that

$$\mathcal{Q} = \mathcal{Q}^* - \frac{2-\alpha_h}{x_h} \frac{\partial}{\partial x_h} + \frac{\alpha_h}{x_h} \frac{\partial}{\partial x_h} = \mathcal{Q}^* - \frac{2(1-\alpha_h)}{x_h} \frac{\partial}{\partial x_h}$$

Applying $\mathcal{Q}$ to both sides of $w = x_h^{1-\alpha_h} w^*$, we obtain

$$\begin{aligned} \mathcal{Q} \, w &= \mathcal{Q} \, x_h^{1-\alpha_h} \, w^* \\ &= \left(\mathcal{Q}^* - \frac{2(1-\alpha_h)}{x_h} \frac{\partial}{\partial x_h}\right) (x_h^{1-\alpha_h} \, w^*) \end{aligned}$$

$$= x_h^{1-\alpha_h} \, \mathcal{Q}^* \, w^* - x_h^{1-\alpha_h} \frac{\partial 2 \, w^*}{\partial \, x_h 2} - (2 - \alpha_h) \, x_h^{-\alpha_h} \frac{\partial 2 \, w^*}{\partial \, x^2_h}$$

$$+ \left(\frac{\partial 2}{\partial \, x_h 2} + \frac{\alpha_h}{x_h} \frac{\partial}{\partial x_h}\right) (x_h^{1-\alpha_h} \, w^*) = x_h^{1-\alpha_h} \, \mathcal{G}^* W^*$$

Now let us take $p = 2$, that is

$$\begin{aligned} \mathcal{Q}^2 \, w &= \mathcal{Q}^2 \, (x_h^{1-\alpha_h} \, w^*) \\ &= \left(\mathcal{Q}^* - \frac{2(1-\alpha_h)}{x_h} \frac{\partial}{\partial \, x_h}\right) (x_h^{1-\alpha_h} \, \mathcal{Q}^* \, w^*) \end{aligned}$$

By calculating the right hand side of this equality, as in the case of $p = 1$, we get

$$\mathcal{D}^2 \text{ w} = x_h^{1-\alpha_h} \mathcal{D}^{*2} \text{ w}^*.$$

So it is proved that relation (18) is valid in the cases p = 1 and p = 2. Now let us assume that the same relation holds for p - 1 operators. We will show that it also holds for p operators. If

$$\mathcal{D}^{p-1} \text{ w} = x_h^{1-\alpha_h} \mathcal{D}^{*(p-1)} \text{ w}^*$$

then

$$\mathcal{D}^{p} \text{ w} = \mathcal{D} \left(x_h^{1-\alpha_h} \mathcal{D}^{*(p-1)} \text{ w}^* \right)$$

$$= \left(\mathcal{D}^* - \frac{2(1-\alpha_h)}{x_h} \frac{\partial}{\partial x_h} \right) \left(x_h^{1-\alpha_h} \mathcal{D}^{*(p-1)} \text{ w}^* \right) \tag{19}$$

Calculating the right hand side of (19), we obtain

$$\mathcal{D}^{p} \text{w} = x_h^{1-\alpha_h} \mathcal{D}^{*p} \text{w}^* - x_h^{1-\alpha_h} \frac{\partial^2}{\partial x_h^2} \mathcal{D}^{*(p-1)} \text{w}^*$$

$$- (2 - \alpha_h) x_h^{-\alpha_h} \frac{\partial}{\partial x_h} \mathcal{D}^{*(p-1)} \text{w}^*$$

$$+ \left(\frac{\partial^2}{\partial x_h^2} + \frac{\alpha_h}{x_h} \frac{\partial}{\partial x_h} \right) \left(x_h^{1-\alpha_h} \mathcal{D}^{*(p-1)} \text{w}^* \right)$$

$$= x_h^{1-\alpha_h} \mathcal{D}^{*p} \text{w}^*.$$

This completes the establishment of the lemma.

