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On Some Planes Of Lenz-Barlottı Class I

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On Some Planes Of Lenz-Barlottı Class I

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SUMMARY

We have presented some non-desarguesian planes and partially discussed their basic geometries in [4]. The purpose of the present paper is to complete this discussion by finding the corresponding ternary rings and the possible collineation groups of these planes, from which it is inferred that they are planes of classes I.1, I.2, and I.4 in the Lenz-Barlotti Classification of projective planes.

1 INTRODUCTION

A (P,L)–*perspectivity* is a collineation of a projective plane which fixes every line through the point P and every point on the line L. The perspectivity is an *elation* if P is on L, a *homology* if P is not on L; in each case P is its centre and L its axis. A plane is said (P,L)–*transitive* if and only if for each pair of points X, Y, distinct from P, not on L, and collinear with P, there exists a (P, L)–perspectivity such that it maps X on Y. It is known that, for the pair (P,L), the set of all (P,L)–perspectivities is a transformation group which may contain only the identity. The (P,L)–transitivity is equivalent to the (P,L)–Desarques Theorem, that is, to the existence of all possible Desargues configurations with centre P and axis L. Lenz [5] and Barlotti [1] have classified the projective planes according to the set of the pairs (P,L), for which planes are (P,L)–transitive. This set is usually denoted by F and called the "*figure*" of plane to which it belongs. The figure of a projective plane of class I.1 is the empty set, while the figure of a projective plane of class I.2 contains only one fixed pair (P,L); the figure of a projective plane of class I.4 is F = $\{ (P_i, L_i)$:

$i = 1,2,3,\ P_i \notin L_i$ and $P_i = L_j \cap L_k, j \neq i \neq k\}$. There is also a very close relationship between transitivity and the algebraic properties of the corresponding ternary ring of a projective plane. (An account of all related subjects is given in Dembowski [**2**].)

A *division neo-ring* is a triple $(S, \oplus, \odot)$ consisting of a set S with at least two distinct elements O and 1, and two binary operations, addition $\oplus$ and multiplication $\odot$ such that

(i) $(S, \oplus)$ is a loop with unit element O;

(ii) $(S-\{O\}, \odot)$ is a loop with unit element 1;

(iii) $O \odot x = x \odot O = O$, for all $x \in S$;

(iv) Multiplication is distributive with respect to addition,

i. e., for all $a,b,c \in S$

$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$

and $(a \oplus b \odot c = (a \odot c) \oplus (b \odot c)$.

Just as with rings, a modifying adjective such as commutative, associative preceding the phrase "division neo-ring" (DNR) refers to multiplication. Associative division neo-rings have been studied by Paige [**6**] under the name of neo-fields and it was shown that, in general, a DNR is not a planar ternary ring. A DNR $(S, \oplus, \odot)$ is a linear planar ternary ring (with ternary operation $T(a,b,c) = a \odot b \oplus c)$ if and only if

(v) $a \odot x \oplus b = c \odot x \oplus d$ has a unique solution x for all $a,b,c,d \in S$, $a \neq c$, and,

(vi) $x \odot a \oplus y = b$, $x \odot c \oplus y = d$ has a unique solution x,y for all $a,b,c,d \in S$, $a \neq c$.

A DNR which satisfies (v) and (vi) is called a *planar* division neo-ring (PDNR). The planar division neo-rings have been studied extensively by Hughes [**3**].

2. PLANES OF CLASS I

In this paper the non-desarguesian planes given in [**4**] are presented in a different notation by giving the set of points and the set of lines of the planes, and the incidence relation between the points and the lines. Throughout the paper R denotes the field of real numbers unless some other binary operations are defined on the set of real numbers. The set of non-zero elements of R is denoted by R^*. The symbol $<O,p>$, $p \in R$, stands for either the set $\{x: x \in R, O<x<p\}$ or the set $\{x: x \in R, p<x<O\}$ according as $p > O$ or $p<O$.

a. The extended plane, π, given in [4], can be represented by elements of the set $R \cup \{\infty\}$ as follows:

Points, $\{(x,y);\ x,y \in R\} \cup \{(m):\ m \in R\} \cup \{(\infty)\}$;

