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by

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Polynomial Moulton Planes

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ABSTRACT

In this paper a family of affine planes is defined. Any plane in the family is determined by a triple $(F, \varnothing, n)$ consisting of a pseudo-ordered field F, a one-to-one and order reversing or order preserving function $\varnothing$ of F onto itself, and an element n of the set $N_2 = \{2x: x \in N$ the set of positive integers$\}$ if $\varnothing$ is order reversing or an element n of either of the sets $N_1 = \{2x-1 : x \in N\}$ and $N_3 = \{(2x-1)^{-1} : x \in N\}$ if $\varnothing$ is order preserving. In the case where F is a finite field of order q if $n \in N_2$ then $(q-1, n)=2$ and the elements α and $-\alpha$ are not both square or non-square elements in the field F; if $n \in N_1$ or $n \in N_3$ then $(q-1, n)=1$ or $(q-1, n^{-1})=1$ respectively. These planes are non-desarguesian for every n and every F unless $\varnothing(x) = \alpha x + \beta$, where $\alpha \in F$ but $\alpha \notin P \cup \{0\}$ or $\alpha \in P$ according as $n \in N_2$ or $n \in N_1 \cup N_3$; $\beta \in F$, (P is the multiplicative subgroup of index 2 of F). For $n=0$ the planes in the family are the so-called Moulton planes.

1. *INTRODUCTION*

Early in this century, Moulton [5] gave a construction of a non-desarguesian plane. The points of this plane are points of the euclidean plane, that is, ordered pairs of (x,y) of the field of real numbers. Its lines are i) ordinary lines of the form $x=a$, or $y=m\ yx+b$ if $m \leq 0$; ii) broken lines of the form $y=mx+b$ or $y=c\ (mx+b)$ depending on whether $y<0$ or $y \geq 0$, for all $m>0$ and a constant c, $1 \neq c > 0$. Levenberg [4] has generalized the Moulton plane by consideration of the broken line geometries defined by an arbitrary function $\varnothing$ between the upper and lower half-plane angles. All of the Leverberg-Moulton planes have been constructed by means of broken lines of the euclidean plane. Pickert [6] has replaced the field of real numbers by an ordered skew-field and exchanged the roles of the x and y axes in the Moulton Construction for certain reasons. The following generalization of the Moulton plane is due to Pierce [7]: Let F be a pseudo-ordered field and $\varnothing$, a one-to-one and order preserving function on F such that $x \to \varnothing(x)$ and $x \to \varnothing(x) - n_0 x$, $n_0 \in F$, $n_0 < 0$, both carry F onto itself. The points of any Pierce-Moulton plane are ordered pairs (x, y), and its lines are given by the equations $x=a$ and $y=m \circ x+b$, where $m \circ x$ denotes $\varnothing(m)\, x$ or mx according as $x<0$ or $x \geq 0$, for all a, b, m, x, y $\in$ F. For additional references pertaining to pseudo-ordered fields see Pierce [7]. In [3], I have given another generalization for the Moulton plane by replacing the broken half-lines with particularly chosen polynomial curves of odd degree. In such a plane a deformed line has the equation $y=m(x-a)$ or $y=cm(x-a)^n$ depending on whether $y<0$ or $y \geq 0$, where all a, c, m, x, y are real numbers with $c>0$, $m>0$ and n is a positive odd integer; c and n are constants for a plane.

The present paper generalizes certain Pierce-Moulton planes [7] and the Levenberg-Moulton planes [4], and also generalizes the planes given in [3] in many ways. In both the Levenberg-Moulton planes and the Pierce-Moulton planes the half-lines refracted by an arbitrary function $\varnothing$ are replaced by certain curves over the pseudo-ordered field F, which are also refracted by the same function $\varnothing$. However, it is not possible to generalize some of the Pierce-Moulton planes in this way, for instance the plane on the field of rational numbers. Generalized desarguesian and non-desarguesian planes are determined by finding the functions $\varnothing$ which define them. The y-axis is taken as the refraction axis for all planes. In the last part of the paper the fields which are used to construct a PM(F, $\varnothing$, n) with $n \in N_2$ are determined. The connection between polynomial Moulton planes and the other planes known as Moulton planes are discussed.

