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by

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Multiplication Theorems for Strong Functional Nörlund Summability

By

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1. *Introduction.* In [1], the author has introduced the idea of strong functional Nörlund summability $[N, p]_\lambda$ and has investigated some of its properties.

In the present paper, we establish some theorems concerning strong functional Nörlund summability of the Cauchy product of two integrals. The analogue of our Theorem 3 for functional Nörlund summability (N, p) is [3, Theorem 6].

2. *Preliminaries.* Let S be the class of (complex valued) functions a (t) of the real variable t defined for all positive t, bounded and measurable in every finite interval (0, T), $T > 0$. Let P be the class of all real valued functions p (t) defined for all $t > 0$ and Lebesgue integrable in any (relevant) finite interval such that $p_1(t) \neq 0$ for all $t > 0$, where

$$p_1(t) = \int_0^t p(u)\, du.$$

As $p_1(t)$, being an integral, is continuous and $\neq 0$, there is no loss of generality to suppose it positive for all $t > 0$. Then p (t) shall be called a *weight function.* Similar notations and definitions will be used for other weight functions q (t), r (t) etc. It is convenient to define all our functions to be zero if their argument is negative. As usual we define the convolution $(a*b)_t$ of any two given functions a (t) and b (t) as

$$(a^*b)_t = \int_0^t a\ (t-u)\ b\ (u)\ du;$$

and we shall make use of the fact that the operation of convolution is commutative and associative.

Given two integrals

$$\int_0^\infty a\ (u)\ du \text{ and } \int_0^\infty b\ (u)\ du$$

with a (t), b (t) ε S, we set c (t) = $(a^*b)_t$ and call the integral

$$\int_0^\infty c\ (u)\ du$$

the Cauchy product of the given two integrals.

Let σ (t) be the integral transform of a (t) ε S defined by

$$\sigma\ (t) = \int_0^\infty a\ (t,u)\ a\ (u)\ du. \tag{2.1}$$

The transformation (2.1) with kernel a (t,u) is said to be *regular* over the set S, if a (t) → A implies σ (t) → A as t → ∞, and it is called *null-preserving* if a (t) → 0 implies σ (t) → 0 as t → ∞. The necessary and sufficient conditions for the regularity of the transformation (2.1) over the set S are [2, p. 50, 61]:

(i) $$\int_0^\infty |\ a\ (t,\ u)\ |\ du = 0\ (1), \tag{2.2}$$

(ii) $$\int_0^\infty a\ (t,u)\ du \to 1 \text{ as } t \to \infty, \tag{2.3}$$

(iii) $$\int_E a\ (t,u)\ du \to 0 \text{ as } t \to \infty, \tag{2.4}$$

for every bounded and measurable set E of u-axis. But if a (t,u) is non-negative, then (iii) is equivalent to (iii′) $$\int_0^c a(t,u)\,du \to 0 \text{ as } t \to \infty \qquad (2.5)$$

for every finite $c > 0$.

The conditions (i) and (iii) are necessary and sufficient for the transformation (2.1) to be null-preserving.

The conditions (i), (ii) and (iii) are respectively called the norm-, row-, and column-condition.

Definitions.

(a) *Functional Nörlund Summability* (N, p) (see [4]).

Let $\int_0^\infty a(u)\,du$ ($a(t) \varepsilon S$) be the given integral. Write

$$a_1(t) = \int_0^t a(u)\,du.$$

If

$$\sigma(t) = \frac{(p^* a_1)_t}{p_1(t)} \to A \text{ as } t \to \infty, \qquad (2.6)$$

we say that $\int_0^\infty a(u)\,du$ is summable (N,p) to the value A, and we denote this by

$$\int_0^\infty a(u)\,du = A\ (N,p) \quad \text{or} \quad a_1(t) \to A\ (N,p).$$

If $p(t) \geq 0$, the Nörlund method (N,p) is called positive.