Lines, $\{[p,q,1]: p,q \in R^*\} \cup \{[p,q,O]: p,q \in R^*\} \cup \{[p,\infty]: p \in R\} \cup \{[\infty,q]:\ q \in R\} \cup \{[\infty]\}$;

Incidence, $[p,q,1] = \{(x,y): x^2/p^2 + y^2/q^2 =$ if $x \in <O,p>$ and $y \in <O,q>$, and $x/p + y/q = 1$ if $x \notin <O,p>\} \cup \{(-q/p\}$,

$$[p,q,O] = \{(x,y) : x/p+y/q = O\} \cup \{(-q/p)\},$$
$$[p,\infty] = \{(x,y) : x = p\} \cup \{(\infty)\},$$
$$[\infty,q] = \{(x,y) : y = q\} \cup \{(O)\},$$
$$[\infty] = \{(m): m \in R\} \cup \{(\infty)\}.$$

The line $[\infty]$ and the points which are denoted by only one element of $R \cup \{\infty\}$ such as (m), (∞) will be called the ideal line and ideal points of the plane, while the others will be called ordinary lines and ordinary points. (Notice that the boundary values of $<O,p>$ differ from the orijinal case in [4].)

Theorem 1. (i) The plane π supports non-trivial collineations: $(x,y) \to (kx,hy)$ for arbitrary $k,h \in R^*$, which form an abelian group, G; (ii) it also supports non-trivial collineations: $(x,y) \to (ky, hx)$ for arbitrary, $k, h \in R^*$.

Proof. (i) Let $\varnothing$ be a transformation on π such that $\varnothing((x,y)) = (kx,hy)$, $k, h \in R^*$. The following can be easily satisfied: $\varnothing([p,q,1]) = [kp,hq,1]$, $\varnothing([p,q,O]) = [kp,hq,O]$, $\varnothing([p,\infty]) = [kp,\infty]$ and $\varnothing([\infty,q]) = [\infty, hq]$. It follows immediately that $\varnothing((\infty)) = (\infty)$ and $\varnothing((-q/p)) = (-hq/kp)$ so $\varnothing([\infty]) = [\infty]$. Since $\varnothing$ is a one-to-one correspondence of π onto itself and preserve the collinearity (and therefore incidence), it is a collineation. It is trivial to show that the set $G = \{\varnothing_i: \varnothing_i((x, y)) = (k_i x, h_i y),\ k_i,h_i \in R^*\}$ is an abelian group.

(ii) Let ψ be a transformation on π such that $\psi((x,y)) = (ky, hx)$, $k,h \in R^*$. In a similar way to the preceding case we get $\psi([p,q,1]) = [kq, hp, 1]$, $\psi([p,q,O]) = [kq, hp, O]$, $\psi([p,\infty]) = [\infty, hp]$ and $\psi([\infty, q]) = [kq, \infty]$, which imply $\psi((-q/p)) = (-hp/kq)$, $\psi((\infty)) = (O)$ and $\psi((O)) = (\infty)$. Therefore $\psi([\infty]) = [\infty]$. It is obvious that ψ is a collineation of π, but $H = \{\psi_i : \psi_i((x,y)) = (k_i y, h_i x),\ k_i, h_i \in R^*\}$ is not a transformation group.

Corollary 1. The plane π is $((O,O), [\infty])$-transitive.

Proof. The only ordinary point which is invariant under each collineation $\varnothing$ in G is (O,O). All lines on the point (O,O), [p,q,O], [O,∞] and [∞, O], are fixed (not pointwise) by $\varnothing$ if and only if $h = k$. The collineation $(x, y) \to (kx, ky)$ with $k \in R^*$ leaves the line [∞] pointwise fixed and maps the lines on any ideal point among themselves. Hence, it is a $((O,O), [\infty])$-homology. For any pair of given points, (x,y) and (x', y'), both different from (O,O) and collinear with (O,O), a k can be uniquely determined from either of the equalities $x' = kx$ and $y' = ky$. Hence, π is $((O,O), [\infty])$-transitive.

Obviously, the $((O,O), [\infty])$-homologies of π form a subgroup G_{kk} of G.

Corollary 2. The plane π is $((O), [O,\infty])$-transitive.