2. POLYNOMIAL MOULTON PLANES

Our terminology will follow in part Pierce [7]. Let N denote the set of all positive integers. For the sake of brevity the following sets will be used very often: $N_1 = \{2x-1 : x \in N\}$, $N_2 = \{2x : x \in N\}$, $N_3 = \{(2x-1)^{-1} : x \in N\}$ and $S=N_1 \cup N_2 \cup N_3$. Throughout the paper F stands for both the field o freal numbers R and the finite field GF(q) of order q, where $q=p^r$, p an odd prime and $r \in N$. Unless otherwise stated, P denotes the multiplicative subgroup of index 2 of F, and $\bar{P} = F-(P \cup \{0\})$. Whenever a finite field is being considered it is assumed that there is a pseudo-order on it, while R ise ordered in the usual way. Let x, y $\in$ F. It is said that x and y have the same sing if x, y $\in$ P or x, y $\in \bar{P}$ and opposite sings if x $\in$ P, y $\in \bar{P}$ or y $\in$ P, x $\in \bar{P}$. A single-valued function $\varnothing$ on a field F is said to be *order preserving* if and only if $(\varnothing(x)-\varnothing(y))/(x-y) \in P$; and *order reversing* if and only if $(\varnothing(x)-\varnothing(y))/(x-y) \in \bar{P}$, for all x, y $\in$ F, $x \neq y$.

DEFINITIONS: Let $\varnothing$ be a one-to-one function of the field F onto itself and n be an element of the set S.A *Polynomial Moulton construction* is a collection of points and lines, in which (i) each point is an ordered pair (x, y), x, y $\in$ F, and (ii) each line is any one of the following sets of the points:

$$[a] = \{(x,y) : x = a,\ x,\ y \in F\}$$

$$[m,b] = \{(x,y) : y = m * x+b,\ m * x = \varnothing(m)x^n \text{ if } x \in \bar{P} \text{ and } m * x = mx \text{ if } x \in P \cup \{0\};\ b,m,x,y \in F\}.$$

Hereafter m and b will be called the *slope* and the y-*intercept* of the line [m, b], respectively. Lines having the form [a] will be called *vertical* lines. A Polynomial Moulton construction will be denoted by the symbol PC(F, $\varnothing$, n).

A construction PC(F, $\varnothing$, n) is called a *Polynomial Moulton plane* if and only if its points and lines form an affine plane. If PC(F, $\varnothing$, n) is such a construction, it will be denoted by the symbol PM(F, $\varnothing$, n).

Let us now give the following preliminary lemmas.

LEMMA 1. *If* $\varnothing$ *is a one-to-one and order reversing function of the field* F *onto itself then* $\varphi: x \to \varnothing(x)-p_0x$, $p_0 \in P$, *is a one-to-one and onto function of* F.

If $\varphi(x) = \varphi(y)$ then $\varnothing(x)-\varnothing(y)=p_0(x-y)$, which is impossible unless x=y because $p_0 \in P$ and $\varnothing$ is order reversing. Hence, φ is one-to-one. If F is finite then the "onto" property of φ follows from the finiteness; if F=R then it can be easily shown that φ is "onto".

LEMMA 2. *If* $\varnothing$ *is one-to-one and order preserving function of the field* F *onto itself then* $\Psi : x \to \varnothing(x)-n_0x$, $n_0 \in \bar{P}$, *is a one-to-one and onto function of* F.

This lemma is indirectly proven for the finite case in Pierce [7], and is obviously valid for $F=R$.

