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A Comparison of Gauss-Markov Estimators and Least Squares Estimators of the Micro and Macro Parameters

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ABSTRACT

A more general linear aggregation model is considered in that we allow for collinearity between the independent or explanatory variables. Thus the analysis is presented in a framework utilizing Moore-Penrose generalized inverses of singular matrices. Gauss-Markov estimators are derived and compared with covariance structure of the micro parameters. The efficiency is obtained of the least squares estimators of the micro parameters.

1. INTRODUCTION

Aggregation theory is concerned with the transformation of individual relations (micro relations) to a relation for the group as a whole (a macro relation). We shall confine ourselves to Linear Relations in order to simplify the exposition.

Theil ([13], [14]) discussed certain relationships between the micro and macro variables. Boot and deWit [3] made important methodological contributions to the problems involved in calculating aggregation bias. The basic framework for Boot and deWit's study was provided by Theil's pioneering work [13]. Theil's approach to the problem of aggregation over individuals in connection with regression models can be briefly stated using the convenient matrix notation introduced by Kloek [7]. The problem also examined in [2] and [6]. Recently, Misra [9], and Lütjohann [8] also utilized matrix notation to study the micro and macro economic relations.

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Our purpose is to derive a relationship between the micro and macro parameters when we take into account the covariance structure of the observations, derive and compare two Gauss-Markov estimators of the micro parameters. The relationship between the micro and macro parameters is used to obtain estimators of the macro relationships by using the least squares estimator of the micro parameters and the Gauss-Markov estimator of the micro parameters. The final section obtains efficiency of the above least squares estimator of the micro parameters.

2. NOTATION AND ASSUMPTIONS

Let the economic relationship for the i–th economic unit, say firm or household, be given by

$$(1) \quad Y_i = X_i \beta_i + u_i, \quad i = 1, 2, \ldots, k$$

where Y_i is a column vector of T values of the dependent variables; X_i is a T x (p + 1) matrix of non-stochastic values of the independent variables, β_i is a column of unknown micro parameters; and u_i is a disturbance vector of T random variables, and k is the number of economic units.

The following assumptions are utilized throughout this paper:

(i) The matrix X_i has rank $r_i \leq p + 1$

(ii) u_i 's have zero means and are independent of $(X_1, X_2, \ldots, X_k)$, i.e., $E(u_i \mid X_i) = 0$

(i) $$E(u_i u_j') = \begin{cases} \Sigma_{ii} & i = j \\ o & i \neq j \end{cases}$$

The model of equation (1) may be expressed as a single model,

$$(2) \quad Y^* = X^* \beta^* + u^*$$

where

$$Y^* = \begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_k \end{bmatrix}, X^* = \begin{bmatrix} X_1 & 0 & \ldots & 0 \\ 0 & X_2 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & X_k \end{bmatrix}, \beta^* = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_k \end{bmatrix}, u^* = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_k \end{bmatrix}$$

and

$$E\ (u^* \mid X^*) = 0; \qquad E\ (u^* u^{*\prime}) = \Sigma.$$

The matrix Σ is block-diagonal matrix, consisting of the matrices Σ_{ii} which have generally unknown elements.

In order to present the macro relations in the convenient matrix notation define the matrices

$$(3) \qquad J_T = I_T \otimes J'_k \quad \text{and} \quad J_{p+1} = I_{p+1} \otimes J'_k$$

where I_T is a T x T identity matrix, j_k a k x 1 column vector of unit elements, and $A \otimes B$ denotes the Kroneckor product of the matrices A and B (Graybill [5]).

The macro relations can be written in matrix form as

$$(4) \qquad Y = X\beta + u,$$

where

$$X = J_T X^* J'_{p+1} = \sum_{i=1}^{k} X_i \text{ and } Y = J_T Y^* = \sum_{i=1}^{k} Y_i$$

and we assume that the rank of X is equal to r ($r \leq p + 1$).