Proof. Let G_k be the subgroup of G, which consists of the collineations $(x,y) \to (kx, y)$ with $k \in R^*$. It can be easilty seen that each $\varnothing$, $\varnothing \in G_k$, fixes the ideal point (O) and maps every line through (O) onto itself, and leaves the ordinary line [O, ∞] pointwise fixed. For two given points, (x,q) and (x',q) on [∞,q], not on [O,∞], a unique k can be found from $kx = x'$. Therefore, every $\varnothing$, $\varnothing \in G_k$, is a $((O), [O, \infty])$-homology, and π is $((O), [O, \infty])$-transitive.

Corollary 3. The plane π is $((\infty), [\infty, O])$-transitive.

Proof is similar to that of preceding corollaries. The $((\infty), [\infty, O])$-homologies are elements of the subgroup G_h of G, where $\varnothing \in G_h$ if and only if $\varnothing((x, y)) = (x, hy)$ with $h \in R^*$.

The group G does not contain any other subroups which consist of the (P, L)-perspevtivities, except G_{kk}, G_k, G_h and G_1, where G_1 is the subgroup which consist of the identity collineation. The other collineation set H contains no (P,L) –perspectivity of π. These corolaries show that π is a (P,L)–Desargues plane for three distinct pairs (P_i, L_i). (For. the definition of the (P, L)–Desargues plane see Pickert [7, p. 74].)

Theorem 2. π is a plane of class I.4.

Proof. By the above corollaries, we have the pairs ((0,0), [∞]), ((O), [O, ∞]) and ((∞), [∞, O]) for which π is transitive. Further (O,O) $\notin$ [∞], (O) $\notin$ [O, ∞], (∞) $\notin$ [∞,O] and (O,O) $\in$ [O, ∞] $\cap$ [∞, O], (O) $\in$ [∞] $\cap$ [∞, O], (∞) $\in$ [O, ∞] $\cap$ [∞]. Combining this result with the information given at the begining of the paper, we can conclude that π is at least of class I.4.To complete the proof it is necassary to show that there is no other pair of (P, L) for which π is transitive. We will not prove it here, but it will be confirmed by theorem 3.

In the following theorem, we will use the coordinate system of Pickert [7].

Theorem 3. The plane π can be represented by coordinates from a commutative and associative division neo–ring (R, $\oplus$,$\odot$). Multiplication $\odot$ for (R, $\oplus$, $\odot$) coincides with that of R itself, but addition $\oplus$ of (R, $\oplus$ $\odot$) is defined as follows: $u \oplus v = (\text{sign } v)(v^2 - u^2)^{1/2}$ or $u+v$ according as $u \in \langle O,-v \rangle$ or $u \notin \langle O,-v \rangle$.

Proof. Let us choose the coordinatizing quadrangle as O = (O,O), E = (1,1), U = (O) and V = (∞). Then we get the ternary operation as $T(m, x, b) = (\text{sign } b)\,(b^2 - m^2x^2)^{1/2}$ or $mx+b$ according as $x \in \langle O,-b/m \rangle$ or $x \notin \langle O,-bm \rangle$. Therefore,

$$(^*)\; u \oplus v = T(1,u,v) = \begin{cases} (\text{sign } v)\ (v^2-u^2)^{1/2} & \text{if } u \in \langle O,-v \rangle \\ u + v & \text{if } u \notin \langle O,-v \rangle, \end{cases}$$

and $u \odot v = T(u,v,O) = uv$ for every $u,v \in R$.

(i) Clearly, $x \oplus O = O \oplus x = x$, for all $x \in R$. Let us show that $a \oplus x = b$ has a unique solution x. The cases in which at least any one of a and b is zero, or they have the same sign are trivial.

Suppose $a>0$ and $b<0$. In this case, by(*), x is to be negative for $a\oplus x = b$. Hence, $a\oplus x = b \Leftrightarrow \{ -(x^2-a^2)^{1/2}$ if $a \in <0,-x>$ and $a \oplus x = b$ if $a \notin <0,-x>\}$. If $a \notin <0,-x>$ then $-a<x<0$; and since $b-a<-a$ the second equation has no solution. If $a \in <0,-x>$ then $x<-a$. Therefore, $x=-(a^2+b^2)^{1/2}$ is the only solution for the first equation.

