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On The Radius Of Starlikeness Of Certain Analytic Functions With Integral Representation

by

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On The Radius Of Starlikeness Of Certain Analytic Functions With Integral Representation

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ABSTRACT

In this paper we study some classes namely $S^*(\lambda)$, $K(\lambda)$, $V(\delta)$ and $S(m, M)$ of functions of the form

$$f(z) = z + a_2 z^2 + \dots$$

regular and univalent in the unit disc $D = \{z: |z| < 1\}$ and also a class $P(\mu)$ of functions of the form

$$p(z) = 1 + a_1 z + a_2 z^2 + \dots$$

regular in D.

For suitable restrictions of real constants α and β we obtain the radius of starlikeness of order η of normalized analytic functions f in D defined by the general integral operator of the form

$$F(z) = \left[\frac{\alpha+\beta}{g(z)^\beta} \int_0^z h(t)^{\beta-1} f(t)^\alpha \, dt \right]^{1/\alpha},$$

where $F \in S^*(\lambda)$, $g \in S^*(\mu)$ and (i) $h \in S(m, M)$ or (ii) $h \in K(\delta)$ or (iii) $h \in V(\delta)$ or (iv) $\frac{h(z)}{z} \in P(\delta)$. Our results are sharp and generalize almost all known results obtained so far in this direction.

INTRODUCTION

Let S denote the class of functions f which are regular and univalent in the unit disc $D = \{z: |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Robertson [7] defined the starlike and convex functions of order λ for functions $f \in S$ such that

$$(1.1) \quad \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \lambda, \; 0 < \lambda < 1, \, z \in D,$$

and

(1.2) $\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\lambda,\ 0\leq\lambda<1,\ z\in D$, respectively.

Jakubowski [3] defined the class S (m, M) of functions $f\in S$ satisfying

(1.3) $\left|\frac{zf'(z)}{f(z)}-m\right|<M,\ z\in D,\ (m,M)\in E,\ E=\left\{(m,M):|m-1|<M\leq m\right\}.$

Evidently.

(1.4) $S(m,M)\subset S^*(m-M)\subset S^*(0)\subset S.$

Let $P(\mu)$ denotes the class of functions p analytic in D having

$\mathrm{Re}\{p(z)\}>\mu,\ 0\leq\mu\leq 1,\ z\in D$ and normalized by $p(0)=1$.

Let $V(\delta)$ denotes the class of functions g, given by

$g(z)=\frac{1}{2}[f(z)+zf'(z)],\ f\in S^*(\delta),\ z\in D,\ 0\leq\delta<1.$

In this paper we obtain the radius of starlikeness of order η for functions $f\in S$ defined by a general integral representation of the form

(1.5) $F(z)^{\alpha}=\frac{\alpha+\beta}{g(z)^{\beta}}\int_0^z h(t)^{\beta-1}f(t)^{\alpha}\,dt,\ \alpha,\beta\in N$

where $F\in S^*(\lambda)$, $g\in S^*(\mu)$, and (i) $h\in S(m,M)$ or (ii) $h\in K(\delta)$ or

(iii) $h\in V(\delta)$ or (iv) $\frac{h(z)}{z}\in P(\delta)$. All powers are principal ones.

Our results are sharp and our first theorem generalize the result of Gupta and Ahmad [2]. In the sequel, it will be convenient to set

$$c=\frac{\alpha(2\lambda-1)+\beta(2\mu-1)}{\alpha+\beta}\tag{1.6}$$

and

$$d=\frac{c(\alpha+\beta)-\alpha\eta}{\alpha+\beta-\alpha\eta}\tag{1.7}$$

Observe that $0\leq c\leq 1$ and $0\leq d\leq 1$.

2. PRELIMINIARY LEMMAS

LEMMA (2.1) If the function w is analytic for $|z| < 1$ and satisfying $|w(z)| < 1$, $w(0) = 0$ then

(2.1) $|w(z)| \leq |z|$ for each z ($|z| < 1$) and $|w'(0)| \leq 1$.

