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On The Geometry of Motion In The Euclidean n-Space

by

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On The Geometry of Motion In The Enclidean n-Space

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ABSTRACT:

In one parameter motion,

$$x = A\,x_0 + C,\ A \in SO\ (n),$$

of Euclidean n-spaces we find a geometrical meaning for the rank $\ddot{A}$ to be n or n-1. Thus we give the geometrical discussions of the 2^{nd} order pole points, pole curves and the axoids in these cases.

I. Introduction.

An one parameter motion of a body in Euclidean n-space is generated by the transformation

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ 1 \end{bmatrix}$$

where $A \in SO(n)$ and x_0 , x , C are nxl real matrices and $SO(n) = \{ A \in O(n):\ \det\ A = 1\}$, $O(n) = \{A \varepsilon | R^n{}_n :\ A = A^{-1}\}$. A and C are C^∞ functions of a real parameter t ; x_0 and x correspond to the position vectors of the same point X, with respect to the orthonormal coordinate systems of the moving space R_0 and the fixed spaçe R, respectively. At the initial time $t = t_0$ we consider the coordinate systems of R_0 and R are coincident.

In Euclidean n-space, H.R. Müller [1] gives a treatment of the 1^{st} order pole points and pole curves of the motion. The same kind of treatment about the higher order pole points and the pole curves depends on the ranks of the derivative matrices $\ddot{A}$, $\dddot{A}$,..., $A^{(v)}$ of A, where(.) indicates $\frac{d}{dt}$. Since we do not know anything about the ranks of these matrices it is not easy to give the geometry of the higher order pole points and pole curves.

In this paper we find a geometrical meaning for the rank $\ddot{A}$ to be n or n–1. Then we obtain, in these cases, the geometry of the 2^{nd} order pole points and pole curves in the manner of Müller [1].

II. *The Ranks of $\dot{A}$ and $\ddot{A}$.*

In the motion represented by (1) since $A \in SO(n)$, the relation

$$A^T A = A\ A^T = I \tag{2}$$

holds identically with respect to t. Hence we can give the following theorem.

Theorem 2.1:

Let $A \in SO(n)$ and n be an odd number. Then rank $\dot{A}$ is an even number.

Proof: Differentiating (2), with respect to t, we have

$$\dot{A}^T A + A^T \dot{A} = 0. \tag{3}$$

If we write

$$\omega = A^T \dot{A} \tag{4}$$

(3) reduces to

$$\omega^T + \omega = 0$$

which shows that ω is a nxn skew matrix. On the otherhand since n is an odd number we have

$$\det \omega = 0.$$

This means that rank $\omega \leq n-1$. If rank $\omega = r$, since ω is a skew matrix all of its r^{th} order minors are rxr skew matrices and so r must be an even number. Since

$$\det\omega = \det A^T.\ \det \dot{A} = 0 \text{ and } \det A^T = \det A = 1$$

we have $\det \dot{A} = 0$ and so

$$\text{rank } \omega = \text{rank } \dot{A}$$

or

$$\text{rank } \dot{A} = r. \blacksquare$$

Theorem 2.2:

Let $A \in SO(n)$. *Then*

$$\textit{rank}\ \ddot{A} = 0 \Leftrightarrow \textit{rank}\ \dot{A} = 0.$$

Proof: By differentiation, with respect to t, (3) gives us

$$\ddot{A}^T A + 2\,\dot{A}^T\dot{A} + A^T\ddot{A} = 0. \tag{5}$$

Since rank $\ddot{A} = 0 \quad \ddot{A} = 0$ and (5) reduces to

$$\dot{A}^T\dot{A} = O. \tag{6}$$

$\forall x \in R$, from (6) we may write that

$$x^T\dot{A}^T\dot{A}\,x = 0$$

or in innerproduct form

$$< \dot{A}\,x, \dot{A}x > = 0$$

which implies that

$$\dot{A}\,x = 0, \quad \forall x \in R$$

or

$$\dot{A} = 0 \quad \Rightarrow \quad \text{rank}\ \dot{A} = 0.$$

The invers case is obvious. ■

III. *Acceleration Pole Points and Acceleration Axoids.*

Derivating (1), with respect to t, we have

$$\dot{x} = \dot{A}\,x_0 + \dot{C} + A\,\dot{x}_0\,, \tag{7}$$

where $\dot{x}$ is the *absolute velocity* (absolutgeschwindgkeit), $\dot{A}\,x_0 + \dot{C}$ is the *sliding velocity* (führungschwindigkeit) and $A\dot{x}_0$ is the *relative velocity* (relativgeschwindigkeit) of the point X whose position vector is x.