In model (4), $X\beta$ is defined as a mathematical expectation of its Gauss-Markov-Estimator (GME)

$$(5) \qquad X\beta = E\ \{X(X'\ \Sigma^{*-1}X)^{-}\ X'\,\Sigma^{*-1}\ Y\} = E\,\{X\tilde{\beta}\}$$

where

$$\Sigma^* = E\,\{\,u - E\ (u)\}\,\{u - E\ (u)\}'; \ \tilde{\beta} = (X'\ \Sigma^{*-1}X)^{-}X)^{-}X'\,\Sigma^{*-1}Y;$$

and A^- denotes the Moore–Penrose generalized inverse of A[1] (Graybill [5]).

3. MICRO AND MACRO RELATIONS

It is of much importance to study the relationship between the micro and the macro parameters, β^* and β, respectively.

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[1]Moore–Penrose generalized inverse of A of order m x n is a matrix A^- of order n x m such that

a) $AA^-A = A$ b) $A^-AA^- = A^-$

c) $(AA^-)' = AA^-$ d) $(A^-A)' = A^-A$

Such an inverse exists and is unique.

It follows from (2) that the correct specification, based on micro theory for the macro dependent variable Y is

$$(6) \qquad Y = J_T X^* \beta^* + J_T u^*$$

or

$$\sum_{i=1}^{k} Y_i = \sum_{i=1}^{k} X_i \beta_i + \sum_{i=1}^{k} u_i \,.$$

The macro relations in (4) are generally different from the true macro relations. The "aggregate" method presumably involves a specification error in (4). Essentially we are trying to describe the vector $J_T X^* \beta^*$ in the best possible way by $X\beta$ weighted by the respective covariance matrices. This best description is the projection of $J_T X^* \beta^*$ onto the vector space spanned by the columns of X denoted by C (X). The projection of $J_T X^* \beta^*$ onto the vector space weighted by Σ^* is

$$(7) \qquad X\beta = X\,(X'\,\Sigma^{*-1}X)^{-}\,X'\,\Sigma^{*-1}\,J_T X^* \beta^* \\ = P^* J_T X^* \beta^*$$

where $P^* = X\,(X'\,\Sigma^{*-1}X)^{-}\,X'\,\Sigma^{*-1}$ is the orthogonal projector onto C (X); which is unique for any choice of the generalized inverse involved in (7), (see Rao [11]).

Thus, the specification error u^+ is in the vector space orthogonal to the vector space spanned by the columns of a matrix X, i. e.,

$$u^+ = (I - X\,(X'\,\Sigma^{*-1}X)^{-}\,X'\,\Sigma^{*-1})\,J_T X^* \beta^*.$$

It is shown that equations (5) and (7) are equivalent. To demonstrate the construction of the specification error, we will use a Geometrical illustration in Figure 3.1 where

$$X = X_1 + X_2 + X_3 \quad X^* = \begin{bmatrix} X_1 & o & o \\ o & X_2 & o \\ o & o & X_3 \end{bmatrix}$$

$$J_T = (I_T \otimes J'_3) \quad (k = 3) \quad \text{and} \quad \beta^* = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

The specification error is the vector between $J_T X^* \beta^*$ and its projection onto C (X). It is also called the "Aggregation bias" of u (see [6]).

Lemma 3.1: The system of equations relating to β and β^*, $X\beta = P^* J_T X^* \beta^*$, is consistent [1].

