

COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A₁: Mathématiques

TOME 29

ANNÉE 1980

**Linear Restrictions on Parameters in Linear Regression Model
With non-spherical Disturbances**

by

FİKRİ AKDENİZ

3

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences de l'Université d'Ankara

Comité de Rédaction de la Série A₁

B. Yurtsever H. Hacısalihođlu M. Oruđ

Secrétaire de publication

Ö. Çakar

La Revue "Communications de la Faculté des Sciences de l'Université d'Ankara" est un organe de publication englobant toutes les disciplines scientifiques représentées à la Faculté.

La Revue, jusqu'à 1975 à l'exception des tomes I, II, III, était composée de trois séries:

Série A: Mathématiques, Physique et Astronomie.

Série B: Chimie.

Série C: Sciences naturelles.

A partir de 1975 la Revue comprend sept séries:

Série A₁: Mathématiques

Série A₂: Physique

Série A₃: Astronomie

Série B: Chimie

Série C₁: Géologie

Série C₂: Botanique

Série C₃: Zoologie

En principe, la Revue est réservée aux mémoires originaux des membres de la Faculté. Elle accepte cependant, dans la mesure de la place disponible, les communications des auteurs étrangers. Les langues allemande, anglaise et française sont admises indifféremment. Les articles devront être accompagnés d'un bref sommaire en langue turque.

Adres: Fen Fakóltesi Tebliđler Dergisi Fen Fakóltesi, Ankara, Turquie.

Linear Restrictions on Parameters in Linear Regression Model With non-spherical Disturbances

FIKRİ AKDENİZ

University of Ankara, Faculty of Science
Department of Mathematics
Ankara, Turkey

(Received, June 10, 1980, accepted September, 24, 1980)

ABSTRACT

In this paper we are concerned with the estimation of parameters subject to linear constraints. No rank conditions are imposed on R, X and V . The results are applied to obtain the generalized inverse of $X'V+X$ which yields a solution of the normal equations subject to nonestimable constraints on the parameters.

1. INTRODUCTION

Consider the linear model

$$Y = X \beta + u, u \sim N(0, \sigma^2 V) \quad (1.1)$$

where Y is $N \times 1$, X an $N \times k$ matrix of fixed quantities, and β a $k \times 1$ vector of unknown parameters, u is an $N \times 1$ vector of jointly normal disturbances with mean vector zero and nonnegative definite covariance matrix V . Further suppose that one is contemplating appending the model with $m < k$ independent linear restrictions on β ,

$$R \beta = r \quad (1.2)$$

where R is an $m \times k$ matrix of known constants, and a r known $k \times 1$ vector. The equations $R \beta = r$ are assumed to have at least one solution in β .

2. MATRIX RESULTS

In this section we establish some of the matrix results, we shall use in the sequel. In particular, we shall establish those properties of the generalized inverse. We write $A^-(n \times m)$ to denote

a generalized inverse or g-inverse of a matrix A ($m \times n$), that is, any matrix A^- satisfying $A A^- A = A$ and we write A^+ to denote the Moore-Penrose inverse, defined as the unique matrix G satisfying each of the following four conditions:

$$A G A = A \quad (1)$$

$$G A G = G \quad (2)$$

$$(AG)' = AG \quad (3)$$

$$(GA)' = GA \quad (4)$$

The following notation and results will be used throughout the paper.

$$Q_A = I - A^+A$$

$$(A^+)^+ = (A^+)'$$

$$(A'A)^+ = A^+(A^+)'$$

$$A'(AA^+)^+ = (A'A)^+A' = A^+$$

$$A (A'A)^+A'A = A$$

$R R^+r = r$ provided $R \beta = r$ are consistent.

$$(XQ_R)' = Q_R X'$$

$$Q_R [Q_R X' V^{-1} X Q_R]^+ Q_R = (Q_R X' V^{-1} Q_R)^+$$

If P is orthogonal matrix, then $P^+ = P'$.