The solution vector x_0 of the system

$$\dot{A}\,x_0 + \dot{C} = 0 \tag{8}$$

is the position vector of the point may be considered as a fixed point of R_0 and R at the same time t. These points are called instantaneous pole points at the time t. The discussion of these solutions had been given by H.R. Müller [1].

The velocity of a fixed point $x_0 \in R_0$ is

$$\dot{x} = \dot{A}\,x_0 + \dot{C} \tag{9}$$

and for the 2^{nd} order velocity (or the 1^{st} order acceleration) of this point $x_0 \in R_0$, (9) gives us

$$\ddot{x} = \ddot{A}\, x_0 + \ddot{C}\,.$$

The point $X \in R$, which corresponds to the fixed point $X_0 \in R_0$, derives its orbit in R, under this acceleration $\ddot{x}$. The discussion of existency of the acceleration poles and the acceleration axoids is the discussion of the solution of the system

$$\ddot{A}\, x_0 + \ddot{C} = 0\,. \tag{10}$$

The solutions of the system (10) depend on rank $\ddot{A}$.

a) *Instantaneous Screw Axis (I.S.A).*

We denote the one parameter motion (1) by R_0/R. In R_0/R we will try to find the points hawing, in a given position, a velocity-vector with minimal norm.If n=2 this kind points are exist[2]. In the case n=2, this point coincides with the pole, in a position corresponding to an instantaneous rotation; in a position corresponding to an instantaneous translation all of the points of the moving 2–space can be regarded as having minimal velocity.

In the case n=3, in any position of R_0 the locus of the points in R_0 having a velocity-vector with stationary norm is a line [2].

We show that, in the general case n, in any position of R_0 the locus of that points in R_0 is a line If $X_0 \in R_0$ has the coordinates

$$(x_1, x_2, \dots, x_n)$$

in an orthonormal frame of R_0 then for these points we can write

$$\frac{\partial}{\partial x_k} < \dot{x}, \dot{x} > = 0, \quad 1 \le k \le n, \tag{11}$$

and replacing (9) in (11) we have

$$< \frac{\partial}{\partial x_k} (\dot{A}\, x_0),\ \dot{A}\, x_0 + \dot{C} > = 0$$

or

$$< \dot{A} \frac{\partial}{\partial x_k} (x_0),\ \dot{A}\, x_0 + \dot{C} > = 0. \tag{12}$$

Since we know that

$$\frac{\partial}{\partial x_k}(x_o) = [\delta_{1k}] \in |R_n^1$$

(12) reduces to

$$< \mathring{A}_k, \mathring{A}\, x_o + \dot{C} > = 0$$

where $\mathring{A}_k$ is the k^{th} column vector of $\mathring{A}$. In the matrix notation the last relation is

$$\mathring{A}^T{}_k(\mathring{A}\, x_o + \dot{C}) = 0, \; 1 \leq k \leq n$$

or

$$\mathring{A}^T\,(\mathring{A}\, x_o + \dot{C}) = 0\,. \tag{13}$$

According to Theorem 2.1, if n is an odd number then the rank $\mathring{A}$ can be n–1. Suppose that rank $\mathring{A}$ = n–1, since Equation (13) is in the form

$$\mathring{A}^T Y = 0\,, \; Y = \mathring{A}\, x_o + \dot{C} \tag{13'}$$

it has a solution space whose dimension is 1. Let E* be a basis of the solution space such that

$$< E^*, E^* > = 1.$$

Then all of the solutions of (13') are in the form

$$Y = \lambda\, E^*$$

and then the solutions of (13) satisfy the relation

$$\mathring{A}\, x_o + \dot{C} - \lambda\, E^* = 0. \tag{14}$$

Since we suppose that rank $\mathring{A}$ = n–1

$$\text{rank}\,(\mathring{A}, \dot{C} - \lambda\, E^*) = n\text{–}1$$

is necessery and sufficient for (14) to have a solution. If $\lambda = \lambda^*$ verifies this condition then we have

$$\text{rank}\,(\mathring{A}, \dot{C} - \lambda^* E^*) = n\text{–}1.$$

Since E* is the general solution of the homogen part of the system, to have the general solution of (14) we must add a special solution of (14) to E*. For this special solution λ^* can be determined from the equation

