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On Higher Curvatures Of A Curve

by

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On Higher Curvatures Of A Curve

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ABSTRACT

The higher curvatures of a Curve in n-dimensional Euclidean space E^n are derived by Herman Gluck [1]. In this paper we give a new method to define and calculate the higher curvatures of a curve which is supposed a differentiable 1- manifold in E^n. We also prove that these curvatures are invariant under the rigid motions.

I. INTRODUCTION

In E^n n-dimensional Euclidean space, a curve α is a diffeomorphic image of an open segment I of a straight line. Therefore in differential geometry a curve of E^n can be regarded as a 1-mainold. Hence for a curve also we can define the Riemanian metric which is known for the submanifolds of E^n. This permits us to have the tangent space, at any point $\alpha(s)$ of the curve α, as an inner product space. We can also apply the theory of the vector fields, the vector-valued forms and the frame bundles over E^n. Thus we can accept that the properties of the equations of structure and the cross section are known also for the curves of E^n.

The Frenet frame of a curve in E^n is an oriented orthonormal frame which determines a cross section

$$C: \alpha(I) \longrightarrow \mathcal{F}_0$$

and hence we may consider the pull back 1-forms

$$C^* (\varnothing_i) = \varnothing^c_i \, , \; 1 \leq i \leq n$$
$$C^* (\varnothing_{ij}) = \varnothing^c_{ij} \, , \; 1 \leq i, j \leq n \, ,$$

where $\mathcal{F}_0$ denotes the orthonormal frame bundle over E^n, $\varnothing_i$, $\varnothing_{ij}$ are the connection forms which appear in the equations of structure of the Lie groups.

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Because of the cotangent space to $\alpha(s)$ is spanned by ds for all $s \varepsilon I$ we may find that

$$\alpha^* (\varnothing^c{}_i) = f_i(s)ds \ , \ 1 \leq i \leq n$$
$$\alpha^* (\varnothing^c{}_{ij}) = k_{ij}(s) \ ds \ , \ 1 \leq i, j \leq n$$

where $k_{ji} = -k_{ij}$. We also show that the functions $f_i(s)$ and $k_{ij}(s)$ are differential geometric invariants of the curve α under the rigid motions.

In differential geometry the role of higher curvatures of a curve is importent for the higher dimensions. To show this importance we can mention the papers [3] and [4]. The differential geometry of higher dimensions is about the p-forms, equations of structures on a manifold. These needs the new techniques of modern differential geometry. So it will be very valuable to treat the paper [1] of Herman Gluck in this new style.

II. BASIC CONCEPTS

a) Tangent and Cotangent Spaces

If M is an r-dimensional submanifold of E^n n-dimensional Euclidean space. Then we denote the tangent space to M at its point m by $T_M(m)$. We consider $T_M(m)$ as a linear subspace of the tangent space $T_{E^n}(m)$ of E^n at the same point m. Then, because of $T_{E^n}(m)$ has an Euclidean inner product, $T_M(m)$ inherits an inner product from $T_{E^n}(m)$. This inner product is called the induced Riemannian metric and denoted by ds^2 in classical books on diffe rential geometry.

Let (Ψ, U) be a coordinate neigborhood for the submanifold M. This means that the mapping

$$\Psi : U \longrightarrow M$$

is a differeomorphism. Then

(II. 1) $$\Psi_* : T_{E^r}(u) \longrightarrow T_M (\Psi(u))$$

is a linear transformation which corresponds to the Jacobian matrix of Ψ. We denote the adjoint of Ψ_* by Ψ^* which is a transformation

(II. 2) $\Psi^*: T^*_M(\Psi(u)) \longrightarrow T^*_{E^r}(u)$

where $T^*_M(\Psi(u))$ and $T^*_{E^r}(u)$ are the dual spaces of $T_M(\Psi(u))$ and $T_{E^r}(u)$, respectively. The vector spaces $T^*_M(\Psi(u))$ and $T^*_{E^r}(u)$ are the cotangent spaces at the corresponding points.

b) Vector fields and forms

Definition II. 1:

Let U be a Euclidean neigborhood. A 1-form w is a mapping

$$w: U \longrightarrow \cup T^*_U(x) \ , \ x \in U$$

where the union is taken over all $x \in U$, such that $p \circ w: U \longrightarrow U$ is the identity mapping;

$$p: \cup T^*_U(x) \longrightarrow U \ , \ t_x \in T^*_U(x),$$

$$p(t_x) = x.$$

Definition II . 2:

