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Vekua's method for solving the equation

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by

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13

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and a Cauchy problem for an ultrahyperbolic equation*

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SUMMARY

In this article a Riemann function associated with the equation $\Delta_\Sigma u = 0$ is established by using a suitable transformation. Also the solution of a Cauchy problem for a class of ultrahyperbolic equations is obtained by a special method of the mean value of a function on the spheres when the parameters are integers.

0. INTRODUCTION

This paper is composed of two sections. In the first section we will establish a Riemann function associated with the equation

$$\Delta_\Sigma u = \sum_{j=1}^{n} \left(\frac{\partial^2 u}{\partial x^2_j} + \frac{\partial^2 u}{\partial x^2_{j+n}} + \frac{k_j}{x_{j+n}} \frac{\partial u}{\partial x_{j+n}} \right) = 0 \qquad (1)$$

In the second section we will discuss the solutions of a Cauchy problem for an equation of ultrahyperbolic type.

I. RIEMANN FUNCTION ASSOCIATED WITH $\Delta_\Sigma u = 0$.

We are already familiar with the method defined by Vekua [1] in investigations of the differential equations of elliptic type.

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In this section we will assume that k_j are integers. Let n_j be the corresponding integer to the parameter k_j. The solution of the problem will be investigated for each of the cases $k_j = n_j - 1$, $k_j > n_j - 1$ and $k_j < n_j - 1$.

a) The case of $k_j = n - 1,\ j = 1, 2, \ldots, m$

Let us assume that the function f, given by the initial conditions, is twice continuously differentiable where it is defined. Now let us define the mean value of f as

$$M(x, t; f) = \left(\prod_{j=1}^{m} \omega_{n_j}\right)^{-1} \int\int_{\alpha^2_1=1} \ldots \int\ldots\int_{\alpha^2_m=1} f(x_{11} + \alpha_{11} t_1, x_{21} + \alpha_{21} t_1, \ldots, x_{n_1} + \alpha_{n_1} t_1, x_{12} + \alpha_{12} t_2, \ldots, x_{n_m m} + \alpha_{n_m m} t_m)\, d\omega_{n_1} \ldots d\omega_{n_m} \tag{8}$$

where

$$\alpha^2_j = \sum_{i=1}^{n_i} \alpha^2_{ij}$$

and ω_{n_j} is the surface area of n_j – dimensional unit sphere, $d\omega_{n_j}$ is the surface element of n_j–dimensional unit sphere. Taking into account the paper by Weinstein [3], it can easily be verified that the function M (x, t; f) is a solution of the equation (6) with the initial conditions (7); that is,

$$u\ \{k_1, \ldots, k_m\} = M(x, t; f)$$

b) The case of $k_j > n_j - 1$

If some of the parameters satisfy $k_j > n_j - 1$ then modifying the mean value (8) by the method of descent provides us the solution of the initial value problem defined by (6) and (7). Just for the simplicity let us assume that $k_m > n_m - 1$. The integral

$$\int\ldots\int_{\alpha^2_m=1} f\, d\omega_{n_m}$$

which is related with k_m can be transformed into

$$\int\ldots\int_{\alpha^2_m \leq 1} \omega_{k_m - n_m + 1} (1 - \alpha^2_m)^{(k_m - n_m - 1)/2} f\, d\tau_{n_m}$$

by using Hadamard's method as in [3]. The equation (8) is brought into the form of

$$M(x, t; f) = (\omega_{k_m - n_m + 1} \prod_{j=1}^{m} \omega^{-1} k_j + 1).$$

$$\underset{\alpha_1^2 = 1}{\int \ldots \int} \ldots \underset{\alpha^2_{m-1} = 1}{\int \ldots \int} \underset{\alpha_m^2 \leq 1}{\int \ldots \int} (1 - \alpha_m^2)^{(k_m - n_m - 1)/2} f \, d\tau_{n_m} \, d\omega_{n_{m-1}} \ldots d\omega_{n_1}$$

This is the solution of the initial value problem for $k_m > n_m - 1$.