Suppose $a<0$ and $b>0$. In this case(*) implies that x is to be positive for $a\oplus x=b$, and that $a\oplus x = b \Leftrightarrow \{+(x^2-a^2)^{1/2} = b$ if $a \in <0,-x>$ and $ax+b$ if $a \notin <0,-x>\}$. If $a \notin <0,-x>$ then $0<x<-a$; and since $b-a>-a$ the second equatinon has no solution. If $a \in <0,-x>$ then $x>a$. Since $+(a^2+b^2)^{1/2}>-a$, it is the only solution for the first equation.

Now, let us show that $x\oplus a = b$ has a unique solution x. The cases in which at least one of a and b ise zero, or they have opposite signs are obvious. Notice that x and a are to have opposite signs when $x \in <0,-a>$.

Suppose $a>0$ and $b>0$. Then $x\oplus a = b \Leftrightarrow \{+(a^2-x^2)^{1/2} = b$ if $x \in <0,-a>$ and $x+a=b$ if $x \notin <0,-a>\}$. If $a>b$ then $-(a^2-b^2)^{1/2} \in <0,-a>$, and therefore $x=-(a^2-b^2)^{1/2}$ is a solution for $x\oplus a = b$; since $b-a \in <0,-a>$, it can not be a solution. If $a<b$ then $b-a \notin <0,-a>$. Hence, $x=b-a$ is the only solution for $a\oplus x = b$.

Suppose $a<0$ and $b<0$. Similarly to the preceding case, $x\oplus a = b \Leftrightarrow \{x = +(a^2-b^2)^{1/2}$ if $x \in <0,-a>$ and $x=b-a$ if $x \notin <0,-a>\}$. If $a>b$ then $(a^2-b^2)^{1/2} \notin R$. However, $b-a \notin <0,-a>$ and therefore $x=b-a$ is the only solution for the equation. If $a<b$ then $+(a^2-b^2)^{1/2} \in <0,-a>$ and $b-a \in <0,-a>$. Therefore $x=+(a^2-b^2)^{1/2}$ is the only solution for $x\oplus a = b$.

As a result $(R,\oplus)$ is a loop. Further, each x in $(R,\oplus)$ has the same additive inverse just as in $(R,+)$.

Since multiplication is the same with that of R itself, (ii) and (iii) are obvious (see the deinition of a division neo-ring), and the following is enough for the validity of both the distrubitive laws:

$$w\,(u\oplus v) = \begin{cases} w\,(\text{sign } v)\,(v^2-u^2)^{1/2} & \text{if } u \in <0,-v> \\ (\text{sign } wv)(w^2v^2-w^2u^2)^{1/2} & \text{if } (uw) \in <0,-wu> \end{cases}$$
$$= wu \oplus wv .$$

Let [m, k] denote any line which is not on V, in the new coordinate system. It can be checked that (x,y) $\in$ [m,k] if and only if $y=mx\oplus k$, for all x,y,m,k $\in R$. Since the ternary operation is linear and new binary operations can be associated with the plane π, (v) and (vi) follow from the postulates of being a projective plane, which have been proven for π in [4]. This completes the proof of the theorem.

The above theorem shows that π is a (V, UV; U, OV)– and (U,UV;V,OU)–Desargues plane, which are equivalent to certain properties of the coresponding planar ternary ring of the plane (see Pickert [7, pp. 80 and 98–99]); and that π satisfies the (OU, OV, UV) –Pappus theorem which is equivalent to R^*, $\odot$) being a commutative group. Further, it also confirms theorem 2.

b. In the rest of this article, we shall state a number of conclusions without proof, similar to those have been proven so far about π, which are about some other planes derived from π. In all these planes, the symbol [p,q] will denote the line $\{(x,y): x/p+y/q=1,\ p,q \in R^*\} \cup \{(-q/p)\}$; however, some extra conditions will be imposed on p or q, for each plane. The coordinatizing quadrangle will be OEUV as in theorem 3.

The original plane given in [4], say π_1, can be derived from the plane π in the following manner: Points of π_1 are the points of π. All lines of π which have been denoted by $[\infty]$, $[\infty, q]$, $[p, \infty]$ and [p,q, O] are also lines of π_1; but the lines which have been denoted by [p,q, 1] are replaced by the lines [p,q] if $p<O$ and $q<O$, or $p>O$ and q $\in R^*$; or they remain unchanged if $p<O$ and $q>O$.