(b) If $\sigma(t) = 0(1)$, we shall say that $\int_0^\infty a(u)\,du$ is bounded (N,p) and shall denote this by

$$\int_0^\infty a(u)\,du = 0\ (N,p).$$

(c) *Strong Functional Nörlund Summability* $[N,p]_\lambda$, $\lambda > 0$.

Let p (t) be a weight function which is such that, for given $T > 0$, there exists $\upsilon = \upsilon(T) > 0$ such that

$$| p(t) | \geq \upsilon \qquad (0 \leq t \leq T). \tag{2.7}$$

We describe the integral $\int_0^\infty a(u)\,du$ $(a(t) \varepsilon S)$ as strongly summable (N,p) with index $\lambda > 0$ to A, and write

$$\int_0^\infty a(u)\,du = A\ [N,p]_\lambda$$

if

$$\int_0^t | p(u) \| \frac{(p^*a)_u}{p(u)} - A |^\lambda\,du = o(p_1(t)). \tag{2.8}$$

We remark that p (t) should be assumed to satisfy (2.7) only in the case when $\lambda > 1$. For, if $\lambda > 1$ and if p (t) does not satisfy (2.7) (and even if we assume that $p(t) \neq 0$ for all t) then we might still have (for example) $p(u) \to 0$ as $u \to u_0$ and the integral in (2.8) might then diverge at u_0. We allow υ to depend on T, since we want to include the case of Cesàro summability (c,k) for which

$$p(t) = t^{k-1} \qquad (k > 0).$$

(d) Let p (t) satisfy (2.7). We say that $\int_0^\infty a(u)\,du$ is strongly bounded (N,p) with index $\lambda > 0$, if

$$\int_0^t | p(u) | \ | \frac{(p^*a)_u}{p(u)} |^\lambda\,du = 0(p_1(t)); \tag{2.9}$$

and we denote this by

$$\int_0^\infty a(u)\,du = 0\ [N,p]_\lambda.$$

(e) Let r (t) = $(p^*q)_t$, p (t), q (t) ε P. Then, if r (t) ε P, we call the Nörlund method (N,r) the symmetric product of (N, p) and (N,q) and write

$$(N, r) = (N, p) * (N, q).$$

(f) We say that the method E includes the method D if every function summable D is also summable E to the same sum and write D $\subseteq$ E.

Note. Whenever we shall be concerned with strong functional Nörlund summability, it will htroughout be assumed that the generating weight functions satisfy (2.7) and will not be stated explicitly.

3. *The Lemmas.* In order to prove our theorems we require the following lemmas.

Lemma 1. ([1, Theorem 2. 5]) . If

$$\int_0^t | p (u) | du = 0 (p_1(t)),$$

then

$$[N,p]_\lambda \subseteq [N,p]_\mu \text{ for } \lambda > \mu > 0.$$

In particular, conclusion holds if $\lambda > \mu > 0$ and (N,p) is positive.

Lemma 2. Let p (t) > 0 for all t, and (N,q) be positive and regular. If, for $\lambda \geq 1$,

$$\int_0^\infty a (u) du = 0 \ [N,p]_\lambda \text{ and } \int_0^\infty b (u) du = 0 \ (N,q)$$

then

$$\int_0^\infty c (u) du = 0 (N,r).$$

Proof. By Lemma 1, it suffices to prove the Lemma when $\lambda = 1$. Let

$$X(t) = \frac{(1^* |\varphi|)_t}{p_1(t)} \quad \text{where } \varphi(t) = (p^*a)_t,$$

$$Y(t) = \frac{(1^*q^*b)_t}{q_1(t)} \quad \text{and } W(t) = \frac{(1^*r^*c)_t}{r_1(t)}.$$

Since

$$(1^*r^*c)_t = (q_1 Y * \varphi)_t ,$$

therefore

$$| W(t) | \leq \frac{1}{r_1(t)} \int_0^t q_1(u) \, | Y(u) | \, | \varphi(t-u) | \, du.$$