LEMMA 3. *Let* P *be the multiplicative subgroup of index* 2 *of the field* $GF(q)$, $q=p^r$, p *an odd prime number*, $r \in N$, *and let* $\bar{P}=GF(q)-(P\cup\{0\})$. *Then the binomial equation*

$$\text{(I)} \qquad x^n-a=0 \quad a \in P,\ n \in N_2$$

has a unique solution in $\bar{P}$ *if and only if* i) *the greatest common divisor of* n *and* $q-1$ *is* 2, *that is*, $(q-1, n)=2$, *and* ii) *each* $x \in P$ *implies* $-x \in \bar{P}$.

Let $x^n-a=0$ with $a \in P$ and $n \in N_2$ has a unique solution $\alpha \in \bar{P}$. Then each $a \in P$ is necessarily an n-th power in the GF (q), which implies that $(q-1, n)=2$, see Dickson [2, §. 63]. Hence, equation (I) has exactly two roots in the GF(q), and the second root is $-\alpha$ which can be in $\bar{P}$ unless ii) is satisfied.

Conversely, if $(q-1, n)=2$ then $(q-1)/2$ elements in the GF(q) are n-th powers, which are elements of P because of $n \in N_2$ and P consists of the even powers of a primitive root of the GF(q). It follows from ii) that only one of the two roots of equation (I) is in $\bar{P}$.

THEOREM 1. *A construction* $PC(F, \varnothing, n)$ *with* $n \in N_2$ *forms a Polynomial Moulton plane* $PM(F, \varnothing, n)$ *if and only if* i) *the function* $\varnothing$ *is order reversing; and for* $F=GF(q)$ *additionally* ii) $(q-1,n) = 2$ *and* iii) *each* $x \in P$ *implies* $-x \in \bar{P}$.

A construction $PC(F, \varnothing, n)$ *with* $n \in N_1$ *or* $n \in N_3$ *forms a Polynomial Moulton plane if and only if* i_1) *the function* $\varnothing$ *is order preserving; and for* $F=GF(q)$ *additionally* ii_2) $(q-1, n)=1$ *if* $n \in N_1$, *or* $(q-1, n^{-1})=1$ *if* $n \in N_3$.

The first axiom for an affine plane is that for any two distinct points there exists exactly one line which is on both points. Let (u_o, v_o) and (u_1, v_1) be two distinct points of a given $PC(F, \varnothing, n)$. Suppose $u_o \in \bar{P}$, $u_1 \in P\cup\{0\}$, then we have $v_o= \varnothing(m)\, u_o^n+b$ and $v_1 =mu_1 +b$. By eliminating b between these two equations we get

$$\text{(II)} \qquad \varnothing(m) - (u_1/u_o^n)\, m+(v_1 -v_o)/u_o^n=0.$$

If $u_1 = 0$ then as $\varnothing$ is one-to-one and onto, $m = \varnothing^{-1}((v_o - v_1)/u_o^n)$. Let $u_1 \neq 0$. If $n \in N_2$ and $\varnothing$ is order reversing, by lemma 1, (and if $n \in N_1$ and $\varnothing$ is order preserving, by lemma 2) there exists a unique solution m in F for equation (II). Therefore, if $n \in N_2$, then the order reversing (or, if $n \in N_1$, the order preserving) property of $\varnothing$ is sufficient to prove that only one line joins any two distinct points. In the case where $u_o \in \bar{P}$ and $u_1 \in P \cup \{0\}$. Suppose $u_o \in \bar{P}$, $u_1 \in \bar{P}$ with $u_o \neq u_1$. Then we have $\varnothing(m)(u_o^n - u_1^n) = v_o - v_1$. Since $u_o \neq u_1$, $u_o^n - u_1^n \neq 0 \Leftrightarrow (u_o u_1^{-1})^n \neq 1$ for $u_o u_1^{-1} \neq 1 \Leftrightarrow a^n \neq 1$ for every $a \in P$, $a \neq 1$. If $n \in N_2$ and $(q-1, n) = 2$ then there exist integers h and k such that $(q-1)h + nk = 2$. Hence $a^2 = a^{(q-1)h+nk} = (a^n)^k$. Clearyly $a^2 \neq 1$ for $a \neq 1$ if and only if $a \neq -1$. But $a \neq -1$ is always true since $a \in P$, $1 \in P$ and by iii) $-1 \in \bar{P}$. It follows that $1 \neq a^2 = (a^n)^k$, that is, $(a^n)^k \neq 1$. Therefore we have $a^n \neq 1$. If $n \in N_1$ and $(q-1, n) = 1$ then there exist integers s and t such that $1 = (q-1)s + nt$. Therefore $1 \neq a = a^{(q-1)s+nt} = (a^n)^t$. But $1 \neq (a^n)^t \Rightarrow a^n \neq 1$ Consequently, in this case ii) and iii) or the first part of ii_1) are sufficient for the the first axiom of an affine plane to be satisfied according as $n \in N_2$ or $n \in N_1$, respectively. Clearly, in all other possible cases the axiom is satisfied without any extra condition for $\varnothing$, F and n.