3. THE BEST LINEAR ESTIMATE OF β WITH A POSITIVE DEFINITE V .

Now, consider the general linear model (1.1) with β subjected to the constraint (1.2). It is well known that every β satisfying (1.2) is specified by

$$\beta = R^+r + (I - R^+R) w \quad (3.1)$$

where w is an arbitrary vector of appropriate dimensions. Substituting (3.1) into (1.1) for β , we obtain

$$Y = X R^+r + X (I - R^+R) w + u$$

or

$$Y - X R^+r = X Q_R w + u. \quad (3.2)$$

Therefore, the best linear estimate (BLE) of w in (3.2) is given by

$$\hat{w} = [(XQ_R)'V^{-1}XQ_R]^+ (XQ_R)'V^{-1} (Y - X R+r). \tag{3.3}$$

Substituting (3.3) into (3.1) for \hat{w} , we get

$$\tilde{\beta} = R+r + Q_R \hat{w}$$

and

$$\tilde{\beta} = R+r + Q_R [Q_R X' V^{-1} X Q_R]^+ Q_R X' V^{-1} (Y - X R+r)$$

or

$$\tilde{\beta} = R+r + (\tilde{X} Q_R)^+ (\tilde{Y} - \tilde{X} R+r). \tag{3.4}$$

where $\tilde{X} = V^{-1/2} X$ and $\tilde{Y} = V^{-1/2} Y$.

The problem we consider is the estimation of linear function of β by means of linear functions of Y . We use the following preliminary results.

LEMMA 3.1 There is a $\bar{\beta}$ of the form

$$\bar{\beta} = AY + c$$

such that $E(p'\bar{\beta}) = p'\beta$ for satisfying the consistent equations $R\beta = r$ if and only if there are vectors δ and ρ such that $p' = \delta'X + \rho'R$.

Proof: The proof of this lemma is given by Rao and Mitra (1971) and Gerig and Gallant (1975).

COROLLARY 3.1 Consider the model (1.1) subject to the linear equality constraints $R\beta = r$. The best linear unbiased estimator of $p'\beta$ is $p'\beta = p'R+r + p'(\tilde{X} Q_R)^+ (\tilde{Y} - \tilde{X}R+r)$, where

$$p' = \delta'X + \rho'R.$$

4. SQUARE ROOTS OF POSITIVE SEMI-DEFINITE MATRICES

Consider the positive semi-definite matrix C . Let it be diagonalized as follows: $T'CT = D$, where T is orthogonal, Graybill (1969 p. 19). Any matrix B such that $B^2 = C$ will be called a

square root of C . Since $C = T D^{1/2} D^{1/2} T' = T D^{1/2} T' T D^{1/2} T'$, obviously $T D^{1/2} T'$ will be such a square root, where $D^{1/2}$ is a diagonal matrix having the square roots of the corresponding diagonal elements of D as diagonal elements. Let C^+ be the Moore-Penrose inverse of C . Then C^+ can be written as $T D^+ T'$, where D^+ is a diagonal matrix having the reciprocal values of the corresponding nonzero diagonal elements of D and zeros as diagonal elements. The matrix $(C^{1/2})^+ = T (D^+)^{1/2} T'$ will then be a square root of C^+ . Since

$$(C^{1/2})^+ = (T D^{1/2} T')^+ = T (D^{1/2})^+ T' = T (D^+)^{1/2} T'$$

we find that $(C^{1/2})^+ = (C^+)^{1/2}$. When C is positive definite we have

$$C^{1/2} = T D^{1/2} T'$$

and

$$C^{-1/2} = T D^{-1/2} T'.$$

5. SOLVING NORMAL EQUATIONS SUBJECT TO NON ESTIMABLE CONSTRAINTS

In this section, when $\sigma^2 V (\neq \sigma^2 I)$ is non-negative matrix, we would like to extend the theorem, given by Gerig and Gallant (1975)