A vector field is a function

$$X : U \longrightarrow \cup T_U(m)$$

such that

$$p \circ X : U \longrightarrow U$$

is the identity mapping and

$$p: \cup T_U(x) \longrightarrow U,$$

$$p(t_x) = x \ , \ t_x \in T(x).$$

Let $x = (x_1, \ldots, x_n)$ be a Euclidean coordinate system in E^n then $\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}$ is a basis of the vector space $\varkappa_p$ of all the parallel vector fields on E^n and $\{dx_1, \ldots, dx_n\}$ is the dual basis of dual space Ω of $\varkappa_p$. Let M be a r-dimensional submanifold of E^n with local coordinates $(u_1, \ldots, u_r)$ given by

$$x_i = x_i(u_1, \ldots, u_r) \quad , \ 1 \leq i \leq n.$$

Then a 1-form on M has the analytic expression

$$\text{(II. 3)} \qquad w = \sum_{i,j}^{n,r} \frac{\partial x_i}{\partial u_j} du_j \otimes \frac{\partial}{\partial x_i}$$

where $\otimes$ denotes the tensor product.

c) Vector-Valued Forms

Definition II.3 :

Let M be a submanifold of E^n and W be a q-dimensional vector space. A vector-valued form on M with values in W is a mapping A defined as

$$A(m) : T_M(m) \xrightarrow{\text{linear}} W.$$

According to Definition II. 3 the 1-form w in the definition II. 1 is a vector valued form. Now we need the following theorem [2]:

Theorem II.1:

Let A be a vector-valued form on M with values in W, and let $w_1, \dots, w_q$ be a basis of W. Then, there exists a uniquely determined ordered set of q 1-forms $\theta_1, \dots, \theta_q$ on M such that

$$A = \sum_{i=1}^{q} \theta_i \otimes w_i \tag{II. 4}$$

which is defined by

$$A(t_m) = \sum_{i=1}^{q} <t_m, \theta_i(m)> w_i \ , \ \forall t_m \in T_M(m). \tag{II. 5}$$

d) Reparameterization of a Curve in E^n

Let α: [a, b] $\longrightarrow E^n$, where [a, b] is the interval $a \leq t \leq b$, be a parameterized curve with parameter t such that $\alpha_* \left(\frac{\partial}{\partial t}\right) \neq 0$ for all $t \in [a,b]$ i. e. α is an immersion.

Definition II. 4:

Let α: [a, b] $\longrightarrow E^n$ be a parameterized curve with parameter t. Let [a′, b′] be an interval with parameter s, and let

$$[a', b'] \xrightarrow{\eta} [a, b] \xrightarrow{\alpha} E^n,$$

where η has a nonvanishing Jacobian. Then, $\alpha o \eta$ is a parameterized curve with parameter s. The curve $\alpha o \eta$ is called a reparameterization of the curve α.

Definition II. 5:

Let α: $[a,b] \longrightarrow E^n$ be a parameterized curve with parameter s. The parameterization of α is called the arc-lenth parameter if $\alpha^*\left(\frac{\partial}{\partial s}\right)$ has length one in $T_M(\alpha(s))$ for all $a \leq s \leq b$.

Now, we can give a well-known theorem [2].

Theorem II. 2:

A parameterized curve can always be reparameterized by an arc-length parameter.

e) Equations of Structure for $\mathcal{F}_0$

Let $\mathcal{F}_0$ be the orthonormal frame bundle over E^n. If $(x_1, \ldots, x_n)$ is a Euclidean coordinate system for E^n, we will choose $f_0 \in \mathcal{F}_0$

$$f_0 = \left(\frac{\partial}{\partial x_1}\Big|_0, \ldots, \frac{\partial}{\partial x_n}\Big|_0 ; O\right).$$

Let r denotes a rigid motion in E^n, then r has the matrix representation

$$r = \begin{bmatrix} g & a \\ 0 & 1 \end{bmatrix}, \quad g = [g_{ij}],$$

where g is an orthogonal $n \times n$ matrix and a is a translation i. e $n \times 1$ matrix .