A proof of lemma (2.1) which is due to Schwarz's may be found in Nehari [6].

LEMMA (2.2) If w is analytic in D satisfying $|w(z)| < 1$ and $w(0) = 0$ then

$$(2.2) \quad |w'(z)| \leq \frac{1-|w(z)|^2}{1-r^2} .$$

A proof of lemma (2.2) may be found in Nehari [6].

LEMMA (2.3) If $h \in S(m,M)$ for $|z| \leq r < 1$ then

$$(2.3) \quad \left|\frac{zh'(z)}{h(z)}\right| \leq Q(r) = \frac{1+ar}{1-br} \quad (|z| = r)$$

where $a = \frac{M^2-m^2+m}{M}$, $b = \frac{m-1}{M}$ and $(m,M) \in E$.

Equality occurs for the function $h(z) = \frac{z}{(1-bz)^{(a+b)/b}}$

A proof of lemma (2.3) which is due to Silverman may be found [9].

LEMMA (2.4) If $h \in K(\delta)$ for $|z| \leq r < 1$ then

$$(2.4) \quad \left|\frac{hz'(z)}{h(z)}\right| \leq B(r,\delta) = \begin{cases} \dfrac{(2\delta-1)r}{(1-r)^{2(1-\delta)}[1-(1-r)^{2\delta-1}]}, & \delta \neq \frac{1}{2} \\ \dfrac{-r}{(1-r)\log(1-r)}, & \delta = \frac{1}{2} \end{cases}$$

Equality occurs for the functions

$$h(z) = \begin{cases} \dfrac{1-(1+z)^{2\delta-1}}{2\delta-1} & \delta \neq \frac{1}{2} \\ \log(1+z) & \delta = \frac{1}{2} \end{cases}$$

A proof of lemma (2.4) which is due to MacGregor may be found in [5].
LEMMA (2.5) If $h \in V(\delta)$ for $|z| \leq r < 1$ then

$$(2.5)\quad \operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} \leq \theta_1(r,\delta) = \frac{1+2(1-2\delta)r+\delta(2\delta-1)r^2}{(1-r)(1-\delta r)}$$

Equality occurs for the function $h(z) = \dfrac{z(1-\delta z)}{(1-z)^{3-2\delta}}$.

A proof of lemma (2.5) which due to Singh and Goel may be found in [10].

LEMMA (2.6) If $\dfrac{h(z)}{z} \in P(\delta)$ for $|z| \leq r < 1$ then

$$(2.6)\quad \operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} \leq \theta_2(r,\delta) = \frac{1-2(2\delta-1)r+(2\delta-1)r^2}{(1-r)\{1-(2\delta-1)r\}}$$

Equality occurs for the function $h(z) = \dfrac{z\{1+(2\delta-1)z\}}{(1+z)}$

A proof of lemma (2.6) which is due to Libera may be found in [4].

3. STATEMENT AND PROOF OF THEOREMS

THEOREM (3.1): Let $F \in S^*(\lambda)$, $g \in S^*(\mu)$, $h \in S(m,M)$, $\alpha(\eta-1) > 2\beta$, $\beta = 1,2,\ldots$ then the function f defined by (1.5) belongs to $S^*(\eta)$ for $|z| < r_0$ where r_0 is the smallest positive root of the equation.

$$(3.1.1)\quad (1+r)(1+cr)\{-1-(\beta-1)Q(r)\}-(\alpha+\beta-\alpha\eta)$$
$$-\{(c+d)(\alpha+\beta-\alpha\eta)+(1-c)\}r-cd(\alpha+\beta-\alpha\eta)r^2=0$$

where c and d are defined by (1.6) and (1.7) respectively.
The result is sharp.