Let us now verify the second axion, that is, existence of a unique line which is on a given point and parallel to a given line. Two lines are considered parallel if and only if they coincide or have no point in common. It is easily seen that the lines which pass through the given point (u_o, v_o) and parallel to the lines [a] and [m, b] are $[u_o]$ and $[m, v_o - m * u_o]$, respectively. Further, every pair of the lines [a] and [m, b] have the common point $(a, m * a + b)$. I f$m_1 \neq m_2$ then the lines $[m_1, b_1]$ and $[m_2, b_2]$ meet at the point $(x_o, m_1 * x_o + b_1)$ with $x_o = (b_1 - b_2)/(m_2 - m_1)$ or $x_o^n = (b_1 - b_2)/(\varnothing(m_2) - \varnothing(m_1))$ according as $x_o \in P \cup \{0\}$ or $x_o \in \bar{P}$. In the case where $n \in N_2$ and $\varnothing$ is order reversing, by using lemma 3 and also the fact that no element in $\bar{P}$ is an even power in the GF(q), it is seen that the unique existence of x_o is equivalent to the conditions ii) and iii) of the theorem; in the case where $n \in N_1$ and $\varnothing$ is order preserving, $(b_1 - b_2)/(m_2 - m_1)$ and $((b_1 - b_2))/(\varnothing(m_2) - \varnothing(m_1))^{1/n}$ have the same sign and therefore if F=GF(q), then the existence and uniqueness of

x_0 is equivalent to q–1 and n being relatively prime, see Dickson [2, § 63].

Conversely if $n \in N_2$ but $\varnothing$ fails to reverse the order for some m_1, m_2 in F, $m_1 \neq m_2$, then $(m_2\text{-}m_1)$ and $(\varnothing(m_2)\text{-}\varnothing(m_1))$ could have the same sign and therefore $[m_1, b_1]$ and $[m_2, b_2]$ could have had either no point or two points in common depending on whether $(b_1\text{-}b_2)/(m_2\text{-}m_1)$ is in $\bar{P}$ or P. It follows that the order reversing property of $\varnothing$ is necessary in order that any two non-parallel lines meet on a unique point. Similarly, if $n \in N_1$ but $\varnothing$ fails to preserve the order for some m_1 and m_2 in F, $m_1 \neq m_2$, then $(m_2\text{-}m_1)$ and $\varnothing(m_2)\text{-}\varnothing(m_1)$ could have different signs and therefore the lines $[m_1, b_1]$ and $[m_2, b_2]$ couldn't meet on exactly one point, that is, order preserving property is necessary for the second axiom to be satisfied.

In the above proof the case where $n \in N_3$ was skipped. In fact, it is more or less the same with the case where $n \in N_1$. However, the condition $(q-1, n^{-1})=1$ is also needed for the first axiom to be satisfied, as well as for the binary operation $*$ to be a single valued symbol. The presence of three noncollinear points is obvious so the proof of the theorem is now complete.