Let $f = r(f_0) = (e_1, \ldots, e_n ; a)$

where

$$e_i = \sum_{j=1}^{n} g_{ji} \frac{\partial}{\partial x_j}\Big|_a.$$

A vector field X_i over $\mathcal{F}_0$ is defined by

$$X_i(f) = e_i.$$

Then a vector-valued form defined by

$$\text{(II. 6)} \qquad dX_i = \sum_{k=1}^{n} \varnothing_{ik} \otimes X_k$$

can be expressed in the form [2]

(II. 7) $$dX = \sum_{k=1}^{n} \varnothing_k \otimes X_k$$

where $1 \leq i , k \leq n$

(II. 8) $$dX_i = \sum_{k=1}^{n} \varnothing_{ik} \otimes X_k$$

(II.9) $$\begin{cases} d\varnothing_i = \sum_{s=1}^{n} \varnothing_s \Lambda \varnothing_{si} \\ d\varnothing_{ij} = \sum_{s=1}^{n} \varnothing_{is} \Lambda \varnothing_{sj}, \quad 1 \leq i, s, j \leq n, \\ \varnothing_{ij} = - \varnothing_{ji}. \end{cases}$$

Formulas II.6, II. 7, II.8 and II.9 are called the equations of structure of $\mathcal{F}_0$.

III. HIGHER CURVATURES OF A CURVE IN E^n

a) Frenet Frame Along A Curve.

Let (α, S) be a local coordinat system for an arc-length curve of E^n with the arc-length s as parameter, such that S is an open interval, i. e,

$$S = \{s: a < s < b, s \in \mathcal{R}\}.$$

We further restrict our attention to curves

$$\alpha: S \longrightarrow E^n$$

which are immersions, i.e, which satisfy $\alpha_* \left(\frac{\partial}{\partial s}\right) \neq 0$. Then clearly α has the property that

$$\| \alpha_* \left(\frac{\partial}{\partial s}\right) \| = 1 \qquad , \forall s \in S.$$

Let define the vector field X_1 as follows

(III.1) $$X_1 = \alpha_* \left(\frac{\partial}{\partial s}\right) .$$

Then X_1 is the unit tangent vector to the curve α in the direction of increasing s.

If $\alpha(s) = (x_1(s), \ldots, x_n(s))$ is Euclidean coordinate system of E^n then from [2, Theorem II. 6. 4] we have that

$$\text{(III.2)} \qquad X_1 = \sum_{i=1}^{n} \frac{dx_i}{ds} \frac{\partial}{\partial x_i} ,$$

and the corresponding vector-valued form is

$$\text{(III.3)} \qquad dX_1 = \sum_{i=1}^{n} d\left(\frac{dx_i}{ds}\right) \otimes \frac{\partial}{\partial x_i}$$

where $\otimes$ denotes the tensor product. By means of Theorem II.1 we may write that

$$dX_1\left(\alpha_*\left(\frac{\partial}{\partial s}\right)\right) = \sum_{i=1}^{n} < \alpha_*\left(\frac{\partial}{\partial s}\right), d\left(\frac{dx_i}{ds}\right)\left(\frac{\partial}{\partial s}\right) > \frac{\partial}{\partial x_i}$$

$$= \sum_{i=1}^{n} < \alpha_*\left(\frac{\partial}{\partial s}\right), \frac{d^2x_i}{ds^2} ds\left(\frac{\partial}{\partial s}\right) > \frac{\partial}{\partial x_i}$$

or since we have that $ds\left(\frac{\partial}{\partial s}\right) = 1$

$$= \sum_{i=1}^{n} < \alpha_*\left(\frac{\partial}{\partial s}\right), \frac{d^2x_i}{ds^2} > \frac{\partial}{\partial x_i} .$$

or since we have that

$$dX_1\left(\alpha_*\left(\frac{\partial}{\partial s}\right)\right) = \alpha^*\left(dX_1\left(\frac{\partial}{\partial s}\right)\right)$$

and α^* is a linear transformation

$$dX_1\left(\frac{\partial}{\partial s}\right) = \sum_{i=1}^{n} (\alpha^*)^{-1} < \alpha_*\left(\frac{\partial}{\partial s}\right), \frac{d^2x_i}{ds^2} > \frac{\partial}{\partial x_i}$$

$$\text{(III.4)} \qquad dX_1 = \sum_{i=1}^{n} (\alpha^*)^{-1} \frac{d^2x_i}{ds^2} ds \otimes \frac{\partial}{\partial x_i} .$$