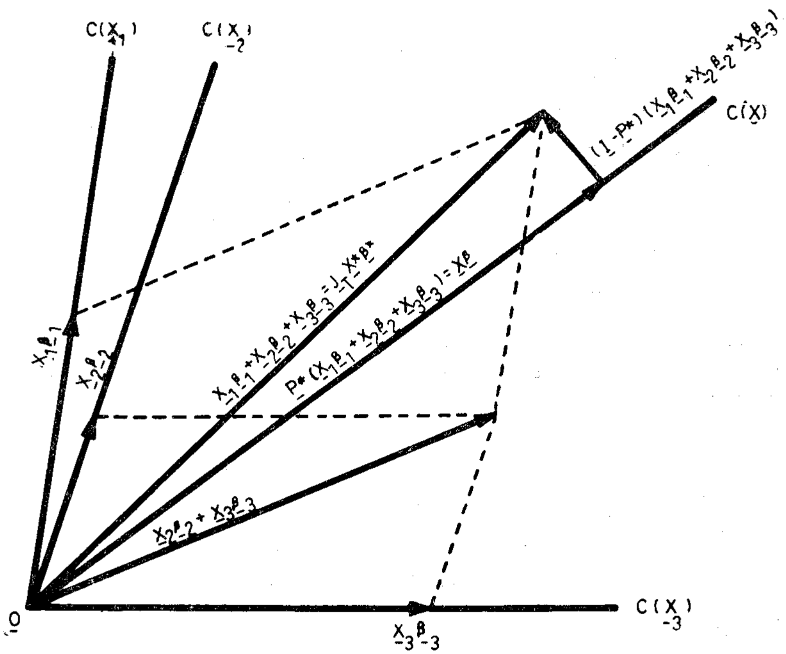

FIGURE 3.1: Geometrical representation of the ralationship between the true macro-relations and usual aggregated macro relations.

The next section is a discussion of various properties the estimators of the micro and macro parameters.

4. A COMPARISON OF TWO ESTIMATORS OF THE MACRO PARAMETERS.

By utilizing the relationship between β and β^* two estimators of $X\beta$ can be obtained. The GME of $X\beta$ from (4) is

$$X\tilde{\beta} = X\,(X'\,\Sigma^{*-1}\,X)^{-}\,X'\,\Sigma^{*-1}Y$$

so that $X\tilde{\beta}$ is uniquely determined. But another estimator exists by utilizing the relationship

$$X\beta = P^{*}J_{T}X^{*}\beta^{*}.$$

Let $\tilde{\beta}^{*}$ be the vector of GME of the micro parameters obtained from (2), then an estimate of $X\beta$ is

$$X\mathrm{b} = P^{*}J_{T}X^{*}\tilde{\beta}^{*}.$$

The expectation of Xb is

$$\begin{aligned} \mathrm{E}\,(X\mathrm{b}\,|X^{*}) &= \mathrm{E}\,[P^{*}J_{T}X^{*}\,(X^{*\prime}\,\Sigma^{-1}\,X^{*})^{-}\,X^{*\prime}\,\Sigma^{-1}\,(X^{*}\beta^{*}+u^{*})\,] \\ &= P^{*}J_{T}X^{*}\,(X^{*\prime}\,\Sigma^{-1}\,X^{*})^{-}\,X^{*\prime}\,\Sigma^{-1}\,X^{*}\beta^{*} \\ &= P^{*}J_{T}X^{*}\beta^{*} \\ &= X\beta. \end{aligned}$$

Lemma 4.1: The two estimators of $X\beta$, $X\tilde{\beta}$ and Xb are equal if and only if C (X_1) = C (X_2) = ... = C (X_k) = C (X), where C (X_i) denotes the column vector space spanned by the columns of X_i.

Theorem 4.1. The variances of the components of $X\tilde{\beta}$ are greater than or equal to the corresponding variances of the components of Xb, thus Xb is the better of the two estimators.

Proof: The proof consists of deriving the covariance matrices of $X\tilde{\beta}$ and Xb and then examining the diagonal elements of the difference of the two covariance matrices.

The covariance matrix of $X\beta$ is denoted by Σ_1 and is defined to be

$$(8) \qquad \Sigma_1 = \mathrm{E}\;[X\tilde{\beta} - \mathrm{E}\,(X\tilde{\beta})]\;[X\tilde{\beta} - \mathrm{E}\,(X\tilde{\beta})]'.$$

Considering (4), (6) and (7) then the macro disturbance vector u can be written as

$$(9) \qquad u = J_{T}u^{*} + u^{+}$$

or

$$u = J_{T}u^{*} + (I - X\,(X'\,\Sigma^{*-1}X)^{-}\,X'\,\Sigma^{*-1})\,J_{T}X^{*}\beta^{*}.$$