Since, $Y(t) = 0\,(1)$ by hypothesis, therefore we can find a suitable constant $K(\pm)$, so that

$$| W(t) | \leq \frac{K}{r_1(t)} \int_0^t q(t-u) \left\{ \int_0^u | \varphi(v) | dv \right\} du$$

$$= \int_0^t a(t,u) \, X(u) \, du \tag{3.1}$$

where

$$a(t,u) = \frac{K\, q(t-u)\, p_1(u)}{r_1(t)} \text{ for } 0 \leq u \leq t \text{ and } = 0 \text{ for } u>t.$$

We assert that the transformation (3.1) with Kernel $a(t,u)$ is null-preserving. For, the norm-condition is clearly satisfied. The column-condition (2.5) requires that

$$\frac{1}{r_1(t)} \int_0^c q(t-u)\, p_1(u)\, du \to 0 \text{ as } t \to \infty \tag{3.2}$$

for every finite $c > 0$. Now, since

$$\int_0^c q(t-u)\, p_1(u)\, du \leq p_1(c) \, \{q_1(t) - q_1(t-c)\},$$

($\pm$) In what follows K, K_1 etc. will denote positive constants which may be different at each occurrence.

and

$$r_1(t) \geq \int_0^{t-c} p_1(t-u)\, q(u)\, du \geq p_1(c)\, q_1(t-c),$$

therefore the left side of (3.2) is

$$\leq \frac{q_1(t) - q_1(t-c)}{q_1(t-c)} \to 0 \text{ as } t \to \infty$$

by the regularity of (N,q); which proves our assertion that (3.1) is null-preserving. Hence, since X (t) = o (1) by hypothesis, (3.1) gives W (t) = o (1) and so

$$\int_0^{\infty} c(u)\, du = 0\ (N,r)$$

as required.

Lemma 3. Let p (t) > 0, q (t) > 0 for all t, and (N,q) be regular. If, for $\lambda \geq 1$,

$$\int_0^{\infty} a(u)\, du = 0\ [N,p]_\lambda \quad \text{and} \quad \int_0^{\infty} b(u)\, du = 0\ [N,q]_\lambda,$$

then

$$\int_0^{\infty} c(u)\, du = 0\ [N,r]_\lambda.$$

Proof. Write

$$F(t) = \frac{(p^*a)_t}{p(t)},\ G(t) = \frac{(q^*b)_t}{q(t)} \text{ and } H(t) = \frac{(r^*c)_t}{r(t)}.$$

Thus, we are given that

$$\xi(t) = \frac{1}{p_1(t)} \int_0^t p(u)\ |F(u)|^\lambda\, du = o(1), \tag{3.3}$$

$$\eta(t) = \frac{1}{q_1(t)} \int_0^t q(u)\ |G(u)|^\lambda\, du = 0(1), \tag{3.4}$$

and we have to show that

$$\frac{1}{r_1(t)} \int_0^t r(u) \; |H(u)|^\lambda \, du = o(1). \tag{3.5}$$

Since

$$r(t) \, H(t) = (pF * qG)_t,$$

therefore, by Hölder's inequality,

$$[r(t) \, |H(t)|]^\lambda \leq [\int_0^t p(u) q(t-u) \; |F(u)| \; |G(t-u)| du]^\lambda$$

$$\leq [\int_0^t p(u) \; q(t-u) \; du]^{\lambda-1} \; [\int_0^t p(u) \, q(t-u) \; |F(u)|^\lambda \; |G(t-u)|^\lambda du]$$

or

$$r(t) \; |H(t)|^\lambda \leq \int_0^t p(u) \; q(t-u) \; |F(u)|^\lambda \; |G(t-u)|^\lambda \; du.$$