From (III.4) we that

$$dX_1\left(\alpha_*\left(\frac{\partial}{\partial s}\right)\right) = \left(\sum_{i=1}^{n} (\alpha^*)^{-1} \frac{d^2x_i}{ds^2} ds \otimes \frac{\partial}{\partial x_i}\right)\left(\alpha_*\left(\frac{\partial}{\partial s}\right)\right)$$

$$= \sum_{i=1}^{n} (\alpha^*)^{-1} \frac{d^2x_i}{ds^2} ds \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) \frac{\partial}{\partial x_i}$$

$$= \sum_{i=1}^{n} \frac{d^2x_i}{ds^2} (\alpha^*)^{-1} (ds) \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) \frac{\partial}{\partial x_i}$$

$\alpha_* \left(\frac{\partial}{\partial s} \right)$ and $(\alpha^*)^{-1}$ (ds) are dual basis for each other we have that

$$(\alpha^*)^{-1} (ds) \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) = 1$$

and so

(III.5) $$dX_1 \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) = \sum_{i=1}^{n} \frac{d^2x_i}{ds^2} \frac{\partial}{\partial x_i}.$$

The vector field $\frac{dX_1}{ds}$ along the curve α is completely determined by the curve and the concept of parallel displacement in E^n.

If we let

(III.6) $$Y_2 (\alpha (s)) = dX_1 \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right)$$

Y_2 is a vector field on $\alpha(S)$ and then by means of (III.5) we have that

(III.7) $$Y_2 (\alpha (s)) = \sum_{i=1}^{n} \frac{d^2x_i}{ds^2} (s) \frac{\partial}{\partial x_i} \Big|_{\alpha(s)} .$$

Let Y_j be defined by

(III. 8) $$Y_j = \sum_{i=1}^{n} \frac{d^jx_i}{ds^j} \frac{\partial}{\partial x_i} \quad , \quad 1 < j \leq n.$$

Then we suppose that the system

$$\{X_1 , Y_2 ,..., Y_n\}$$

of vector fields is linearly independent. Since $T_{E^n}(\alpha (s))$ is an inner-product vector space with a basis $\{X_1, Y_2 ,..., Y_n\}$, there exists an algorithm, called the Gram-Schmidt process, for converting $\{X_1, Y_2 , ..., Y_n\}$ into a coherently oriented orthonormal frame $\{X_1, X_2, ..., X_n\}$ which is an orthonormal basis of $T_{E^n}(\alpha(s))$.

Since $\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}$ is also another positive oriented orthonormal basis of $T_{E^n}(\alpha(s))$ these two bases $\{X_i\}$ and $\left\{ \frac{\partial}{\partial x_i} \right\}$ are related by

$$(\text{III. 9}) \qquad X_i = \sum_{j=1}^{n} k_{ji} \frac{\partial}{\partial x_j} \quad , \quad 1 \leq i \leq n.$$

We have thus proven the following theorem.

Theorem III.1:

Let α be a curve in E^n with arc - length parameter s and

$X_1 = \alpha_* \left(\frac{\partial}{\partial s} \right)$. If $\frac{dX_1}{ds} = dX_1 \left(\frac{\partial}{\partial s} \right) \neq 0$, where dX_1 is the vector - valued form determined by the vector field X_1 and Euclidean parallelism, at any point of α, we have a positive oriented orthonormal frame $X_1, X_2, \ldots, X_n$ at each point of α.

Definition III. 1:

Using the above notation, $\{X_1, \ldots, X_n\}$ is called the Frenet frame along α.

b) Frenet Formulas And The Higher Curvatures.

Definition III. 2:

A mapping

$$C_f : \alpha(S) \longrightarrow \mathcal{F}_o^+$$

defined by

$$C_f(\alpha(s)) = (X_1(\alpha(s)), \ldots, X_n(\alpha(s)) \; ; \; \alpha(s)) \in \mathcal{F}_o^+$$

is called the Frenet cross section.

If $\varnothing_{ij}$, $\varnothing_i$, $1 \leq i, j \leq n$, are the 1-forms defined on $\mathcal{F}_o^+$ then the 1-forms defined on S are

$$(\text{III. 10}) \qquad \begin{cases} (C_f)^* (\varnothing_{ij}) = \varnothing^{Cf}_{ij} \\ (C_f)^* (\varnothing_i) = \varnothing_i^{Cf}. \end{cases}$$

The basis of the vector space of all 1-forms on S is $\{ds\}$. Therefore there are differentiable functions

$$t_{ij} : S \longrightarrow R$$

such that

(III.11) $\alpha^*(\varnothing_{ij}{}^{c_f}) = t_{ij}\, ds \;,\; 1 \leq i, j \leq n.$