$$< E^*, \dot{C} - \lambda^* E^* > = 0$$

and then

$$\lambda^* = < \dot{C}, E^* >.$$

Therefore (14) reduces to

$$\mathring{A}\, x_0 + \dot{C} - < \dot{C}, E^* > E^* = 0$$

or

$$\mathring{A}\, x_0 + B = 0 \tag{15}$$

where

$$B = \dot{C} - < \dot{C}, E^* > E^*. \tag{16}$$

Hence if a solution (15) is

$$x_0 = P \tag{17}$$

then the general solution of (15) is

$$x_0 = P + \lambda E \tag{18}$$

where E corresponds to E^* such that

$$\mathring{A}^T E^* = O, \quad \mathring{A}\, E = 0.$$

To verify that (18) is a solution of (15) we can replace (18) in (15) and see that

$$\mathring{A}\,(P + \lambda E) + B = \mathring{A}\, P + B + \lambda\, \mathring{A}\, E,$$

where $\mathring{A}\, P + B = 0$ and $\mathring{A}\, E = 0$.

The geometrical meaning of (18) is a straight line which passes through P and parallel to E.

Hence we proved the following theorem.

Theorem 3.1:

Let the motion R_0/R be represented by the equvtion (1) If rank $\mathring{A}=n-1$, at any time, the locus of the points in R_0 having a velocity-vector with stationary norm is a straight line.

Definition 3.1 :

At any time t of the motion R_0/R, the straight line, which is the locus of the points in R_0 having a velocity –vector with stationary, is called the instantaneous screw-axis and denoted by I.S.A for short.

The vector-representation of I.S.A in R_0 is given by (18). At the same time the vector-representation of the same line in R is

$$x = A\ P + C + \lambda\ A\ E \tag{19}$$

which is obtained (1) and (18).

b) *The Relation Between E*and E.*

In consequence of (3) we have

$$\mathring{A}^T = - A^T \mathring{A}\ A^T$$

and so the relation $\mathring{A}^T E = 0$ reduces to

$$A^T \mathring{A}\ A^T E^* = 0$$

or

$$\mathring{A}\ A^T E^* = 0.$$

Since we have that

$$\mathring{A}\ E = 0$$

we can write that

$$A^T E^* = \mu\ E \quad \text{or} \quad E^* = \mu\ A\ E.$$

On the otherhand

$$\langle E, E\rangle = \langle E^*, E^*\rangle = 1$$

leads to have

$$\mu = 1$$

and then we obtain the result

$$E^* = A\ E. \tag{20}$$

c) *The Velocity of the Points on I.S.A*

The velocity of a fixed point of R_0 is

$$\dot{x} = \mathring{A}\ x_0 + \dot{C},$$

on the other hand if this point is on I.S.A we have

$$x_0 = P + \lambda\ E$$

and

$$\mathring{A}\ E = 0$$

then the velocity of a point on I.S.A is

$$\dot{x} = \mathring{A}\ P + \dot{C}$$

or by means of (15) and (17) since we have

$$\mathring{A}\, P = - B$$

and then

$$\bar{x} = S - B$$

or from (16) and (20)

$$\hat{x} = < \dot{C}, A\, E > A\, E \tag{21}$$

which shows us that $\hat{x}$ does not depend on x_0 and x. Therefore we have proved the following theorem.

Theorem 3.2:

At any time t of the motion R_0/R, all of the points of the I.S.A have the same velocity which is directed along the I.SA.

d) *The Sliding–Rolling of The Acceleration Pole Curves Upon Each Other.*

We need the following two theorems to see that whether the acceleration pole curves have the sliding-rolling upon each other.

Theorem 3.3:

If $A \in SO(n)$ and rank $\mathring{A} = n-1$, then

(i) $\ddot{A}\, E = 0 \Leftrightarrow 0$ rank $\ddot{A} = n-1$;

(ii) $\ddot{A}\, E \neq 0 \Leftrightarrow$ rank $\ddot{A} = n$.