Hence

$$\frac{1}{r_1(t)} \int_0^t r(u) \; |H(u)|^\lambda \; du \leq \frac{1}{r_1(t)} \int_0^t p(v) \; |F(v)|^\lambda \int_0^{t-v} q(w) \; |G(w)|^\lambda \; dw \; dv$$

$$= \frac{1}{r_1(t)} \int_0^t p(t-v) \; | \; F(t-v) \; |^\lambda \; q_1(v) \; \eta(v) \; dv$$

$$\leq \frac{K}{r_1(t)} \int_0^t p(t-v) \; |F(t-v)|^\lambda \; q_1(v) \; dv \quad (\text{by } (3.4))$$

$$= \frac{K}{r_1(t)} \int_0^t q(t-u) \; p_1(u) \; \xi(u) \; du \tag{3.6}$$

$$= o(1)$$

by (3.3) and the fact that the transformation defined by (3.6) is null-preserving (cf. the proof of Lemma 2). This establishes (3.5) and the proof is thus complete.

Lemma 4. ([3, Hilfssatz p. 51]). If (N,p) is positive and regular and if $\int_{t-1}^{t} s(u)\, du = s_1(t)$, then $s(t) \to s$ (N,p) implies $s_1(t) \to s$ (N,p).

Lemma 5. Suppose that $p(t) > 0$, $q(t) > 0$** for all t and (N,q) is regular. Then

(a) $(N,p) \subseteq (N,r)$

(b) $[N,p]_\lambda \subseteq [N,r]_\lambda$ for $\lambda \geq 1$.

Proof. The first part of the lemma is [4, Theorem 2]. To prove the second part, suppose that $\int_0^\infty a(u)\, du = A\ [N,p]_\lambda$.

Write

$$M(t) = \frac{(p^*a)_t}{p(t)} - A \text{ and } N(t) = \frac{(r^*a)_t}{r(t)} - A. \tag{3.7}$$

Thus, we are given that

$$\gamma(t) = \frac{1}{p_1(t)} \int_0^t p(u)\ |M(u)|^\lambda\, du = o(1)$$

and we are required to show

$$\frac{1}{r_1(t)} \int_0^t r(u)\ |N(u)|^\lambda\, du = o(1)\ . \tag{3.8}$$

Since

$$r(t)\ N(t) = (q^*\ pM)_t$$

therefore, using Hölder's inequality, we obtain

$$r(t)\ |N(t)|^\lambda \leq \int_0^t q(t-u)\ p(u)\ |M(u)|^\lambda\, du$$

(**) If we replace the assumption $q(t) > 0$ for all t by the weaker assumption $q(t) > 0$ for $0 \leq t \leq 1$ and $q(t) \geq 0$ for $t > 1$, even then the conclusion of the lemma holds.

Thus

$$\frac{1}{r_1(t)} \int_0^t r(u)\, |N(u)|^\lambda \, du \le \frac{1}{r_1(t)} \int_0^t q(v) \int_0^{t-v} p(w)\, |M(w)|^\lambda \, dw\, dv$$

$$= \frac{1}{r_1(t)} \int_0^t q(t-v)\, p_1(v)\, \gamma(v)\, dv \qquad (3.9)$$

$$= o(1)$$

since $\gamma(t) = o(1)$ and the transformation defined by (3.9) can easily be seen to be regular. This establishes (3.8) and the lemma is thus proved.

Lemma 6. Let $p(t) > 0$ for all t and (N,p) be regular. Define

$$\bar{a}(t) = \begin{cases} a(t) - A & \text{for } 0 \le t \le 1 \\ a(t) & \text{for } \quad t > 1. \end{cases} \qquad (3.10$$

If either $p(t)^* \uparrow$ or $p(t) \downarrow$, then

$$\int_0^\infty a(u)\, du = A \; [N,p]_1 \text{ implies } \int_0^\infty \bar{a}(u)\, du = 0 \; [N,p]_1.$$

Proof. By definition

$$(p^*\bar{a})_t = \begin{cases} (p^*a)_t - A \int_0^t p(t-u)du = (p^*a)_t - A\, p_1(t) \text{ for } 0 \le t \le 1 \\ (p^*a)_t - A \int_0^1 p(t-u)du = (p^*a)_t - A\, \{p_1(t) - p_1(t-1)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } t > 1. \end{cases}$$

Thus, for $t > 0$,

$$(p^*\bar{a})_t = (p^*a)_t - A\, \{p_1(t) - p_1(t-1)\} \qquad (3.11)$$

since $p_1(t) = 0$ for $t < 0$.