This expression is then used to evaluate (8). The expectation of $X\tilde{\beta}$ is

$$\begin{aligned} E(X\tilde{\beta}) &= E\,[X(X'\Sigma^{*-1}X)^{-}X'\Sigma^{*-1}Y] \\ &= E\,[X(X'\Sigma^{*-1}X)^{-}X'\Sigma^{*-1}X\beta + X(X'\Sigma^{*-1}X)^{-}X'\Sigma^{*-1}u] \\ &= X\beta + X(X'\Sigma^{*-1}X)^{-}\;X'\Sigma^{*-1}\;(I - X(X'\Sigma^{*-1}X)^{-} \\ &\quad X'\Sigma^{*-1})\;J_T X^{*}\beta^{*} \\ &\quad + X(X'\Sigma^{*-1}X)^{-}X'\Sigma^{*-1}J_T E(u^{*}) \\ &= X\beta. \end{aligned}$$

Then

$$\begin{aligned} \Sigma_1 &= E\,[X\tilde{\beta} - X\beta]\,[X\tilde{\beta} - X\beta]' \\ &= X(X'\Sigma^{*-1}X)^{-}X'\Sigma^{*-1}E(uu')\Sigma^{*-1}X(X'\Sigma^{*-1}X)^{-}X', \end{aligned}$$

where

$$E(uu') = [I - X(X'\Sigma^{*-1}X)^{-}X'\Sigma^{*-1}]\,WW'\,[I - \Sigma^{*-1}X(X'\Sigma^{*-1}X)^{-}X'] + J_T\Sigma J_T' \tag{10}$$

and

$$W = J_T X^{*}\beta^{*} = \sum_{i=1}^{k} X_i\beta_i.$$

Thus, Σ_1 becomes

$$\Sigma_1 = P^{*}J_T\Sigma_T'P^{*\prime} = P^{*}\,[\sum_{i=1}^{k}\Sigma_{ii}]\,P^{*\prime}. \tag{11}$$

The covariance matrix of Xb is

$$\begin{aligned} \Sigma_2 &= E\,(Xb - X\beta)(Xb - X\beta)' \\ &= E\,(P^{*}J_T X^{*}\tilde{\beta}^{*} - P^{*}J_T X^{*}\beta^{*})\,(P^{*}J_T X^{*}\tilde{\beta}^{*} - P^{*}J_T X^{*}\beta^{*})' \\ &= P^{*}J_T X^{*}(X^{*\prime}\Sigma^{-1}X^{*})^{-}X^{*\prime}J_T'P^{*\prime}. \end{aligned} \tag{12}$$

To compare the two estimators, we examine $\Sigma_1 - \Sigma_2$ or

$$\Sigma_1 - \Sigma_2 = P^{*}J_T\,(\Sigma - X^{*}(X^{*\prime}\Sigma^{-1}X^{*})^{-}X^{*\prime})\,J_T'P^{*\prime}. \tag{13}$$

In order to substantiate the theorem, it is sufficient to show that $\Sigma_1 - \Sigma_2$ is a nonnegative matrix. Since Σ is positive definite, it can be written as $\Sigma = \Delta \Delta'$ where Δ is a nonsingular matrix. Thus (13) becomes

$$\begin{aligned}\Sigma_1 - \Sigma_2 &= P^* J_T \Delta \; [I - \Delta^{-1} X^* (X^{*\prime} \Delta'^{-1} \Delta^{-1} X^*)^- X^{*\prime} \\ &\quad \Delta'^{-1}] \; \Delta' \; J_T' \; P^{*\prime} \\ &= P^* J_T \Delta \; [I - V \; (V'V)^- \; V' \;] \; \Delta' \; J_T' P^{*\prime} \\ &= P^* J_T \Delta \; M \Delta' \; J_T' \; P^{*\prime},\end{aligned}$$

where $V = \Delta^{-1} X^*$, $M = I - V (V'V)^- V'$ and $\Sigma = (\Delta')^{-1} \Delta^{-1}$. It is well known that M is positive semi-definite, then it may be written as

$$M = BB'.$$

Thus,

$$\Sigma_1 - \Sigma_2 = P^* J_T \Delta B \; (P^* J_T \Delta B)'$$

is positive semi definite (or nonnegative). This completes the proof.