Now we will prove the following theorem:

THEOREM III. Under the hyphothesis $1 \leqq \nu \leqq n - 1$, $h = i$ $(i = 1, 2, \ldots, \nu)$ the solution w of the equation

$$\mathcal{D}^p \text{ w} = 0$$

admits the decomposition

$$\text{w} = \sum_{i=1}^{p} x_h^{2(p-i)} \text{u} \left\{ \alpha_1, \alpha_2, \ldots, \alpha_h + 2 (p - i), \ldots, \alpha_\nu \right\}$$

Proof By the above lemma the equation $\mathcal{Q}^p\, w = 0$ gives $\mathcal{Q}^{*p}\, w^* = 0$. Then let us take $\alpha_h{}^1 = \alpha_h{}^2 = \ldots = \alpha_h{}^p = \alpha_h$ in Theorem II. So, the decomposition established there takes the form

$$w = \sum_{i=1}^{p} u\{\alpha_1 . \alpha_2, \ldots, \alpha_h - 2\,(p - i), \ldots, \alpha_\nu\} \tag{20}$$

In order to apply the decomposition to the function w^*, $\alpha_h - 2\,(p - i)$ should be substituted by $2 - [\alpha_h - 2\,(p - i)]$. Thus we get

$$w^* = \sum_{i-1}^{p} u\{\alpha_1, \alpha_2, \ldots, 2 - [\alpha_h - 2\,(p - i)], \ldots, \alpha_\nu\} \tag{21}$$

On the other hand, for $p = 1$ or $w = u$ the relation

$w = x_h^{1-\alpha_h}\, w^*$ gives $u = x_h^{1-\alpha_h}\, u^*$ or

$$u\{\alpha_1,{}_2\, \alpha, \ldots, 2 - [\alpha_h - 2\,(p - i)], \ldots, \alpha_\nu\} \tag{22}$$
$$= x_h^{1+\alpha_h+2(p-i)-2}\, u\{\alpha_1, \alpha_2, \ldots, \alpha_h + 2\,(p - i), \ldots, \alpha_\nu\}$$

is deduced. Substituting the result (22) in (21) and writing $w^* = w\, x_h^{\alpha_h - 1}$, we conclude with

$$w\, x_h^{\alpha_h-1} = \sum_{i=1}^{p} x_h^{1+\alpha_h+2(p-i)-2}\, u\{\alpha_1, \alpha_2, \ldots, \alpha_h + 2\,(p - i), \ldots, \alpha_\nu\}$$

or

$$w = \sum_{i=1}^{p} x_h^{2(P-i)}\, u\{\alpha_1, \alpha_2, \ldots, \alpha_h + 2\,(p - i), \ldots \alpha_\nu\}. \tag{23}$$

This is the formula which was to be established. So the proof of the theorem is completed for the case more than one parameter. The decomposition (23) is the generalization of the decomposition due to Payne; if we choose $\alpha_h = \alpha$ and the rest zero, it gives Payne's decomposition.

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ÖZET

Bu yazıda çift mertebeden kısmi türevli eliptik lineer bir denklem sınıfının çözümleri için aşağıda işaret edeceğimiz bazı dekompozisyon formülleri kurulmuştur.

1. $\mathcal{Q} = L_{\alpha_1, \alpha_2, \ldots, \alpha_\nu} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{\nu} \frac{\alpha_i}{x_i} \frac{\partial}{\partial x_i}$ olmak üzere

$\mathcal{Q}^p$ w = 0 denkleminin w çözümü için

$$w = \sum_{j=o}^{p-1} x_{hj}\, u^j \{\alpha_1, \alpha_2, \ldots, \alpha_\nu\}$$

2. $\prod_{i=1}^{p} L_{\alpha_1, \alpha_2, \ldots, \alpha_h{}^i, \ldots, \alpha_\nu}\, w = 0$ denkleminin çözümü için

$$w = \sum_{i=1}^{p} u\{\alpha_1, \alpha_2, \ldots, \alpha_h{}^i - 2\,(p - i), \ldots, \alpha_\nu\}.$$

3. $\mathcal{Q}^p$ w = 0 denkleminin çözümü için

$$w = \sum_{i=1}^{p} x_h^{2(p-i)}\, u\, \{\alpha_1, \alpha_2, \ldots, \alpha_h + 2\,(p - i), \ldots, \alpha_\nu\}$$

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