If we write

$$\bar{F}(t) = \frac{(p^*\bar{a})_t}{p(t)}$$

(*) We use the symbols $\uparrow$ and $\downarrow$ for non-decreasing and non-increasing respectively.

then (3.11) gives

$$p(t)\,\bar{F}(t) = p(t)\,M(t) + A\,[p(t) - \{p_1(t) - p_1(t-1)\}\,] \tag{3.12}$$

where M(t) is given by (3.7). Also, by hypothesis,

$$\int_0^t p(u)\ |M(u)|\ du = o\ (p_1(t)\ ). \tag{3.13}$$

(i) If $p\ (t)\ \uparrow$, then

$$p_1(t) - p_1(t-1) = \int_{t-1}^t p\ (u)\ du \geq p\ (t-1)$$

and so (3.12) gives

$$p(t)\ \bar{F}(t) \leq p(t)\ M(t) + A\,\{p(t) - p(t-1)\ \}.$$

Thus, using (3.13), we obtain

$$\int_0^t p(u)\ |\ \bar{F}(u)\ |\ du \leq o\ (p_1(t)\ ) + |A|\ \{p_1(t)-p_1(t-1)\ \}$$

$$= o\ (p_1(t)\ ) \tag{3.14}$$

since (N,p) is regular.

(ii) If $p\ (t) \downarrow$, then $\{p_1(t) - p_1(t-1)\ \} \geq p\ (t)$ and so the second term on the right side of (3.12) is ≤ 0. Thus, again using (3.13), we find that (3.14) holds. Thus, in any event, (3.14) holds and hence

$$\int_0^\infty \bar{a}\ (u)\ du = 0\ \ [N,p]_1$$

as required.

Lemma 7. Assume that $p\ (t) > 0$ for all t and (N,p) is regular. Define

$$\triangle\ a_1(t) = a_1(t) - a_1(t-1) \text{ where } a_1(t) = \int_0^t a\ (u)\ du. \tag{3.15}$$

If either $p\ (t)\ \uparrow$ or $p\ (t) \downarrow$, then

$$\int_0^\infty a(u)\ du = A\,[N,p]_1 \text{ implies } \int_0^\infty \triangle a_1(u)\ du = A\,[N,p]_1.$$

Proof. If we write

$$\bar{q}(t) = \begin{cases} 1 \text{ for } 0 \leq t \leq 1 \\ 0 \text{ for } \quad t > 1, \end{cases}$$

then

$$(\bar{q}*a)_t = \int_{t-1}^{t} a(u)\,du = \triangle a_1(t),$$

$$\bar{r}(t) = (p*\bar{q})_t = \int_{t-1}^{t} p(u)\,du = p_1(t) - p_1(t-1),$$

$$\bar{r}_1(t) = \int_0^t \bar{r}(u)\,du = \int_{t-1}^{t} p_1(u)\,du \tag{3.16}$$

and

$$(\bar{r}*a)_t = (p*\bar{q}*a)_t = (p*\triangle a_1)_t. \tag{3.17}$$

It follows from footnote (**) to Lemma 5 that $[N,p]_1 \subseteq [N,\bar{r}]_1$ and thus

$$\int_0^\infty a(u)\,du = A \; [N,\bar{r}]_1. \tag{3.18}$$

If we write

$$\bar{M}(t) = \frac{(p*\triangle a_1)_t}{p(t)} - A \text{ and } \bar{N}(t) = \frac{(\bar{r}*a)_t}{\bar{r}(t)} - A, \tag{3.19}$$