Proof: (i) By means of Theorem 2.2 we have that

$$\text{rank } \ddot{A} \neq 0.$$

Thus rank $\ddot{A}$ must satisfy the inequality

$$0 < \text{rank } \ddot{A} \leq n$$

and then we have the following two cases:

(1°). $0 < \text{rank } \ddot{A} < n,$

or

(2°). $0 < \text{rank } \ddot{A} = n.$

In the first case we can find an unit vector $\mathcal{E} \in R$ such that

$$\ddot{A}\, \mathcal{E} = 0.$$

Therefore from (5) we may wtite

$$\mathcal{E}^T (\ddot{A}^T A + 2\, \mathring{A}^T \mathring{A} + A^T \ddot{A})\, \mathcal{E} = 0$$

or by means of $\ddot{A}\,\mathcal{E} = 0$

$$\mathcal{E}^T \mathring{A}^T \mathring{A}\, \mathcal{E} = 0$$

or

$$(\mathring{A}\,\mathcal{E})^T (\mathring{A}\, \mathcal{E}) = 0$$

or in the form of inner product of vectors

$$< \mathring{A}\mathcal{E}, \mathring{A}\mathcal{E} > = 0.$$

Since the inner product is positive definite in Euclidean n-space, the last expression induces that

$$\mathring{A}\mathcal{E} = 0.$$

On the other hand, since we have

$$\text{rank } \mathring{A} = n-1$$

we can write that

$$\mathcal{E} = + E$$

an then

$$\ddot{A}\ E = 0,$$

where E is the unit vector directed along the I.S.A. Therefore the dimension of the solution of the system is 1. Thus according to the theory of homogen linear system [3] we have

$$\text{rank } \ddot{A} = n-1.$$

The second case does not hold because of the hypothesis $\ddot{A}E=O$. (ii) If $\ddot{A}\,E \neq O$ then rank $\ddot{A}$ does not satisfy the inequality

$$0 < \text{rank } \ddot{A} < n$$

and so we may have

$$\text{rank } \ddot{A} = n$$

in this case. ■

As a consequence of Theorem 3.3 we have the following one.

Theorem 3.4:

If $A \in SO(n)$ *and rank* $\mathring{A} = n-1$ *then:*

(i) *The direction of the* I.S.A *is stationary* ⇔

$$\text{rank } \ddot{A} = n-1;$$

(ii) *The direction of the I.S.A is not stationary* ⇔

$$\textit{rank } \ddot{A} = n.$$

Proof: (i) For the unit vector E directed along the I.SA we know that

$$\dot{A}\, E = 0$$

and

$$\langle E, E \rangle = 1.$$

From these equations we obtain, respectively,

$$\ddot{A}\, E + \dot{A}\, \dot{E} = 0 \tag{22}$$

and

$$\langle \dot{E}, E \rangle = 0. \tag{23}$$

Since E is stationary we may have that

$$\dot{E} = 0$$

and then (22) reduces to

$$\ddot{A}\, E = 0$$

which means that

$$\text{rank } \ddot{A} = n-1.$$

Conversley, rank $\ddot{A} = n-1$ implies that

$$\ddot{A}\, E = 0.$$

and then (22) becomes

$$\dot{A}\, \dot{E} = 0.$$

On the other hand we have that

$$\dot{A}\, E = 0.$$

Therefore we may write that

$$E = \lambda \dot{E}.$$

Thus (23) reduces to

$$\langle \dot{E}, \lambda \dot{E} \rangle = 0$$

or

$$\lambda \langle \dot{E}, \dot{E} \rangle = 0$$

or

$$\langle \dot{E}, \dot{E} \rangle = 0$$

$$\dot{E} = 0.$$

(ii) If E is not stationary then

$$\dot{E} \neq 0$$

which means that

$$E \neq \lambda\, \dot{E}.$$

Therefore $\dot{A}\ \dot{E} \neq O$ and so, from (22),

$$\ddot{A}\ E \neq 0.$$

Thus by means of Theorem 3.3

$$\text{rank } \ddot{A} = n$$

in this case. ■

Now we can give the following theorem.

Theorem 3.5:

If $A \in SO(n)$, *rank* $\dot{A} = n-1$ *and* I.S.A *is not stationary then in the motion* (1): $\dot{A}\ (\ddot{A}^{-1}\ \ddot{C}) + \dot{C} = 0 \Leftrightarrow$ *The acceleration pole curves roll, without sliding, upon each other.*

Proof: In this case by means of the Theorem 3.4 rank $\ddot{A} = n$, and so the motion (1) has an only one instantaneous point $J \in R$ whose sliding velocity is zero. The point J corresponds to another instantaneous point $J_0 \in R_0$ which is the solution of the system

$$\ddot{A}\ x_o + \ddot{C} = 0$$

and so

$$J_o = - \ddot{A}^{-1}\ \ddot{C}. \tag{24}$$

The equation of acceleration pole curve (moving) in R_0 is (24). From (1) and (24) the expression of the point $J \in R$ is

$$J = A\ J_o + C. \tag{25}$$

During the motion (1) the point J has an orbit which is the acceleration pole curve (fixed) in R.