On the other hand from (II.8) we have that

(III. 12) $dX_i = \sum_{j=1}^{n} \varnothing_{ij}{}^{c_f} \otimes X_j \qquad 1 \leq i \leq n.$

Replacing (III.11) in (III.12) we obtain that

(III. 13) $dX_i = \sum_{j=1}^{n} (\alpha^*)^{-1} (t_{ij}\, ds) \otimes X_j$

and so applying the same computation for (III.5) we obtain from (III.13) that

(III. 14) $dX_i\left(\alpha_* \left(\frac{\partial}{\partial s}\right)\right) = \sum_{j=1}^{n} t_{ij}(s)\, X_j.$

If we have the notation

$$dX_i \left(\alpha_* \left(\frac{\partial}{\partial s}\right)\right) = X'_i$$

according to Herman Gluck, the expressions (III. 13) and (III.14) are the Frenet Formulas. Since we know [1] that

(III.15) $\begin{cases} X'_i(s) = - t_{i-1}(s)\; X_{i-1}(s) + t_i(s)\; X_{i+1}(s) \;, & 2 \leq i \leq n\text{-}1 \\ X'_n(s) = - t_{n-1}(s)\; X_{n-1}(s) \end{cases}$

then the matrix $[t_{ij}]$ has the form

(III.16) $$\begin{bmatrix} 0 & t_{12} & 0 & \ldots & 0 & 0 & 0 \\ -t_{12} & 0 & t_{23} & \ldots & 0 & 0 & 0 \\ 0 & -t_{23} & 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & t_{(n-2)\,(n-1)} & 0 & t_{(n-1)n} \\ 0 & 0 & 0 & & 0 & t_{(n-1)n} & 0 \end{bmatrix}.$$

The 1-forms $\alpha^*(\varnothing_{ij}{}^{c_f})$ have the expressions [1, pp: 105, formula V]:

(III.17) $\alpha^* (\varnothing_{ij}{}^{c_f}) = \sum_{\lambda=1}^{n} k_{\lambda j} \frac{dk_{\lambda i}}{ds}\, ds\;.$

Equations (III. 11) and (III.17) give us that

$$\text{(II.18)} \qquad t_{ij} = \sum_{\lambda=1}^{n} k_{\lambda j} \frac{dk_{\lambda i}}{ds}.$$

Definition III.3:

Using the above notation the coefficients t_{ij} are called the higher curvatures of the curve α.

c) *Special Case*:

If n = 3 then the matrix (III.16) becomes

$$\begin{bmatrix} 0 & t_{12} & 0 \\ -t_{12} & 0 & t_{23} \\ 0 & -t_{23} & 0 \end{bmatrix}$$

where t_{12} is from (III.18)

$$t_{12} = \sum_{\lambda=1}^{3} k_{\lambda 2} \frac{dk_{\lambda 1}}{ds}.$$

On the otherhand, from (III.2), we may write that

$$X_1 = \sum_{\lambda=1}^{3} k_{\lambda 1} \frac{\partial}{\partial x_\lambda} = \sum_{\lambda=1}^{3} \frac{dx_\lambda}{ds} \frac{\partial}{\partial x_\lambda}$$

and from (III.8).

$$X_2 = \sum_{\lambda=1}^{3} k_{\lambda 2} \frac{\partial}{\partial x_\lambda} = \frac{1}{\|Y_2\|} \sum_{\lambda=1}^{3} \frac{d^2x_\lambda}{ds^2} \frac{\partial}{\partial x_\lambda}.$$

Thus we can write that

$$\begin{aligned} t_{12} &= \sum_{\lambda=1}^{3} \frac{1}{\|Y_2\|} \frac{d^2x_\lambda}{ds^2} \frac{d^2x_\lambda}{ds^2} \\ &= \frac{1}{\|Y_2\|} \sum_{\lambda=1}^{3} \left(\frac{d^2x_\lambda}{ds^2}\right)^2 \\ &= \frac{1}{\|Y_2\|} \cdot \|Y_2\|^2 \\ t_{12} &= \|Y_2\| \end{aligned}$$

$$\text{(III.19)} \quad t_{12} = \sqrt{\sum_{\lambda=1}^{3} \left(\frac{d^2x_\lambda}{ds^2}\right)^2}$$

Then t_{12} is the first curvature of the curve. In a similar way we see that

$$\text{(III.20)} \quad | t_{23} | = \| dx_3 \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) \| = \| X'_3 \|$$

and so t_{23} is the second curvature (torsion) of the curve.

d) The Invariance Of The Curvatures t_{ij}.