The above theorem implies that the better estimator of the macro relations is via the equation expressing the relationship between β and β^*.

5. THE GAUSS–MARKOV THEOREM

Suppose, as above, $Y^{*\prime} = (Y'_1, Y'_2, ..., Y_k')$ is a Tk-dimensional observation vector, $\beta^{*\prime} = (\beta'_2, ..., \beta_k')$ a $(p + 1)$ k – dimensional parameter vector, and $X^* = \text{diag} \; \{ X_1, X_2, ..., X_k \}$ a Tk x $(p + 1)$ k non-stochastic matrix of rank

$\sum_{i=1}^{k} r_i < (p + 1)$ k. Let

$$\text{(5.1)} \qquad E\,(Y^*) = X^* \beta^*$$

and

$$\text{(5.2)} \qquad E\,(Y^* - X^* \beta^*)\,(Y^* - X^* \beta^*)' = \Sigma$$

where Σ is a Tk x Tk positive definite covariance matrix. We call $X^* \beta^*$ the regression function of Y^* on β^* and $u^* = Y^* - X^* \beta^*$ disturbance vector.

The fundamental problem is to find the "best" estimate of the parameter vector β^* based on the observables $Y_1, Y_2, ..., Y_k$. It is well known that

$$(5.3) \qquad G'Y^* = X^* (X^{*\prime} \Sigma^{-1} X^*)^- X^{*\prime} \Sigma^{-1} Y^* = X^* \tilde{\beta}^*$$

is the BLUE of $X^*\beta^*$. The least squares estimate of β^* is given by

$$(5.4) \qquad b^* = X^{*-} Y^*.$$

The estimate X^*b^* is also a linear unbiased estimate (LUE) of $X^*\beta^*$. On the other hand we have the following expression $X\tilde{b} = XX^- J_T X^* \tilde{\beta}^*$, where $X\tilde{b}$ is an estimate of $X\beta$ in model (4). Thus the variance covariance matrix of the GME of $XX^- J_T X^* \beta^*$ is

$$\begin{aligned}(5.5) \qquad W_1 &= \mathrm{E}\ \{X\tilde{b} - \mathrm{E}\,(X\tilde{b})\}\ \{X\tilde{b} - \mathrm{E}\,(X\tilde{b})\}' \\ &= \mathrm{E}\ \{XX^- J_T P^* Y^* - XX^- J_T X^* \beta^*\}\{XX^- J_T P^* Y^* \\ &\quad - XX^- J_T X^* \beta^*\} \\ &= \mathrm{E}\ (Bu^* u^{*\prime} B') \\ &= B \Sigma B',\end{aligned}$$

where

$$B = XX^- J_T X^* (X^{*\prime} \Sigma^{-1} X^*)^- X^{*\prime} \Sigma^{-1} = XX^- J_T P^*.$$

The variance-covariance matrix of $XX^- J_T X^* b^*$ is given by

$$(5.6) \qquad W_2 = XPX^{*-} \Sigma (X^{*\prime})^- P'X'$$

where $P = X^- J_T X^*$. Since $X^* \tilde{\beta}^*$ is BLUE of $X^*\beta^*$ we conclude that

$$(5.7) \quad W_2 - W_1 = XP\ [X^{*-} \Sigma (X^{*\prime})^- - (X^{*\prime} \Sigma^{-1} X^*)^-]\ P'X'$$

or

$$= XPX^{*-}\ [\Sigma - X^* (X^{*\prime} \Sigma^{-1} X^*)^- X^{*\prime}]\ (X^{*\prime})^-\ P'X'$$

is nonnegative definite by theorem 4.1. The next Lemma gives necessary and sufficient conditions for the estimator from (5.4) to be equivalent to the estimator from (5.3).