PROOF: From (1.5) we have

$$(3.1.2)\quad (\alpha+\beta)\frac{zh(z)^{\beta-1}f(z)^{\alpha}}{F(z)^{\alpha}g(z)^{\beta}} = \alpha\frac{zF'(z)}{F(z)}+\beta\frac{zg'(z)}{g(z)}$$

Since $F \in S^*(\lambda)$, $g \in S^*(\mu)$ there exists a function w such that $w(0) = 0$, $|w(z)| < 1$ and

$$(3.1.3)\quad \frac{zF'(z)}{F(z)} = \frac{1+(2\lambda-1)\,w(z)}{1+w(z)}, \quad \frac{zg'(z)}{g(z)} = \frac{1+(2\mu-1)w(z)}{1+w(z)}$$

From (3.1.2) and (3.1.3) we have

$$(3.1.4)\quad \frac{zh(z)^{\beta-1}\,f(z)^{\alpha}}{F(z)^{\alpha}\,g(z)^{\beta}} = \frac{1+cw(z)}{1+w(z)}, \; c = \frac{\alpha(2\lambda-1)+\beta\,(2\mu-1)}{\alpha+\beta}$$

Differentiating logarithmically (3.1.4) with respect to z and using (3.1.3) we get

$$\alpha\,\frac{zf'(z)}{f(z)} - \alpha\eta = -1-(\beta-1)\,\frac{zh'(z)}{h(z)} + (\alpha+\beta-\alpha\eta)\,\frac{1+dw(z)}{1+w(z)}$$

$$-(1-c)\,\frac{zw'(z)}{(1+cw(z))\,(1+w(z))}$$

where $d = \dfrac{c\,(\alpha+\beta)-\alpha\eta}{\alpha+\beta-\alpha\eta}$

$$(3.1.5)\;\alpha\,\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}-\eta\right\} \geq -1-(\beta-1)\left|\frac{zh'(z)}{h(z)}\right| - (\alpha+\beta-\alpha\eta)\left|\frac{1+dw(z)}{1+w(z)}\right|$$

$$-(1-c)\left|\frac{zw'(z)}{(1+c\,w(z))\,(1+w(z))}\right|$$

Using lemmas (2.1), (2.2) and (2.3) in (3.1.5) we get

$$(3.1.6)\;\alpha\,\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}-\eta\right\} \geq -1-(\beta-1)\;Q(r)$$

$$+\;\frac{(\alpha\eta-\alpha-\beta)-\{(c+d)\,(\alpha+\beta-\alpha\eta)+(1-c)\}\,r-cd\,(\alpha+\beta-\alpha\eta)\,r^{2}}{(1+r)\,(1+cr)}$$

Now $\mathrm{Re}\left\{\dfrac{zf'(z)}{f(z)}-\eta\right\} \geq 0$ if

$$(3.1.7)\quad P(r) \equiv -1-(\beta-1)\;Q\,(r)$$

$$+\frac{(\alpha\eta-\alpha-\beta)-\{(c+d)(\alpha+\beta-\alpha\eta)+(1-c)\}r-cd(\alpha+\beta-\alpha\eta)r^2}{(1+r)(1+cr)}$$

$$=\frac{\begin{array}{c}(1+r)(1+cr)\ \{-1-(\beta-1)Q(r)\}+(\alpha\eta-\alpha-\beta)\\ -\{(c+d)(\alpha+\beta-\alpha\eta)+(1-c)\}r-cd(\alpha+\beta-\alpha\eta)r^2\end{array}}{(1+r)(1+cr)}\geq 0$$

For $\alpha(\eta-1)>\beta$ let $r(\alpha,\beta,\eta,c,d)$ be the positive root of the equation

(3.1.8) $(\alpha\eta-\alpha-\beta)-\{(e+d)(\alpha+\beta-\alpha\eta)+(1-c)\}r-cd(\alpha+\beta-\alpha\eta)r^2=0$