Consider the problem of solving the normal equations

$$X'V^{-1}X\beta = X'V^{-1}Y \quad (\text{or } X'V+X\beta = X'V+Y)$$

subject to the non-estimable parametric constraints

$$R\beta = 0.$$

By non-estimable, we mean that there do not exist non-zero vectors δ and ρ such that

$$\delta'V^{-1/2}X = \rho'R, \quad (\text{or } \delta'(V^{1/2})^+X = \delta'R)$$

Our solution consists of showing that

$C_1 = (Q_R X'V^{-1}X Q_R)^+$ and $C_2 = (Q_R X'V+X Q_R)^+$ are generalized inverses of $X'V^{-1}X$ and $X'V+X$ respectively.

THEOREM 5.1 If there do not exist non-zero vectors δ and ρ such that $\delta'(V^{1/2})^+X = \rho'R$, then

$$C_2 = (Q_R X'V+X Q_R)^+$$

is a generalized inverse of $X'V+X$.

Proof: Let

$$U = \begin{bmatrix} I & & - (V^{1/2})+XR^+ \\ \dots & \dots & \dots \\ 0 & & I \end{bmatrix}, Z = \begin{bmatrix} (V^{1/2})+XQ_R \\ \dots \\ R \end{bmatrix} = U \begin{bmatrix} (V^{1/2})+X \\ \dots \\ R \end{bmatrix}$$

Since U is non-singular, the ranks of Z and $\begin{bmatrix} (V^{1/2})+X \\ \dots \\ R \end{bmatrix}$ are

the same. Since the linear spaces generated by the rows of $(V^{1/2})+X$ and R are disjoint by hypothesis we have the

$$\text{rank} \begin{bmatrix} (V^{1/2})+X \\ \dots \\ R \end{bmatrix} = \text{rank} [(V^{1/2})+X] + \text{rank} (R). \text{ Consider}$$

$$\begin{bmatrix} \tilde{X} \\ \dots \\ R \end{bmatrix} (Q_R : R') = \begin{bmatrix} \tilde{X} Q_R : \tilde{X} R' \\ \dots \\ 0 : R R' \end{bmatrix},$$

where $\tilde{X} = (V^{1/2})+X$. Since $(Q_R : R')$ and U have full rank, we write

$$\begin{aligned} \text{rank} (Z) &= \text{rank} \begin{bmatrix} \tilde{X} \\ R \end{bmatrix} = \text{rank} \begin{bmatrix} \tilde{X} Q_R : \tilde{X} R' \\ \dots \\ 0 : R R' \end{bmatrix}, \\ &= \text{rank} \left\{ U \cdot \begin{bmatrix} \tilde{X} Q_R : \tilde{X} R' \\ \dots \\ 0 : R R' \end{bmatrix} \right\} = \text{rank} \begin{bmatrix} \tilde{X} Q_R : 0 \\ \dots \\ 0 : RR' \end{bmatrix}, \\ &= \text{rank} (\tilde{X} Q_R) + \text{rank} (R R'), \\ &= \text{rank} \{ (V^{1/2})+XQ_R \} + \text{rank} (R). \end{aligned}$$

Thus, $\text{rank} \{ (V^{1/2})+XQ_R \} = \text{rank} \{ (V^{1/2})+X \}$. Since the columns of $(V^{1/2})+XQ_R$ are linear combinations of the columns of $(V^{1/2})+X$ and the ranks of the two matrices are equal, there is a non-singular matrix S such that $(V^{1/2})+XQ_R = (V^{1/2})+X S$. Thus, we can write

$$X'V+X (Q_R X'V+XQ_R)+X'V+X = (S')^{-1}S'X'(V^{1/2})+(V^{1/2})+X [Q_R X'V+XQ_R]^+ \cdot X'(V^{1/2})+(V^{1/2})+X S.S^{-1},$$

$$\begin{aligned}
&= (S')^{-1} Q_R X' V + X Q_R [Q_R X' V + X Q_R] + Q_R X' V + X Q_R S^{-1} \\
&= (S')^{-1} Q_R X' V + X Q_R S^{-1}, \\
&= (S')^{-1} Q_R X' (V^{1/2})' + (V^{1/2}) + X Q_R S^{-1}, \\
&= X' (V^{1/2})' + (V^{1/2}) + X, \\
&= X' V + X,
\end{aligned}$$

and hence the theorem is proved.