c) The case of $k_j < n_j - 1$ and $k_j \neq -1, -3, -5, \ldots$

The relations

$$u\{k_1, \ldots, k_m\} = u\{k_1, \ldots, k_{l-i}, 2 - k_l, \ldots, 2 - k_m\} \prod_{j=1}^{m} t_j^{1-k_j} \quad (9)$$

and

$$\frac{\partial}{\partial t_j} u\{k_1, \ldots, k_m\} = t_j^{-1} u\{k_1, \ldots, k_{j-1}, k_j + 2, k_{j+1}, \ldots, k_m\} \quad (10)$$

were obtained before [4,5] for Σ - harmonic functions. It is easy to show that (9) and (10) are also valid for the solutions of Equation (6). And the case of $k_j < n_j - 1$ can be investigated with the aid of these relations.

For the sake of simplicity we will assume that the only parameter satisfying the inequality $k_j < n_j - 1$ is k_m. Now let us choose the smallest integer β satisfying the inequality $k_m + 2\beta \geq n_m - 1$, and consider the equation (6) substituted $k_m + 2\beta$ for k_m:

$$\sum_{j=1}^{m} \sum_{i=1}^{n_j} \frac{\partial^2 u}{\partial x^2_{ij}} = \sum_{i=1}^{m-1} \left(\frac{\partial^2 u}{\partial t_i^2} + \frac{k_i}{t_i} \frac{\partial u}{\partial t_i} \right) + \frac{\partial^2 u}{\partial t^2_m} + \frac{k_m + 2\beta}{t_m} \frac{\partial u}{\partial t_m}$$

We already know by the cases of (a) and (b) that the solution for the above equation with the initial conditions

$$u(x_{12}, x_{21}, \ldots, x_{n_m m}, 0, 0, \ldots, 0) = \frac{f(x_{11}, x_{21}, \ldots, x_{n_m m})}{(k_m + 1)(k_m + 3) \ldots (k_m + 2\beta - 1)}$$

$$\frac{\partial}{\partial t_i} u(x_{11}, x_{12}, \ldots, x_{n_m m}, 0, 0, \ldots, 0) = 0$$

can be obtained and it is represented by

$$u\{k_1, \ldots, k_{m-1}, k_m + 2\beta\}$$

where f has continuous derivatives upto the order $(n_m - k_m + 3)/2$. Applying the correspondence principle (9) to this solution, we get

$$t_m^{k_m+2\beta-1}\, u\{k_1,\ldots, k_{m-1}, k_m+2\beta\} = u\{k_1,\ldots, k_{m-1}, 2-k_m-2\beta\} \quad (11)$$

Now let us apply (10) successively β times to (11):

$$\left(\frac{\partial}{t_m \partial t_m}\right)^{\beta} t_m^{k_m+2\beta-1}\, u\, \{k_1,\ldots, k_{m-1}, k_m+2\beta\} = u\, \{k_1,\ldots, k_{m-1}, 2 - k_m\}$$

Using (9), we obtain from this result

$$u\, \{k_1,\ldots, k_m\} = t_m^{1-k_m} \left(\frac{\partial}{t_m \partial t_m}\right)^{\beta} t_m^{k_m+2\beta-1}\, u\, \{k_1,\ldots, k_{m-1}, k_m + 2\beta\}$$

So this is the solution of the initial value problem defined by (6) and (7), in the case of $k_m < n_m - 1$.

Remark: The exceptional cases of $k_j = -1, -3, \ldots$ can be investigated, for example, as in Diaz and Weinberger [6].

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ÖZET

Bu yazının birinci kısmında uygun bir dönüşüm kullanılarak $\Delta_{\Sigma}\, u = 0$ denklemi ile ilgili bir Riemann fonksiyonu elde edilmiştir. İkinci kısmında ise parametrelerin tam sayılar olması halinde, ultrahiperbolik bir diferensiyel denklem sınıfı için Cauchy probleminin çözümü, fonksiyonların çeşitli küreler üzerindeki ortalama değerlerinden yararlanarak, bulunmuştur.

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