The sliding acceleration of J, at any time t, is

$$\ddot{A}\ J_o + \ddot{C} = O.$$

Therefore the relation between the orbital velocities

$$J = \frac{dJ}{dt} \text{ and } J_0 = \frac{dJ_o}{dt}$$

is

$$\dot{J} = \dot{A}\ J_o + \dot{C} + A\ \dot{J}_o$$

or

$$\dot{J} = \dot{A}\ (\ddot{A}^{-1}\ \ddot{C}) + \dot{C} + A\ \dot{J}_o. \tag{26}$$

If we have

$$\dot{A}\,(\ddot{A}^{-1}\,\ddot{C}) + \dot{C} = 0 \tag{27}$$

then (26) reduces to

$$\dot{J} = A\ \dot{J}_o$$

which gives us that

$$ds = \|\ \dot{J}\ \|\ dt = \|\ \dot{J}_o\ \|\ dt = ds_o, \tag{28}$$

where ds_o and ds are, respectively, the arc elements of the acceleration pole curves (24) and (25). Hence we can say that under the condition (27) the curves (24) and (25) roll, without sliding, upon each other.

Conversely if (24) and (25) roll, without sliding, upon each other then we have (28) and so

$$\dot{J} = A\ \dot{J}_o.$$

By means of this, (26) gives us the condition (27). ■

In the case that the I.S.A is stationary we can give the following theorem.

Theorem 3.6:

A $\in$ SO (n), *rank* $\dot{A}$ = n–1 *and the* I.S.A *is stationary then, in generally, the accerleration oxoids are not tangent to each other.*

Proof: In this case since Theorem 3.4 gives us that rank $\ddot{A}$ = n–1 the solution systems of (10) are not unique. The rank of the generalized matrix $(\ddot{A}, \ddot{C})$ of (10) is, generally,

$$\text{rank}\ (\ddot{A}, \ddot{C}) \geq n-1,\ \det\ (\ddot{A}) = 0.$$

Therefore the homogen part of (10)

$$\ddot{A}V_o = 0$$

has an 1– dimensional solution space V. The space V is spaned by E. In the fixed space R as we see from (20) that

$$E^* = A\ E.$$

Thus we can say that

$$V = (I.S.A).$$

$\ddot{C}$ has the two parts: one is parallel to V and the other is U perpendicular to V:

$$\ddot{C} = U + V, \tag{29}$$

where

$$V = \lambda E, \ <U, E> = 0. \tag{30}$$

Therefore since

$$\text{rank } (\ddot{A}, U) = \text{rank } \ddot{A} = n-1$$

the system

$$\ddot{A} x_o + U = 0 \tag{31}$$

is a solvable system. At an instant t of the motion, the solution vectors of (31) span a space B_o which we called the *acceleration axis space* of the motion. In the moving space the position vector of any point of B_1 is

$$x_0 = q_o + \lambda E \tag{32}$$

where q_o is a special solution vector of (31), i.e.

$$A q_o + U = 0. \tag{33}$$

The parameter λ, in (32), is a rectangular coordinat. During the motion (1) we obtain an one parameter family of these 1–dimensional spaces B_o, in the moving space R_o. These spaces formed the acceleration axoid $M_o = \{B_o\}$. B_o corresponds to another acceleration axis spaces B, in R, and so M_o also corresponds to another acceleration axoid $M = \{B\}$ in the fixed space R.

The Positions M_o *and* *M Upon Each Other:*

Expression (32) can give us the positions of these axoids M_o and M, relative to each other. Eq. (23) has two parameters t and λ.

In the fixed space R, by means of (1), Eq, (32) corresponds to the following equation

$$x = q + \lambda A E \tag{34}$$

where $q = A q_o + C$.

At an instant t, for M_o and M, to be tangent to each other, we must have that

$$\dot{x} \wedge A E = A \dot{x}_o \wedge A E \tag{35}$$

where $\wedge$ denotes Grassmannian product [4]. According to (32) and (34), the relation (35) needs to have the condition

$$\dot{A}\ q_o + \dot{C} = 0$$

which completes the proof of the theorem. ■

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ÖZET

n–boyutlu Öklid uzaylarının

$$x = A\ x_0 + C\ ,\ A \in SO\ (n)$$

ile verilen hareketinde rank Ä nın n ve n–l olmasının geometrik anlamlarını bulduk. Böylece, bu hallere karşılık gelen ikinci mertebeden pol noktalarının, pol eğrilerinin ve aksoidlerin geometrik irdelemisini vermek mümkün oldu.

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