Polynomial Moulton planes are generally non-desarguesian. However, it is notable that a subclass of the family of these planes satisfies the most important theorems of projective plane geometry such as the Pappus theorem and the Desargues theorem. The following theorem provides us with some such planes for any n in the set S.

THEOREM 2. *A Polynomial Moulton plane* PM $(F, \varnothing, n)$ *is isomorphic to* π_F, *the classic affine plane over the field* F, *if and only if* $\varnothing(m) = \alpha m + \beta$, *where* $\beta \in F$, *and* $\alpha \in \bar{P}$ *or* $\alpha \in P$ *according as* $n \in N_2$ *or* $n \in N_1 \cup N_3$.

Assume that π_F and PM $(F, \varnothing, n)$ are isomorphic and T denotes the isomorphism between them. We can without loss of generality assume that T maps 1) the point (x', y') of π_F for $x' \in P \cup \{0\}$ onto (x, y) of PM$(F, \varnothing, n)$, where $x=x'$, $y=y'$, and 2) the line $\{y'=mx'+b\}$ of π_F onto the line $[m, b]$ of PM $(F, \varnothing, n)$. Let $\{y'=m_i x'+b_i\}$, $m_i \neq m_j$, $b_i \neq b_j$, i, j=1, 2, 3, be three concurrent lines of π_F. As T is an isomorphism it has to preserve concurrency. The above three lines are concurrent if and only if

(III) $m_i = \lambda + \mu b_i$ for some $\lambda, \mu \in F, \mu \neq 0, i=1, 2, 3$

while the images of these lines under T are concurrent if and only if

(IV) $\begin{cases} m_i = \lambda + \mu b_i \text{ for some } \lambda, \mu \in F, \mu \neq 0, i=1, 2, 3 \\ \text{or} \\ \varnothing(m_i) = \nu + \eta b_i \text{ for some } \nu, \eta \in F, \eta \neq 0, i=1, 2, 3 \end{cases}$

according as the first coordinate x of the intersection point of the lines is $x \in P \cup \{0\}$ or $x \in \bar{P}$. By eliminating b_i between (III) and (IV) and replacing m_i with m we get $\varnothing(m) = \alpha m + \beta$, where $\alpha = \eta\mu^{-1}$, $\beta = \nu - \eta\lambda\mu^{-1}$. Then $\alpha \in \bar{P}$ or $\alpha \in P$ follows from the order reversing or order preserving property of $\varnothing$ according as $n \in N_2$ or $n \in N_1 \cup N_3$.

Conversely, let PM (F, $\varnothing$, n) be a Polynomial Moulton plane with $\varnothing(m) = \alpha m + \beta$, $\beta \in F$ and $\alpha \in \bar{P}$ if $n \in N_2$ or $\alpha \in P$ if $n \in N_1 \cup N_3$. Then, define the transformation T from π_F to PM (F, $\varnothing$, n) such that $T((x', y')) = (x, y)$, where $(x, y) = ((\alpha^{-1}x')^{1/n}, y' + \beta\alpha^{-1}x')$ or (x', y') according as $x' \in \bar{P}$ or $x' \in P \cup \{0\}$, provided that if $n \in N_2$ then $(\alpha^{-1}x')^{1/n} \in \bar{P}$. It can be easily verified that T is a one-to-one correspondence between the points of π_F and PM(F, $\varnothing$, n), and maps the line $\{y' = mx' + b\}$ onto $[m, b]$; and the line $\{x' = a\}$ onto $[a]$ or $[(\alpha^{-1}a)^{1/n}]$ according as $a \in P \cup \{0\}$ or $a \in \bar{P}$, provided that $(\alpha^{-1}a)^{1/n} \in \bar{P}$. Hence it is an isomorphism between π_F and PM(F, $\varnothing$, n).