Since

$P(0)=\alpha(\eta-1)-2\beta>0$ if $\alpha(\eta-1)>2\beta$ and

$P\{r(\alpha,\beta,\eta,c,d)\}=-1-(\beta-1)Q\{r(\alpha,\beta,\eta,c,d)\}<0,$

the smallest positive root of the equation $P(r)=0$ is less then $r(\alpha,\beta,\eta,c,d)$. Hence from (3.1.7) $P(r)>0$ if $r<r_0<r(\alpha,\beta,\eta,c,d)$. This completes the proof of theorem (3.1). The result is sharp as can be seen from the functions

$$F(z)=\frac{z}{(1-z)^{2(1-\lambda)}},\ g(z)=\frac{z}{(1-z)^{2(1-\mu)}},\ h(z)=\frac{z}{(1-bz)^{\frac{a+b}{b}}}$$

REMARK: The following result of Gupta and Ahmad [2] follows as as corollary bt taking $a=\dfrac{\nu-2\nu N+N}{N}$, $b=\dfrac{N-1}{N}$ and $a\leftrightarrow 1-2\nu$ $b\rightarrow 1$ as $N\leftrightarrow\infty$ in theorem (3.1).

COROLLARY: [Gupta and Ahmad] Let $F\in S^*(\lambda)$, $g\in S^*(\mu)$, $h\in S^*(\nu)$ then the function f defined by (1.5) belongs to $S^*(\eta)$ for $|z|<r_1$ where r_1 is the smallest positive root of the equation

$$\alpha(1-\eta)cDr^2+[(1-c)-\alpha(1-\eta)(c-D)]r-\alpha(1-\eta)=0$$

with

$$c=\frac{\alpha(2\lambda-1)+\beta(2\mu-1)}{\alpha+\beta}$$

$$D=\frac{|d-\eta|}{(1-\eta)}=|d'|$$

$$d = (2\lambda-1) + \left\{ \frac{\beta\,(2\mu-1) + (1-\beta)\,(2\nu-1) - 1}{\alpha} \right\}$$

THEOREM (3.2.): Let $F \in S^*(\lambda)$, $g \in S^*(\mu)$, $h \in K(\delta)$, $\alpha(\eta-1) > 2\beta$, $\beta = 1,2, \ldots$ then the function f defined by (1.5) belongs to $S^*(\eta)$ for $|z| < r_0$ where r_0 is the smallest positive root of the equation

$$(3.2.1)\quad (1+r)(1+cr)\{-1-(\beta-1)\,\beta(r,\delta)\} - (\alpha+\beta-\alpha\eta)$$
$$-\{(c+d)(\alpha+\beta-\alpha\eta) + (1-c)\}\,r - cd\,(\alpha+\beta-\alpha\eta)\,r^2 = 0$$

where c and d are defined by (1.6) and (1.7) respectively. The result is sharp.

PROOF: Here $h \in K(\delta)$ hence using lemma (2.4) and proceeding on the same lines as in theorem (3.1) we have

$$\text{Re}\left\{ \frac{zf'(z)}{f(z)} - \eta \right\} \geq 0 \text{ if}$$

$$(3.2.2)\quad P(r) \equiv -1-(\beta-1)\,B(r,\delta)$$

$$+ \frac{(\alpha\eta-\alpha-\beta) - \{(c+d)(\alpha+\beta-\alpha\eta) + (1-c)\}\,r - cd\,(\alpha+\beta-\alpha\eta)\,r^2}{(1+r)(1+cr)}$$

$$= \frac{(1+r)(1+cr)\{-1-(\beta-1)\,B(r,\delta)\} + (\alpha\eta-\alpha-\beta) - \{(c+d)(\alpha+\beta-\alpha\eta) + (1-c)\}\,r - cd\,(\alpha+\beta-\alpha\eta)\,r^2}{(1+r)(1+cr)} \geq 0$$