then (3.18) implies

$$\int_0^t \bar{r}(u)\,|\bar{N}(u)|\,du = o(\bar{r}_1(t)). \tag{3.20}$$

Now, because of (3.16), $\bar{r}_1(t) = p_1(t_0)$ for suitable t_0 between $t-1$ and t, therefore $r_1(t)$ is asymptotically equivalent to $p_1(t)$. So (3.20) becomes

$$\int_0^t \bar{r}(u)\,|\bar{N}(u)|\,du = o(p_1(t)). \tag{3.21}$$

Now, from (3.17) and (3.19), we have

$$p\ (t)\ \overline{M}\ (t) = (p^{*}\triangle a_1)_t - A\ p\ (t)$$
$$= \bar{r}\ (t)\ \overline{N}\ (t) + A\ [\{p_1(t)-p_1(t-1)\}-p(t)]. \qquad (3.22)$$

(i) If p (t) ↑, then (3.22) gives

$$\int_0^t p(u)\ |\overline{M}(u)|\ du \leq \int_0^t \bar{r}(u)\ |\overline{N}(u)|\ du = o\ (p_1(t)) \qquad (3.23)$$

by (3.21).

(ii) If p (t) ↓, then from (3.22) we have

$$\int_0^t p(u)\ |\overline{M}(u)|\ du \leq o\ (p_1(t)) + |A|\ \{p_1(t-1) - p_1(t)\}\ (\text{by } (3.21))$$
$$= o\ (p_1(t))$$

since (N,p) is regular.

Thus, in any case, (3.23) holds and hence

$$\int_0^{\infty} \triangle a_1(u)\ du = A\ [N,p]_1$$

as desired.

4. *Main Results.*

Theorem 1. If p (t) > 0 for all t, p (t) ↑, $\lambda \geq 1$,

$\int_0^{\infty} a(u)\ du = 0\ [N,p]_{\lambda}$ and $\int_0^{\infty} b(u)\ du$ is absolutely convergent, then

$$\int_0^{\infty} c(u)\ du = 0\ [N,p]_{\lambda}.$$

When $\lambda = 1$, the condition "p (t) ↑" may be dropped.

Proof. If we write

$$J\ (t) = \frac{(p^{*}c)_t}{p\ (t)},$$

then

$$p(t)\, J(t) = (p\, F^{*}b)_t.$$

Using Hölder'ş inequality, we find

$$\{p(t)\ |J(t)|\}^{\lambda} \leq \left\{\int_0^t p(u)\ |b(t-u)|\ |F(u)|^{\lambda}\, du\right\} \left\{\int_0^t p(u)\ |b(t-u)|\ du\right\}^{\lambda-1}. \quad (4.1)$$

Since by hypothesis $p(t) \uparrow$ and $\int_0^{\infty} b(u)\, du$ is absolutely convergent, therefore (4.1) gives

$$p(t)\ |J(t)|^{\lambda} \leq K \int_0^t p(u)\ |b(t-u)|\ |F(u)|^{\lambda}\, du.$$

When $\lambda = 1$, the second term on the right side of (4.1) does not appear and so we need not assume that $p(t) \uparrow$.

Now

$$\int_0^t p(u)\ |J(u)|^{\lambda}\, du \leq K \int_0^t p(v)\ |F(v)|^{\lambda} \int_0^{t-v} |b(w)|\, dw\, dv$$

$$\leq K_1 \int_0^t p(v)\ |F(v)|^{\lambda}\, dv, \quad (4.2)$$

since $\int_0^{\infty} b(u)\, du$ is absolutely convergent. By hypothesis, the right side of (4.2) is $= o(p_1(t))$, and hence

$$\int_0^t p(u)\ |J(u)|^{\lambda}\, du = o(p_1(t))$$

so

$$\int_0^{\infty} c(u) = 0\ [N,p]_{\lambda}$$

as required.