Lemma: The estimate X^*b^* defined from (5.4) equals the estimate $X^*\tilde{\beta}^*$ defined from (5.3) if and only if

$$\sum_{i=1}^{h} \mathrm{rank}\ (X'_j\ P_{ij}) = \mathrm{rank}\ (X_j); \qquad (j = 1, 2, \ldots, k)$$

where the covariance matrix Σ_{jj} in (iii) has h distinct characteristic roots $\lambda_{1j}, \lambda_{2j}, ..., \lambda_{hj}$ with multiplicity $m_1, m_2, ..., m_h$ and corresponding orthonormalized characteristic vector sets $P_{1j}, P_{2j}, ..., P_{hj}$.

The T x T matrix $P_j = (P_{1j}, P_{2j}, ..., P_{hj})$ is an orthogonal matrix, P_{ij} is T x m_i,

$$\sum_{i=1}^{h} m_i = T.$$

The proof is given by Styan [12].

6. EFFICIENCY OF LEAST SQUARES ESTIMATOR

The existing criteria for determining efficiency of estimation is for full rank models. Thus, in order to investigate the efficiency problems, we first reparameterize the original model (2) to a full rank model. Following Graybill [4] we note that there exists a non-singular matrix R such that

$$R'X^{*\prime}X^*R = \begin{bmatrix} S & O \\ O & O \end{bmatrix}$$

where S is non-singular of rank

$$q = \sum_{i=1}^{k} r_i.$$

If we let $R = [R_1 \mid R_2]$ where R_1 is k (p + 1) x q and R_2 is k (p + 1) x $\{$k (p + 1) − q$\}$ it follows that

$$R_1' X^{*\prime}X^*R_1 = S, \qquad X^*R_2 = O.$$

Defining

$$T^* = R^{*-1} = \begin{bmatrix} T_1^* \\ T_2^* \end{bmatrix}$$

where T^*_1 is q x k (p + 1) and T^*_2 is [k (p + 1) − q] x k (p + 1), we get the following relationship

$$X^*\beta^* = X^*RT^*\beta^* = X^*R_1T_1^*\beta^* = Z^*\beta^*.$$

The model (2) can be equivalently expressed as a full rank model in terms of $Z^*\alpha^*$ as

$$Y^* = X^*\beta^* + u^* = Z^*\alpha^* + u^*.$$

where $Z^* = X^*R_1$ is Tk x q of rank q and $\alpha^* = T_1^*\,\beta^*$ is a q x 1 vector of parameters. By the construction of Z^*, then X^* and Z^* have the same column space thus it follows that $X^*b^* = Z^*\hat{\alpha}^*$ and $X^*\,(X^{*\prime}X^*)^-\,X^{*\prime} = Z^*\,(Z^{*\prime}Z^*)^{-1}Z^{*\prime}$, where b^* is any solution to the normal equations $X^{*\prime}X^*b^* = X^{*\prime}Y^*$ and $\hat{\alpha}^*$ is the solution to the normal equations $Z^{*\prime}Z^*\hat{\alpha}^* = Z^{*\prime}Y^*$. Thus the variance covariance matrix of $P^*J_TX^*b^*$ can be written as

$$\begin{aligned} V_1 &= \mathrm{E}\ \{P^*J_TX^*b^* - P^*J_TX^*\beta^*\}\ \{P^*J_TX^*b^* - P^*J_T\,X^*\,\beta^*\}' \\ &= \mathrm{E}\ \{P^*J_TZ^*\hat{\alpha}^* - P^*J_T\,Z^*\alpha^*\}\ \{P^*J_TZ^*\hat{\alpha}^* - P^*J_TZ^*\alpha^*\}' \\ &= P^*J_TZ^*(Z^{*\prime}Z^*)^{-1}Z^{*\prime}\Sigma\ Z^*\ (Z^{*\prime}Z^*)^{-1}\ Z^{*\prime}\ J'_TP^{*\prime}. \end{aligned}$$