Now $P(0) > 0$ if $\alpha(\eta-1) > 2\beta$ and

$$P\{r(\alpha,\beta,\eta,c,d)\} = -1-(\beta-1)\,B\,\{r(\alpha,\beta,\eta,c,d),\delta\} < 0$$

since $\underset{0\leq r\leq 1}{\text{g.l.b.}}\, B(r,\delta) \geq 1$. Thus the smallest positive root of the equation $P(r) = 0$ is less than $r(\alpha,\beta,\eta,c,d)$ where $r(\alpha,\beta,\eta,c,d)$ is the positive root of the equation (3.1.8). Hence from (3.2.2) $P(r) > 0$ if $r < r_0 < r(\alpha,\beta,\eta,c,d)$. This completes the proof of theorem (3.2). The result is sharp as can be seen from the functions

$$F(z) = \frac{z}{(1-z)^{2(1-\lambda)}} \qquad \text{and } h(z) = \begin{cases} \dfrac{1-(1+z)^{2\delta-1}}{(2\delta-1)}, & \delta \neq \frac{1}{2} \\ \log(1+z), & \delta = \frac{1}{2} \end{cases}$$

$$g(z) = \frac{z}{(1-z)^{2(1-\mu)}}$$

THEOREM (3.3): Let $F \in S^*(\lambda)$, $g \in S^*(\mu)$, $h \in V(\delta)$, $\alpha(\eta-1) > 2\beta$, $\beta = 1, 2, \ldots$ then the function f defined by (1.5) belongs to $S^*(\eta)$ for $|z| < r_0$ where r_0 is the smallest positive root of the equation

(3.3.1) $$(1+r)(1+cr)\{-1-(\beta-1)\theta_1(r,\delta)\} - (\alpha+\beta-\alpha\eta) - \{(c+d)(\alpha+\beta-\alpha\eta) + (1-c)\} r - cd(\alpha+\beta-\alpha\eta) r^2 = 0$$

where c and d are defined by (1.6) and (1.7) respectively.

The result is sharp.

PROOF: Here $h \in V(\delta)$ hence using lemma (2.5) and proceeding on the same lines as in theorem (3.1) we have

$$\text{Re}\left\{\frac{zf'(z)}{f(z)} - \eta\right\} \geq 0 \text{ if}$$

(3.3.2) $$P(r) \equiv -1-(\beta-1)\,\theta_1(r,\delta) + \frac{(\alpha\eta-\alpha-\beta) - \{(c+d)(\alpha+\beta-\alpha\eta) + (1-c)\} r - cd(\alpha+\beta-\alpha\eta) r^2}{(1+r)(1+cr)}$$

$$= \frac{(1+r)(1+cr)\{-1-(\beta-1)\theta_1(r,\delta)\} + (\alpha\eta-\alpha-\beta) - \{(c+d)(\alpha+\beta-\alpha\eta) + (1-c)\} r - cd(\alpha+\beta-\alpha\eta) r^2}{(1+r)(1+cr)} \geq 0$$

Since $P(0) > 0$ if $\alpha(\eta-1) > 2\beta$ and

$$P\{r(\alpha,\beta,\eta,c,d)\} = -1-(\beta-1)\,\theta_1\{r(\alpha,\beta,\eta,c,d),\delta\} < 0$$

the smallest positive root of the equation $P(r) = 0$ is less than $r(\alpha,\beta,\eta,c,d)$ where $r(\alpha,\beta,\eta,c,d)$ is the positive root of the equation (3.1.8). Hence from (3.3.2) $P(r) > 0$ of $r < r_0 < r(\alpha,\beta,\eta,c,d)$. This completes the proof of theorem (3.3). The results is sharp as can be seen from the functions

$$F(z) = \frac{z}{(1-z)^{2(1-\lambda)}}, \quad g(z) = \frac{z}{(1-z)^{2(1-\mu)}}$$

and

$$h(z) = \frac{z(1-\delta z)}{(1-z)^{3-2\delta}}.$$

THEOREM (3.4): Let $F \in S^*(\lambda)$, $g \in S^*(\mu)$, $\frac{h(z)}{z} \in P(\delta)$, $\alpha(\eta-1) > 2\beta$

$\beta = 1, 2, \ldots$ then the function f defined by (1.5) belongs to $S^*(\eta)$ for $|z| < r_0$ where r_0 is the smallest positive root of the equation

(3.4.1) $$(1+r)(1+cr)\{-1-(\beta-1)\theta_2(r,\delta)\}-(\alpha+\beta-\alpha\eta)$$
$$-\{(c+d)(\alpha+\beta-\alpha\eta)+(1-c)\}r-cd(\alpha+\beta-\alpha\eta)r^2 = 0$$

where c and d are defined by (1.6) and (1.7) respectively.