COROLLARY 1. *Any* PM(F, $\varnothing$, n) *with* $\varnothing(m) = \alpha m + \beta$ *is a pappian plane, where* $\beta \in F$, $\alpha \in \bar{P}$ *if* $n \in N_2$ *and* $\alpha \in P$ *if* $n \in N_1 \cup N_3$.

The Corollary follows immediately from theorem 2.

There exist affine transformations between the planes π_F and PM(F, $\varnothing$, n) with $\varnothing(m) = \alpha m + \beta$, $\beta \in F$, $\alpha \in \bar{P}$ if $n \in N_2$, $\alpha \in P$ if $n \in N_1 \cup N_3$, which map any line $\{y' = m' x' + b'\}$ of π_F onto any line $[m, b]$ of PM(F, $\varnothing$, n) such that $T((x', y')) = (x, y)$, where $(x, y) = (kx', h_1 x, + h_2 y' + h_3)$ or $((k\alpha^{-1}x')^{1/n}, (k\beta\alpha^{-1} + h_1) x' + h_2 y' + h_3)$ depending on whether $x' \in P \cup \{0\}$ or $x' \in \bar{P}$, provided that $(k\alpha^{-1}x')^{-1/n} \in \bar{P}$; $m = k^{-1}(h_1 + h_2 m')$, $b = h_2 b' + h_3$, $h \neq 0$, $k \in P$, $h_1, h_2, h_3 \in F$. The vertical lines are mapped among themselves.

Although the Polynomial Moulton planes have been presented as affine planes, they can be regarded as projective planes by adding ideal elements. Let V and (m) be the ideal points on the lines [a] and [m,b] respectively. Denote U=(0), and let PM($\bar{F}$, ∅,n) denote the extended polynomial plane.

THEOREM 3. *Every* PM($\bar{F}$, ∅, n) *is a* (V, UV) *–Desargues plane*[1].

Let A=(a, x), A′=(a, x′) on [a] and B=(b, y), B′=(b, y′) on [b], a≠b. First let us examine when AB, A′B′ and UV are concurrent.

Suppose a, b ∈ $\bar{P}$. Then the slope of AB is

m= $\varnothing^{-1}$ ((x-y) /(a^n-b^n)) and the slope of A′B′ is

m′= $\varnothing^{-1}$ ((x′-y′) /(a^n-b^n)). By theorem 1, if AB, A′B′, UV are concurrent then m=m′. As ∅ is one-to-one and onto, m=m′ if and only if x-y=x′-y′.

Suppose a∈P, b∈$\bar{P}$. Then m satisfies ∅(m)-(a /b^n) m=(x-y) /b^n, and similarly m′ satisfies ∅ (m′)-(a /b^n) m′=(x′-y′) /b^n. It follows from lemma 1 and lemma 2 that m=m′ if and only if x-y=x′-y′.

The case where a ∈ $\bar{P}$, b ∈ P is the same as the preceding one; and obviously if a, b ∈ P ∪ {0} then m=m′ if and only if x-y=x′-y′.

Let C= (c, z) and C′= (c,z′) on [c], b≠c≠a. Then the triangles ABC and A′B′C′ are perspective from V. Since any two of the equalities x-y=x′-y′, x-z=x′-z′, y-z=y′-z′ imply the third the triangles are also perspective from the line UV. Thus, PM($\bar{F}$, ∅, n) is a (V, UV) –Desargues plane.

LEMMA 4. *Any* PM(F, ∅, n) *is isomorphic to a* PM(F, Ψ, n) with Ψ (0).

