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On The Mean Values Of an Entire Function represented By a Dirichlet Series II

by

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On The Mean Values Of an Entire Function represented By a Dirichlet Series II

JODH PAL SINGH

SUMMARY

In this note we prove a theorem which gives us information as to how the functions $\log I_\delta(\sigma)$ and $\log J_{\delta,k}(\sigma)$ grow relative to each other as $\sigma \to \infty$.

Theorem. Let f(s) be an entire function represented by a Dirichlet series, then

$$\underline{\lim}_{\sigma\to\infty} \frac{\log I_\delta(\sigma)}{\log J_{\delta,k}(\sigma)} \leqq \bar{\lambda}\left\{(1+k/\bar{\lambda})\right\}^{(1+\bar{\lambda}/k)} \quad (\bar{\lambda}>0)$$

$$\leqq 1, \qquad (\bar{\lambda}=0)$$

where $\bar{\lambda} = \lim_{\sigma\to\infty} \frac{\log\log I_\delta(\sigma)}{\sigma}$

1. In the usual notation,

$$f(s) = \sum_{1}^{\infty} a_n e^{s\lambda_n}, \ (s = \sigma + it), \ 0 < \lambda_n < \lambda_{n+1} \ (n \geqq 1)$$

$$\lim_{n\to\infty} \lambda_n = \infty$$

is an entire function in the sense that the Dirichlet series representing it, is absolutely convergent for all finite s and possesses

$$\overline{\underline{\lim}}_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\sigma} = \frac{\rho}{\lambda}$$

where $0 \leqq \lambda, \rho \leqq \infty$, and $M(\sigma)$ have their usual meanings.

The mean values of f(s) are defined as follows:

$$(1.1)\quad I_\delta(\sigma) = \left\{ \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^\delta\, dt \right\}^{\frac{1}{\delta}},\ (\delta > 0)$$

and

$$(1.2)\quad J_{\delta,k}(\sigma) = \exp\left\{ \frac{1}{e^{k\sigma}}\int_{0}^{\sigma} \log I_\delta(x)\, e^{kx}\, dx \right\}\quad 0 < k < \infty.$$

The mean value (1.1) for $\delta = 2$ was defined by Hadamard [1], and it is known that

$$\frac{\rho}{\lambda} = \frac{\overline{\lim}}{\underline{\lim}}_{\sigma\to\infty} \frac{\log\log I_\delta(\sigma)}{\sigma} \leq \frac{\rho}{\lambda} \qquad [2]$$

Then, following Shah [3], there exists a lower proximate order $\overline{\lambda}(\sigma)$ relative to $\log I_\delta(\sigma)$, satisfying the following conditions:

(i) $\lim_{\sigma\to\infty} \overline{\lambda}(\sigma) = \overline{\lambda}$,

(ii) $\lim_{\sigma\to\infty} \sigma\,\overline{\lambda}'(\sigma) = 0$,

(iii) $\log I_\delta(\sigma) \geq \exp(\sigma\,\overline{\lambda}(\sigma))$ for all large σ,

(iv) $\underline{\lim}_{\sigma\to\infty} \dfrac{\log I_\delta(\sigma)}{\exp(\sigma\,\overline{\lambda}(\sigma))} = 1.$

In this note we prove a theorem which gives us information as to how the functions $\log I_\delta(\sigma)$ and $\log I_{\delta,k}(\sigma)$ grow relative to each other as $\sigma \to \infty$. In what follows we shall prove the following:

2. Theorem. Let f(s) be an entire function represented by Dirichlet series, then

$$\underline{\lim}_{\sigma\to\infty} \frac{\log I_\delta(\sigma)}{\log J_{\delta,k}(\sigma)} \leq \overline{\lambda}\left\{1 + k/\overline{\lambda}\right\}^{(1+\overline{\lambda}/k)},\quad (\overline{\lambda} > 0)$$
$$\leq 1,\quad (\overline{\lambda} = 0).$$

We need the following lemma for our purpose.

Lemma. For $h > 0$,

$$\lim_{\sigma \to \infty} \frac{S(\sigma + h)}{S(\sigma)} = \exp(h\bar{\lambda})$$

where

$$S(\sigma) = \exp(\sigma \bar{\lambda}(\sigma)).$$

Proof. By a simple calculation, we have

$$\frac{S'(\sigma)}{S(\sigma)} = \sigma \bar{\lambda}'(\sigma) + \bar{\lambda}(\sigma).$$

Therefore, using the properties (i) and (ii), we see that for any $\varepsilon > 0$, there is a σ_0 such that for every $\sigma \geqq \sigma_0$

$$(\bar{\lambda} - \epsilon) < S'(\sigma)/S(\sigma) < (\bar{\lambda} + \varepsilon).$$

Integrating the above inequality from σ to $\sigma + h$, we have

$$(\bar{\lambda} - \varepsilon)h < \log \left\{ \frac{S(\sigma + h)}{S(\sigma)} \right\} < (\bar{\lambda} + \varepsilon)h.$$

So that

$$\lim_{\sigma \to \infty} \frac{S(\sigma + h)}{S(\sigma)} = \exp(h\bar{\lambda})..$$

Now, we are in position to prove the theorem.