As an example of the use of the theorem 5.1 in applications, consider a complete randomized block ^{design} with two treatments and two blocks:

$$\begin{bmatrix} w_{11} \\ w_{12} \\ w_{21} \\ w_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{bmatrix}$$

or in matrix notations.

$$W = N \beta + e, \quad e \sim N(0, \sigma^2 I). \quad (5.1)$$

To incorporate the standard assumptions $\sum \alpha_i = \sum \beta_j = 0$, put

$$R = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Premultiplication of (5.1) by

$$M = \begin{bmatrix} I_4 \\ j'_4 \end{bmatrix}$$

we get the following new model (see also Searle (1971))

$$M W = M N \beta + M e,$$

or

$$Y = X \beta + u, \quad (5.2)$$

where $X = M N$, $Y = M W$, $u = M e$, I_4 is a 4x4 identity matrix, j_4 a 4x1 column vector of unit elements, and $u \sim N(0, \sigma^2 V)$. Therefore, V can be written as

$$V = \begin{bmatrix} I_4 & j_4 \\ j'_4 & 4 \end{bmatrix} = M M'.$$

Since rank $(V) = 4$, it is seen that V is a singular variance covariance matrix. Since V symmetric matrix, there exists an orthogonal matrix P such that $P'V P = D$, where D is a diagonal matrix with the characteristic roots of V displayed on the diagonal of D . Since the characteristic roots of V are 1, 1, 1, 5, 0; then we obtain

$$P = \begin{bmatrix} 1/\sqrt{12} & -2/\sqrt{6} & 0 & 1/\sqrt{20} & 1/\sqrt{5} \\ 1/\sqrt{12} & 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{20} & 1/\sqrt{5} \\ 1/\sqrt{12} & 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{20} & 1/\sqrt{5} \\ -3/\sqrt{12} & 0 & 0 & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & 0 & 0 & 4/\sqrt{20} & -1/\sqrt{5} \end{bmatrix}$$

Hence, from section 4, we get

$$(V^{1/2})^+ = \begin{bmatrix} I_4 + \left(\frac{1}{10\sqrt{20}} - \frac{1}{4} \right) J_4 & \frac{4}{10\sqrt{20}} j_4 \\ \frac{4}{10\sqrt{20}} j'_4 & \frac{16}{10\sqrt{20}} \end{bmatrix}$$

and

$$V^+ = \begin{bmatrix} I_4 - \frac{24}{100} J_4 & \frac{4}{100} j_4 \\ \frac{4}{100} j'_4 & \frac{16}{100} \end{bmatrix}$$

where J_4 is a 4x4 matrix of unit elements. The Moore-Penrose g -inverse of R is

$$R^+ = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}$$

As seen in the theorem 5.1, rank $\{(V^{1/2})^+XQ_R\} = \text{rank}\{(V^{1/2})^+X\} = 3$. Furthermore,

$$\begin{aligned} Q_R X'V^+X Q_R &= Q_R (MN)'V^+(MN) Q_R \\ &= Q_R N'M'(MM')^+M N Q_R \\ &= Q_R N'M'(M')^+M^+M N Q_R \\ &= Q_R N'(M^+M)'M^+M N Q_R \\ &= Q_R N'N Q_R, \end{aligned}$$

or clearly

$$Q_R X'V^+X Q_R = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

where $M'V^+M = I_4$ and $M^+M = I_4$.