Let T be a coordinate transformation on PM (F, ∅, n) such that T ((x, y)) = (x′, y′), where (x′, y′) = (x, y-(0 ∗ x)) if x ∈ $\bar{P}$ and (x′, y′) = (x, y) if x ∈ P ∪ {0}. Then T ([a]) = [a], and y= ∅(m)x^n+b x ∈ $\bar{P}$ is mapped onto y′= Ψ(m) x'^n+b, x′ ∈ $\bar{P}$, where Ψ(m)= ∅(m)- ∅(0). The function Ψ is one-to-one, onto and roder reversing (or order preserving) on F if and only if ∅ has the

1 For the definition of the (V, UV) –Desargues plane see Pickert [6, p.74].

same properties. Further $\Psi(0)=0$. Hence, Ψ can be used to define a Polynomial Moulton plane PM $(\overline{F}, \Psi, n)$ isomorphic to PM $(F, \varnothing, n)$.

THEOREM 4. *Any* PM $(\overline{F}, \Psi, n)$ *with* $\Psi(0)=0$ *is a* (U, UV; V, OU) *-Desargues plane*[2] *if and only if* $\Psi(m) = \alpha m$, $\alpha \in \overline{P}$ or $\alpha \in P$ *according as* $n \in N_2$ *or* $n \in N_1 \cup N_3$, *where* O=(0, 0).

If $\Psi(m) = \alpha m$ with $\alpha \in \overline{P}$ and $n \in N_2$ or $\alpha \in P$ and $n \in N_1 \cup N_3$, by Corollary 1, the plane PM $(\overline{F}, \Psi, n)$ is desarguesian. Thus, the condition is sufficient. Let $\Psi(0)=0$. Then PM $(\overline{F}, \Psi, n)$ has lines of the form $\{(x, y)\colon y=b;\ x, y \in F\} \cup \{(0)\}$. The following special case of the (U, UV; V, OU) -Desargues configuration will provide us with the necessity of the condition. Let $A=(u, o)$, $A'=(u', o)$, $B=(v,b)$, $B'=(v', b)$, $C=(v,b')$ and $C'=(v',b')$ such that $u, v \in \overline{P}$, $u', v' \in P$, $b, b' \in F$, $u \neq v$, $u' \neq v'$ and $b \neq b'$. Clearly $BC \cap B'C'=V$. The lines AB, A'B', UV are concurrent if and only if $\Psi(b/(v'-u'))=b/(v^n-u^n)$; AC, A'C', UV are concurrent if and only if $\Psi(b'/(v'-u'))=b'/(v^n-u^n)$. Either of these two equalities implies the other for all b, b', u, u', v, v' with the above conditions if and only if $x/\Psi(x)=y/\Psi(y)$, where $x=b/(v'-u')$ and $y=b'/(v'-u')$. Since always $x \neq y$, then $\Psi(m)=\alpha m$, $\alpha \in F-\{0\}$. The function Ψ reverse or preserve the order if $\alpha \in \overline{P}$ or $\alpha \in P$ respectively.

COROLLARY 2. *Any* PM$(F, \varnothing, n)$ *is desarguesian*[3] *if and only* if $\varnothing(m)=\alpha m+\beta$, $\beta \in F$, $\alpha \in \overline{P}$ if $n \in N_2$ and $\alpha \in P$ if $n \in N_1 \cup N_3$.

The corollary is an immediate consequence of Corollary 1, Lemma 4 and Theorem 4. Clearly if F=GF(q) we assume that $q \geq 9$ for this corollary and the preceding theorem.

Our last theorem will be about determination of the field F= GF(q) which was used to construct a PM $(F, \varnothing, n)$ in the case where $n \in N_2$.

THEOREM 5. *Let* $n \in N_2$. *The field* GF(q), $q=p^r$, p *an odd prime number and* $r \in N$, *can be used to construct* a PM $(F, \varnothing, n)$ *if and only* if i) $(q-1, n)=2$, ii) $p \equiv 3 \bmod 4$, iii) $r \in N_1$.