Proof of Theorem. We have

$$\log J_{\delta,k}(\sigma + h) = \frac{1}{e^{k(\sigma + h)}} \int_0^{\sigma+h} \log I_\delta(x)\, e^{kx}\, dx,$$

$$\geqq \frac{1}{e^{k(\sigma + h)}} \int_\sigma^{\sigma+h} \log I_\delta(x)\, e^{kx}\, dx$$

$$\geqq \frac{\log I_\delta(\sigma)}{k\, e^{kh}} (e^{kh} - 1).$$

So that, by the property (iv), we have

$$\varliminf_{\sigma\to\infty} \frac{\log J_{\delta,k}(\sigma)}{S(\sigma)} \geqq \varliminf_{\sigma\to\infty} \frac{\log I_{\delta}(\sigma)}{S(\sigma)} \left\{\frac{e^{kh}-1}{k\, e^{kh}}\right\}$$

$$\geqq \frac{e^{kh}-1}{k\, e^{kh}} \tag{2.1}$$

Put

$$\frac{\log J_{\delta,k}(\sigma+h)}{S(\sigma)} = \frac{\log J_{\delta,k}(\sigma+h)}{S(\sigma+h)} \quad \frac{S(\sigma+h)}{S(\sigma)}$$

Here $\log J_{\delta,k}(\sigma+h)/S(\sigma+h)$ and $S(\sigma+h)/S(\sigma)$ are non-negative, so we have

$$\varliminf_{\sigma\to\infty} \frac{\log J_{\delta,k}(\sigma+h)}{S(\sigma)} \leqq \varliminf_{\sigma\to\infty} \frac{\log J_{\delta,k}(\sigma+h)}{S(\sigma+h)} \exp(h\bar{\lambda})$$

by the lemma. This inequality with (2.1) will give us

$$\varliminf_{\sigma\to\infty} \frac{\log J_{\delta,k}(\sigma)}{S(\sigma)} \geqq \frac{e^{kh}-1}{k\exp((k+\bar{\lambda})h)}$$

Further, we have

$$\varliminf_{\sigma\to\infty} \frac{\log I_{\delta}(\sigma)}{\log J_{\delta,k}(\sigma)} \leqq \varliminf_{\sigma\to\infty} \left\{\frac{\log I_{\delta}(\sigma)}{S(\sigma)}\right\} \tag{2.2}$$

$$\varlimsup_{\sigma\to\infty} \left\{\frac{S(\sigma)}{\log J_{\delta,k}(\sigma)}\right\},$$

$$\leqq 1 .\frac{1}{\varliminf_{\sigma\to\infty} \dfrac{\log J_{\delta,k}(\sigma)}{S(\sigma)}}$$

$$\leqq \frac{k\exp((k+\bar{\lambda})h),}{e^{kh}-1}, \qquad (\bar{\lambda}>0)$$

$$\leqq \frac{k\exp(kh)}{e^{kh}-1} \qquad (\bar{\lambda}=0)$$

Now, by usual method of calculus we minimize the right hand side of (2.2). We find that its minima is attained for that value of h which satisfies the relation

$$e^{kh} = \frac{k + \bar{\lambda}}{\bar{\lambda}} \qquad \bar{\lambda} > 0$$

Substituting this value of h in (2.2), we get

$$\underline{\lim}_{\sigma \to \infty} \frac{\log I_\delta(\sigma)}{\log J_{\delta,k}(\sigma)} \leqq \bar{\lambda}[(1 + k/\bar{\lambda})]^{(1 + \bar{\lambda}/k)}$$

The case $\bar{\lambda} = 0$, is obvious.

This completes the proof of the theorem.

In conclusion, I wish to thank Dr. Shankar Hari Dwivedi, University of Udaipur and Dr. V. B. Goyal, University of Kurukshetra for useful discussions.

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ÖZET

Bu çalışmada, log I_δ (α) ve log $J_{\delta,k}$ (σ) nın biri diğerine nazaran ne şekilde büyüdüklerini gösterir bir teorem ıspatlanmıştır.

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