The result is sharp.

PROOF: Here $\frac{h(z)}{z} \in P(\delta)$ hence using lemma (2.6) and proceeding

on the same lines as in theorem (3.1) we have

$$\text{Re}\left\{\frac{zf'(z)}{f(z)} - \eta\right\} \geq 0 \text{ if}$$

(3.4.2) $$P(r) \equiv -1-(\beta-1)\theta_2(r,\delta)$$

$$+\frac{(\alpha\eta-\alpha-\beta)-\{(c+d)(\alpha+\beta-\alpha\eta)+(1-c)\}r-cd(\alpha+\beta-\alpha\eta)r^2}{(1+r)(1+cr)}$$

$$=\frac{(1+r)(1+cr)\{-1-(\beta-1)\theta_1(r,\delta)\}+(\alpha\eta-\alpha-\beta)-\{(c+d)(\alpha+\beta-\alpha\eta)+(1-c)\}r-cd(\alpha+\beta-\alpha\eta)r^2}{(1+r)(1+cr)} \geq 0$$

Since $P(0) > 0$ if $\alpha(\eta-1) > 2\beta$ and

$$P\{(\alpha,\beta,\eta,c,d)\} = -1-(\beta-1)\theta_2\{r(\alpha,\beta,\eta,s,d),\delta\} < 0$$

the smallest positive root of the equation $P(r) = 0$ is less than $r(\alpha,\beta,\eta,c,d)$ where $r(\alpha,\beta,\eta,c,d)$ is the positive root of the equation (3.1.8). Hence from (3.4.2) $P(r) > 0$ if $r < r_0 < r(\alpha,\beta,\eta,c,d)$. This completes the proof of theorem (3.4). The result is sharp as can be seen from the functions

$$F(z) = \frac{z}{(1-z)^{2(1-\lambda)}}, \quad g(z) = \frac{z}{(1-z)^{2(1-\mu)}}$$

and

$$h(z) = \frac{z\{(1+(2\delta-1)z\}}{(1+z)} .$$

REFERENCES

[1] **M. W. Causey and W.L. White,** Starlikeness of certain analytic functions with integral representation, J. Math. Anal. Appl. 64 (1978), 458-466.

[2] **V. P. Gupta and Iqbal Ahmad,** On Starlike functions, Bull. Austral. Math. Soc. 22 (1980), 241-245.

[3] **Z. J. Jakubowski,** On the coefficients of starlike functions of some classes, Annal. Polon. Math 26 (1972), 305-313.

[4] **R. J. Libera,** Some radius of convexity problems, Duke. Math. J. 31 (1964), 143-158.

[5] **T. H. MacGregor,** A subordination of convex functions of order α, J. London Math. Soc. (2) 9 (1975), 530-536.

[6] **Z. Nehari,** Conformal Mapping McGraw Hill 1952.

[7] **M. S. Robertson,** On the theory of univalent functions, Annals Math. 37 (2), (1936), 374-408.

[8] **J. S. Ratti,** The radius of convexity of certain analytic functions II, Inter. J. Mathe. Math. Sci. 3 (3) (1980) 483-498.

[9] **H. Silverman,** Subclasses of starlike functions, Rev. Roum. Math. Pures, et. Appl. 23 (1978) 1093-1099.

[10] **V. Singh and R. M. Goel,** On radii of convexity and starlikeness of some classes of functions, J. Math. Soc. Japan 23 (1971), 323-339.

[11] **Vinod Kumar and S. L. Shukla,** Certain integrals for classes of p-valent meromorphic functions, Bull. Austral. Math. Soc. 25 (1982), 85-97.