The plane π_1 supports the collineations: $(x, y) \to (kx, hy)$ with $h,k > O$, and $(x,y) \to (ky, hx)$ with $h,k < O$, but no (P,L) –perspectivity. Thus, π_1 contains no (P, L)–Desargues configuration and is a plane of class I.1. π_1 can be associated with a nonlinear ternary ring in which neither of the distrunitive laws holds. Multiplication of the corresponding ternary ring coincides with that of R itself, and addition for it is defined as follows: $u\oplus v = +(v^2-u^2)^{1/2}$ or $u+v$ according as $-v<u<O$ or unless $-v<u<O$. Four such planes, all isomorphic to π_1, can be derived from π by changing the signs of p and q for the line [p,q].

Although the following planes, π_2 and π_3, have not been mentioned in [4], the theorems proven in [4] are sufficient for any of them being a projective plane.

The plane π_2 can be derived from π in the following manner: Points of π_2 are the points of π. All lines of π, except [p,q,1], are lines of π_2; the lines [p,q, 1] are replaced by the lines [p,q] if $p>O$ or left unchanged if $p<O$. It can be associated with a linear ternary ring of which binary operations are as follows: $u \odot v = uv$; and $u \oplus v = +(v^2-u^2)^{1/2}$ or $u+v$ according as $-v<u<O$ or unless $-v<u<O$. Obviously, $(R,\oplus)$ is a loop, which satisfies neither the associative nor the commutative law. Both the distrubitive laws do not hold. Since the corresponding ternary ring is linear and $(R,\odot)$ is an abelian group, π_2 is (V,OU) -transitive (in former notation $((\infty), [O,\infty])$-transitive), therefore π_2 is a plane of class I.2. The plane π_2 supports the collineations: $(x,y) \to (kx, hy)$ with $k>O$, $h \neq O$, and $((\infty), [O,\infty])$ - homologies: $(x,y) \to (x, hy)$ with $h \neq O$. Four such planes, all isomorphic to π_2, can be derived from π by changing the condition imposed on the line [p,q]; two of them are $((O), [\infty, O])$-transitive planes.

The other plane, π_3, can be described as follows: Its points are the points of π. All lines of π, except [p,q, 1], are lines of π_3; the lines [p,q,1] are replaced by the lines [p,q] if $pq>O$, or left unchanged if $pq<O$. It can be coordinatized by elements from a commutative and associative DNR, but the correponding ternary ring is not linear. The binary operations, $\oplus$ and $\odot$, are the same as those given for π in theorem 3. It supports the collineations: $(x, y) \to (kx, hy)$, and $(x,y) \to (ky, hx)$, where $k, h \in R^*$ and $kh > O$. The set of collineations has the elements: $(x,y) \to (kx, ky)$ with $k \in R^*$ as $((O,O), [\infty])$-homologies. Another plane, isomorphic to π_3, can be derived from π by replacing [p,q,1] with [p,q] if $pq<O$. π_3 is also a plane of class I.2.

Remarks: 1) Notice that the corresponding ternary rings (according to the same coordinatizing quadrangle OEUV) of some of the planes which are isomorphic either to π_1 or π_3 have field properties, but not linear. 2) Besides the planes listed above, another plane can be derived from π by replacing [p,q,1] with [p,q] if $p>O$ and $q>O$, which is not isomorphic to any one of the planes mentioned above. It is also a plane of class I.1.

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ÖZET

Bu makalede [4] ile verilen reel non-desarguesian projektif düzlemlerin ve aynı metotla inşa edilen başkalarının i) üzerlerinde mümkün olan kolinasyon grupları ve onların (P, L) –perspektivitelerden meydana gelen altgrupları yardımıyle mevcut Desargues konfigurasyonları araştırılır; ii) mütekabil ternary halkalarının genel olarak "division neo-ring" yapısında oldukları ispatlanır. Daha sonra bunlar yardımıyla düzlemlerin Lenz-Barlotti sınıflamasında I. 1, I. 2 ve 1.4 tipinde oldukları gösterilir.

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