Now, consider the variance-covariance matrix of GME of $P^*J_T X^*\,\beta^*$:

$$\begin{aligned} V_2 &= \mathrm{E}\ \{P^*J_T\,X^*\,(X^{*\prime}\,\Sigma^{-1}X^*)^-\ X^{*\prime}\,\Sigma^{-1}Y^* - P^*J_TX^*\beta^*\} \\ &\quad .\ \{P^*J_TX^*\,(X^{*\prime}\,\Sigma^{-1}X^*)^-\ X^{*\prime}\,\Sigma^{-1}Y^* - P^*J_TX^*\beta^*\}' \\ &= \mathrm{E}\ \{P^*J_TZ^*\ (Z^{*\prime}\ \Sigma^{-1}Z^*)^-Z^{*\prime}\ \Sigma^{-1}Y^* - P^*J_TZ^*\alpha^*\} \\ &\quad .\ \{P^*J_TZ^*\ (Z^{*\prime}\ \Sigma^{-1}Z^*)^{-1}\ Z^{*\prime}\ \Sigma^{-1}Y^* - P^*J_TZ^*\alpha^*\}' \\ &= P^*J_TZ^*\ (Z^{*\prime}\,\Sigma^{-1}Z^*)^{-1}\ Z^{*\prime}\,J_T{}'P^{*\prime} \end{aligned}$$

where

$$\tilde{\alpha}^* = (Z^{*\prime}\ \Sigma^{-1}Z^*)^{-1}\ Z^{*\prime}\ \Sigma^{-1}Y^*$$

is GME of α^*. Since.

$$X^*\ (X^{*\prime}\ \Sigma^{-1}X^*)^-\ X^{*\prime}\ \Sigma^{-1}Y^* = Z^*\ (Z^{*\prime}\ \Sigma^{-1}Z^*)^{-1}\ Z^{*\prime}\ \Sigma^{-1}Y^*$$

are the BLUES of $X^*\beta^*$ and $Z^*\alpha^*$, respectively, we conclude that

$$\begin{aligned} V_1 - V_2 &= P^*J_TZ^*\ [(Z^{*\prime}Z^*)^{-1}Z^{*\prime}\Sigma\,Z^*\,(Z^{*\prime}Z^*)^{-1} - (Z^{*\prime}\,\Sigma^{-1}Z^*)^{-1}] \\ &\quad (P^*J_TZ^*)' \\ &= P^*J_TZ^*\,(Q_1 - Q_2)\ (P^*J_TZ^*)'. \end{aligned}$$

Since $Q_1 - Q_2$ is a nonnegative matrix, then $V_1 - V_2$ is a nonnegative matrix (see Rao [10]). Thus, we define the efficiency of the least squares estimator $\hat{\alpha}^*$ as compared to the GME $\tilde{\alpha}^*$ as

$$\text{Eff}(\hat{\alpha}^*) = \frac{\det^2 (Z^{*\prime} Z^*)}{\det (Z^{*\prime}\ \Sigma\ Z^*) \det (Z^{*\prime}\ \Sigma^{-1} Z^*)}$$

$$= \frac{\det^2 (R_1{}' X^{*\prime} X^* R_1)}{\det (R_1{}' X^{*\prime}\ \Sigma X^* R_1) \det (R_1{}' X^{*\prime}\ \Sigma^{-1}\ X^* R_1)}$$

Since α^* consists of a set of linearly independent estimable functions of β^*, then the efficiency of $\hat{\alpha}^*$ measures the efficiency of b^*.

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ÖZET

Açıklayıcı değişkenler arasında doğrudaşlık olması halinde daha genel lineer aggregatıon problemi düşünüldü. Böylece singüler matrislerin Moore-Penrose inversi kullanılarak analiz yapıldı. Mikro parametrelerin Gauss-Markov tahmin edicileri belirtildi, kovaryans yapısı göz önüne alınarak karşılaştırıldı ve en küçük kareler tahmin edicilerinin elverişliliği incelendi.

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