The curvatures t_{ij} of the curve α are invariant under the rigid motions of E^n.

The distance-preserving mappings of E^n are called rigid motions. The set of all rigid motions of E^n will be denoted by R(n).

Let $r \in R(n)$ and the Frenet frame of the curve $r(\alpha(S))$ be the frame $\{X^r_1, X^r_2, \ldots, X^r_n\}$. Thus Equation (III.14), for the curve $r(\alpha(S))$, becomes

$$\text{(III.14)}' \quad dX^r_i \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) = \sum_{j=1}^{n} t^r_{ij} X^r_j.$$

We want to show that $t_{ij} = t^r_{ij}$.

Let the local coordinat system of $r(\alpha(S))$ be $(ro\alpha, S)$. Then for this new curve (III.1) reduces to

$$\text{(III.1)}' \quad X^r_1 = (ro\alpha)_* \left(\frac{\partial}{\partial s} \right).$$

Indeed $\| X^r_1 \| = 1$. To show this we use the fact that

$$(ro\alpha)_* = r_* o \alpha_*$$

and so since we know that $r \in R(n)$

$$\| X^r_1 \| = \| r_* \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) \| = \| \alpha_* \left(\frac{\partial}{\partial s} \right) \| = 1.$$

Since $r \in R(n)$ has the matrix form

$$r = \begin{bmatrix} g & a \\ 0 & 1 \end{bmatrix}$$

as we see in the paragraph II we may have that

$$\text{(III.8)' } Y'_j = \sum_{i=1}^{n} \frac{d^j\left(\sum_{\lambda=1}^{n} g_{i\lambda} x_\lambda\right)}{ds^j} \frac{\partial}{\partial x_i}$$

$$= \sum_{i=1}^{n} \sum_{\lambda=1}^{n} g_{i\lambda} \frac{d^j x_\lambda}{ds^j} \frac{\partial}{\partial x_i}$$

$$= \sum_{i=1}^{n} \left(\sum_{\lambda=1}^{n} g_{i\lambda} \frac{d^j x_\lambda}{ds^j}\right) \frac{\partial}{\partial x_i}$$

$$= r^* (Y_j)$$

and so

$$(X^r_1,\dots,X^r_n) = (r_* (X_1),\dots, r_* (X_n)).$$

Definition III.2 gives us the following Frenet Cross section for the curve $r(\alpha(S))$:

$$C^r_f : (ro\alpha)(S) \longrightarrow \mathcal{F}_0$$

which is defined by

$$C^r_f(s) = (r_*(X_1(s)),\dots,r_*(X_n(s));\ r(\alpha(s))).$$

Hence Equation (III.11), for this new curve, reduces to

(III.11)' $(ro\alpha)^* (\varnothing^{C^r_f}_{ij}) = \alpha^* (r^*(\varnothing^{C^r_f}_{ij}) = t^r_{ij} ds.$

And finally as a similar way for Equation (III.17) we have

$$(ro\alpha)^* (\varnothing^{C^r_f}_{ij}) = \sum_{\lambda=1}^{n} (\sum_{t=1}^{n} g_{\lambda t}\ k_{tj}) \frac{d\left(\sum_{\mu=1}^{n} g_{\lambda\mu}\ k_{\mu i}\right)}{ds} ds$$

$$= \sum_{\lambda=1}^{n} \sum_{t=1}^{n} g_{\lambda t} k_{tj} \sum_{\mu=1}^{n} g_{\lambda\mu} \frac{dk_{\mu_i}}{ds} ds$$

$$\sum_{t=1}^{n} k_{tj} (\sum_{\mu,\lambda}^{n} g_{\lambda t}\ g_{\lambda\mu}) \frac{dk_{\mu_i}}{ds} ds$$

$$= \sum_{t=1}^{n} k_{tj}\ \delta_{\lambda\mu} \frac{dk_{\mu_i}}{ds} ds$$

$$= \sum_{t=1}^{n} k_{tj} \frac{dk_{ti}}{ds} ds$$

(III.21) $(ro\alpha)^* (\varnothing_{ij}^{c^r f}) = \alpha^* (\varnothing_{ij}^{c} f)$.

Using the Equations (III.11), (III.11)' and (III.21) we have that

$$t^r_{ij} = t_{ij}$$

which completes the proof.

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ÖZET

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