Theorem 2. Assume that $p(t) > 0$, $q(t) > 0$ for all t, (N,p) and (N,q) regular, and either $p(t) \uparrow$ or $p(t) \downarrow$. If, for $\lambda \geq 1$,

$$\int_0^\infty a(u)\, du = A\ [N,p]_\lambda \text{ and } \int_0^\infty b(u)\, du = B\ (N,q)$$

then

$$\int_0^\infty c(u)\, du = AB\ (N,r).$$

Proof. By Lemma 1, it suffices to prove the Theorem for the case $\lambda = 1$. If $A = 0$, the result is an immediate consequence of Lemma 2. Suppose $A \neq 0$. Define $\bar{a}$ (t) by (3.10). Let

$$\begin{aligned} \bar{c}\ (t) &= (\bar{a}*b)_t \\ &= c\ (t) - A\ \{b_1(t) - b_1(t-1)\}, \end{aligned} \tag{4.3}$$

where

$$b_1(t) = \int_0^t b\ (u)\ du.$$

Now another application of Lemma 2 yields

$$\int_0^\infty \bar{c}\ (u)\ du = 0\ (N,r) \tag{4.4}$$

since, by Lemma 6,

$$\int_0^\infty \bar{a}\ (u)\ du = 0\ [N,p]_1. \tag{4.5}$$

Further, since $(N,q) \subseteq (N,r)$ by Lemma 5 (a) (with p,q interchanged),

$$\int_0^\infty b\ (u)\ du = B\ (N,r).$$

Now, since $b_1(t) \to B\ (N,r)$, we have

$$\begin{aligned} \int_0^t \{b_1(u)-b_1(u-1)\}\ du &= \int_{t-1}^t b_1(u)\ du \\ &\to B\ (N,r) \end{aligned}$$

by Lemma 4. In other words

$$\int_0^\infty \{b_1(u) - b_1(u-1)\} \, du = B \ (N,r). \tag{4.6}$$

Hence, since

$$c\ (t) = \bar{c}\ (t) + A\ \{b_1(t) - b_1(t-1)\},$$

it follows from (4.4) and (4.6) that

$$\int_0^\infty c\ (u)\ du = AB\ (N,r)$$

which completes the proof.

Theorem 3. Suppose that p (t) > 0, q (t) > 0 for all t, and (N,p) and (N,q) are regular. Further suppose that either p (t) $\uparrow$ or p (t) $\downarrow$ and also that either q (t) $\uparrow$ or q (t) $\downarrow$. If

$$\int_0^\infty a(u)\ du = A\ [N,p]_1 \text{ and } \int_0^\infty b(u)\ du = B\ [N,q]_1$$

then

$$\int_0^\infty c(u)\ du = AB\ [N,r]_1.$$

Proof. If A = B = 0, the result follows from Lemma 3. Suppose that $A \neq 0$, $B = 0$. Define $\bar{a}$ (t) and $\bar{c}$ (t) as in (3.10) and (4.3) respectively. Again, by Lemma 3, since (4.5) holds, we have

$$\int_0^\infty \bar{c}(u)\ du = 0\ [N,r]_1. \tag{4.7}$$

Since, by hypothesis and Lemma 5 (b) (with $\lambda = 1$ and p,q interchanged),

$$\int_0^\infty b(u)\ du = 0\ [N,r]_1,$$

therefore, from Lemma 7 (with a_1 and p replaced by b_1 and q respectively) it follows that

$$\int_0^\infty \{b_1(u) - b_1(u-1)\}\, du = 0 \quad [N,r]_1. \tag{4.8}$$

Hence, since

$$c(t) = \bar{c}(t) + A\ \{b_1(t) - b_1(t-1)\},$$

from (4.7) and (4.8), we obtain

$$\int_0^\infty c(u)\, du = 0 \quad [N,r]_1$$

as desired.

Finally, when $A \neq 0$, $B \neq 0$, we define $\bar{b}(t)$ similar to $\bar{a}(t)$ and a similar argument yields

$$\int_0^\infty c(u)\, du = AB \quad [N,r]_1.$$

This completes the proof.

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