Thus

$$\begin{aligned} C_2 = (Q_R X'V^+X Q_R)^+ &= \begin{bmatrix} 1/4 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & -1/4 & 0 & 0 \\ 0 & -1/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & -1/4 \\ 0 & 0 & 0 & -1/4 & 1/4 \end{bmatrix} \\ &= (X'V^+X)^+. \end{aligned}$$

On the other hand, since

$$\begin{aligned} X'(I - V V^+) &= N'M'(I - MM'(MM')^+) \\ &= N'(M' - M'MM'(M')^+M^+) \\ &= N'(M' - M') = 0, \end{aligned}$$

we say that, X is contained in the column space of V .

If X is contained in the column space of V , the BLUE of $p_1'\beta$ is given by $p_1'\bar{\beta}$, where $p_1' \bar{\beta} = \delta'(V^{1/2})^+X + \rho'R$ and $\bar{\beta}$ is given in the following theorem.

THEOREM 5.2 (Hallum, Lewis and Boullion (1973)) The best linear estimate of β in (1.1) subject to (1.2) with X, R , and V assumed to be arbitrary ranks, is given by $\bar{\beta}$, where

$$\bar{\beta} = R^+r + (A_1 + A_2)(Y - X R^+r)$$

with

$$A_1 = [(Q_R \ X'V+X \ Q_R)] +X'V+,$$

$$A_2 = [Q_R \ X'(I - V V^+) \ X \ Q_R] +X'(I - V V^+).$$

COROLLARY 5.2 In the model $Y = X\beta + u$, $u \sim N(O; \sigma^2I)$, wherein the only restrictions on β are equations $R\beta = r$, the BLUE of a parametric function $p'_2\tilde{\beta} = p'_2R+r + p'_2(X Q_R)^+ (Y - X R+r)$, where $p'_2 = \delta'X + \rho'R$. This result is fairly well known, see for example Gerig and Gallant (1975) and Baksalary and Kala (1979).

ACKNOWLEDGMENT. I wish to express my appreciation to Dr. C. Yapar for assistance with computations.

REFERENCES

- 1- Baksalary, J.K. and Kala, R. (1979) Best Linear Unbiased Estimation in the Restricted General Linear Model, Math. Operationsforsch. Statist., Ser. Statistics. Vol. 10,27-35.
- 2- Gerig, T.M. and Gallant, A.R. (1975) Computing Methods for Linear Models Subject to linear Parametric Constraints, J. Statist. Comput. Simul. Vol 3, 283-296.
- 3- Graybill, F.A., (1969) "Introduction to Matrices with Applications in Statistics" Wadsworth Publishing Co. Belmont, California.
- 4- Hallum, C.R., Lewis, T.O. and Boullion, T.L., (1973) Estimation in the Restricted General Linear Model with a Positive Semidefinite Covariance Matrix, Communication in Statistics 1 (2), 157-166.
- 5- Rao, C.R., and S.K. Mitra (1971): "Generalized Inverse of Matrices and Its Applications, "John Wiley and Sons, Inc., New York.
- 6- Searle, S.R., (1971) "Linear Models" John Wiley and Sons, Inc. New York.

ÖZET

Bu çalışmada lineer kısıtlamalar altında parametrelerin kestirimi incelenmiştir. R , X ve V üzerinde rank kısıtlaması yapılmamıştır. Elde edilen bulgulara dayanarak parametreler üzerindeki kestirilebilir olmayan kısıtlamalar altında normal denklemlerin çözümünü veren $X'V+X$ nin geliştirilmiş tersi bulunmuştur.

Prix de l'abonnement annuel

Turquie : 15 TL; Étranger: 30 TL.

Prix de ce numéro : 5 TL (pour la vente en Turquie).

Prière de s'adresser pour l'abonnement à : Fen Fakültesi
Dekanlığı Ankara, Turquie.