Proof will be obtained by combining some theorems. The ondition $(q-1, n)=2$ appeared in Th.1. Clearly the condition that

2 For the definition of the (U, UV; V, OU) -Desargues plane see Pickert [6, p.80].

3 Carlitz showed in [10] that order preserving (or reversing) polynomials over a finite field should be of the form $\varnothing(m)=a\, m^{p^j}+b$ with $0 \leq j < r$. This shows that the set of non-Desarguesian polynomial planes is not empty.

each $x \in P$ implies $-x \in \bar{P}$, in the same theorem, is equivalent to $-1 \in \bar{P}$. But $-1 \in \bar{P}$ shows that -1 is not a quadratic residue of p in GF(p), and therefore $p \equiv 3 \mod 4$, see Hardy [9, p. 69, Th.82]. On the other hand -1 is also a non-square element in $GF(p^r)$ if and only if $r \in N_1$, see Dickson [2, §. 62].

The connection between the plynomial Moulton planes and the other Moulton planes can be summarized as follows:

1. Any plane PM (F, $\varnothing$, n) with F=R, n=1 and $\varnothing$ an order preserving function is a Levenberg-Moulton plane [4] under the condition that the roles of the coordinate axes are exchanged.

2. Any plane PM (F, $\varnothing$, n) with n=1 and $\varnothing$ an order preserving function is a Pierce-Moulton plane [7].

3. The plane originally given by Moulton (see Pickert [6, p. 93]) is a PM (F, $\varnothing$, n) with F = R, n = 1 and $\varnothing(m)$ = km or m according as $m < 0$ or $m \geq 0$. Where k is a positive constant.

4. Each generalized Moulton plane given in Kaya [3] is a PM (F, $\varnothing$, n) with F=R, $n \in N_1$ and $\varnothing(m)$=cm or m according as $m < 0$ or $m \geq 0$ under the condition that the roles of coordinate axes are exchanged. Where c is a positive constant.

Furthermore, a PM(F, $\varnothing$, n) with n=1 and $\varnothing(m)$=m is the classic affine plane defined over the field F.

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ÖZET

Bu makalede klasik Moulton düzleminin K.Leverberg, W.A.Pierce ve R.Kaya tarafından verilen çeşitli genelleştirilmişleri yeniden daha da genelleştirilmektedir. Bunun için verilen pseudo-sıralı bir F cismi ile bu cisim üzerinde birebir örten bir $\varnothing$ fonksiyonu ve $N_1 = \{2x-1: x \in N\}$, $N_2 = \{2x: x \in N\}$, $N_3 = \{(2x-1)^{-1}: x \in N\}$ cümlelerinden herhangi birinin bir n elemanından meydana gelen her $(F, \varnothing, n)$ üçlüsü için bir afin düzlem tanımlanır: Böyle bir düzlem için.

1) $n \in N_2$ iken $\varnothing$ nin sıralamayı ters çeviren bir fonksiyon olmasının; ayrıca $F = GF(q)$, $q = p^r$, iken $(q-1, n) = 2$, $p \equiv 3 \bmod 4$, $r \in N_1$ şartlarının sağlanmasının gerek ve yeter olduğu,

2) $n \in N_1 \cup N$ iken $\varnothing$ nin sıralamayı koruyan bir fonksiyon olmasının; ayrıca $F = GF(q)$, $q = p^r$ iken $n \in N_1$ yada $n \in N_3$ olmasına göre sırayla $(q-1, n) = 1$ yada $(q-1, n^{-1}) = 1$ şartının sağlanmasının gerek ve yeter olduğu ispatlanır. Bu tip her afin düzlemin projektif düzleme genişletilmişinin daima (V, UV) –Desargues düzlemi olduğu gösterilir. α, $n \in N_2$ iken F nin tamkare olmayan bir elemanı ve fakat $n \in N_1 \cup N_3$ iken F nin bir tamkare elemanı olmak üzere $\varnothing$ fonksiyonu $\varnothing(x) = \alpha x + \beta$ şeklinde ise bunun yardımiyle tanımlanan her düzlemin Desargues düzlemi olduğu gösterilir.

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