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by

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The Interior and Exterior Problems For The Generalized Poly-Axially Symmetric Helmholtz Equation

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SUMMARY

In this article, the interior and exterior problems for the Generalized Helmholtz equation with poly-axially symmetric are studied. It consists of three sections. In the first one the interior Dirichlet problem is examined by defining an inner-product and a kernel function. In the second section the solution of the exterior Dirichlet problem is obtained. The uniqueness of the solution is also proved. Examination of the exterior Naumann problem by means of a system of integral equation, forms the last section.

I. INTERIOR DIRICHLET PROBLEM

I. Introduction

Let us consider the Bi-axially symmetric Helmholtz equation with two independent variables:

$$h_{\Sigma}\,[U] \equiv U_{xx} + U_{yy} + \frac{2\mu}{x} U_x + \frac{2\nu}{y} U_y + k^2 U = o \qquad (I.I)$$

Here μ,ν and k are real positive constants. For the values $\mu \neq o$, $\nu \neq o$ the equation (I-I) has the singular coefficients for $x = o$, $y = o$.

We will define a Dirichlet problem with sufficiently smooth boundary data in the first quarter of the xy-plane for the equation (I-I). Let P (x, y) be a point on xy-plane and R be an arbitrary positive constant. Let us make the definitions below;

$$D = \{ (x, y) \mid x^2 + y^2 < R^2,\ x > o,\ y > o \},$$

$$C_R = \{ (x,y) \mid x^2 + y^2 = R^2,\ x \geq o,\ y \geq o \},$$

$$\Gamma_x = \{(x, y) \mid o \leq x < R, \ y = o\},$$
$$\Gamma_y = \{(x, y) \mid x = o, \ o \leq y < R\},$$
$$C = C_R \cup \Gamma_x \cup \Gamma_y.$$

We will denote the arc-lenght of C, by s and the outward normal at the boundary of the region D by n.

2. *The Kernel Function*

Let U (x, y) and V (x, y) be two functions in the region D which have continuous derivatives up to and including the second order. For these functions the divergence formula

$$\iint_D x^{2\mu} y^{2\nu} V h_\Sigma [U] \, dx\, dy = \int_C x^{2\mu} y^{2\nu} V \frac{\partial U}{\partial n} ds - \iint_D x^{2\mu} y^{2\nu} \times (U_x V_x + U_y V_y) \, dx\, dy + k^2 \iint_D x^{2\mu} y^{2\nu} \times U V \, dx\, dy \tag{1.2}$$

and the Green identity

$$\iint_D x^{2\mu} y^{2\nu} (V h_\Sigma [U] - U h_\Sigma [V]) \, dx\, dy = \int_C x^{2\mu} y^{2\nu} \left(V \frac{\partial U}{\partial n} - U \frac{\partial V}{\partial n}\right) ds, \tag{1.3}$$

are known.

It is easy to see that

$$(U, V) = \iint_D x^{2\mu} y^{2\nu} (U_x V_x + U_y V_y) \, dx\, dy \tag{1.4}$$

defines an inner-product and

$$(U, U) = o \Leftarrow U = o.$$

Thus we can define a semi-norm by the inner product (U, V). If U is a solution of the equation $h_\Sigma [U] = o$ then (1.4) can be written as

$$(U, V) = \int_C x^{2\mu} y^{2\nu} V \frac{\partial U}{\partial n} ds + k^2 \iint_D x^{2\mu} y^{2\nu} U V \, dx\, dy \tag{1.5}$$

$\mathcal{J}$ (D) will denote the class of the functions satisfying the equation (I.I) in the region D such that $(u, u) = \| u \|^2 < \infty$. Ediventy $\mathcal{J}$ (D) does not contain constant functions except the zero function.

In polar coordinates, the set of complete solutions of (I.I) which are regular at the origin is:

$$f_n(r, \theta) = r^{-(\mu+\nu)} J_{\mu+\nu+2n}(kr) \quad P_n^{(\nu-\frac{1}{2},\, \mu-\frac{1}{2})}(\cos 2\theta), \quad n = 0, 1, 2, \ldots \tag{1.6}$$

Here $J_{\mu+\nu+2n}(kr)$ is the Bessel function of the first kind and $P_n^{(\nu-\frac{1}{2},\, \mu-\frac{1}{2})}(\cos 2\theta)$ is the Jacobi polynomial. This set of functions form an orthogonal system with respect to the variable θ under the inner product (1.5).

$\{f_n\}$ can be brought into an orthonormal set as

$$U_n(r, \theta) = \left(\frac{2^{\mu+\nu+1}}{C_{n,\, \nu-\frac{1}{2},\, \mu-\frac{1}{2}}} \right)^{\frac{1}{2}} \frac{r^{-(\mu+\nu)} J_{\mu+\nu+2n}(kr)}{J_{\mu+\nu+2n}(kR)} P_n^{(\nu-\frac{1}{2},\, \mu-\frac{1}{2})}(\cos 2\theta), \tag{1.7}$$

where

$$C_{n,\alpha,\beta} = \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} \left[P_n^{(\alpha,\beta)}(x) \right]^2 dx$$

$$= \frac{2^{\alpha+\beta+1}\, \Gamma(n+\alpha+1)\, \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\, \Gamma(n+1)\, \Gamma(n+\alpha+\beta+1)} \tag{1.8}$$

It is obvious that $C_{n,\alpha,\beta}$ is bounded for every n ($\alpha > -1$, $\beta > -1$). Let two arbitary points in the region D be P (ρ, $\varnothing$) and Q (r, θ). Let us define the infinite series;

$$K(P, Q) = K(\rho, \varnothing; r, \theta) = \sum_{n=0}^{\infty} U_n(\rho, \varnothing)\, U_n(r, \theta)$$

$$= \sum_{n=0}^{\infty} \frac{2^{\mu+\nu+1}}{C_{n,\, \nu-\frac{1}{2},\, \mu-\frac{1}{2}}} \; \frac{r^{-(\mu+\nu)} \rho^{-(\mu+\nu)} J_{\mu+\nu+2n}(kr)\, J_{\mu+\nu+2n}(k\rho)}{\left[J_{\mu+\nu+2n}(kR) \right]^2}$$

$$\times P_n^{(\nu-\frac{1}{2},\, \mu-\frac{1}{2})}(\cos 2\theta)\; P_n^{(\nu-\frac{1}{2},\, \mu-\frac{1}{2})}(\cos 2\varnothing) \tag{1.9}$$

We know that [7] the Jacobi polynomials have the properties

$$\max_{-1\leq x\leq 1} \left| P_n^{(\alpha,\beta)}(x) \right| = \begin{cases} \binom{n+q}{n} \sim n^q, \ q = \max(\alpha, \beta) \geq -\frac{1}{2} \\ \left| P_n^{(\alpha,\beta)}(x') \right| \sim n^{-\frac{1}{2}}, \ q = \max(\alpha, \beta) < -\frac{1}{2} \end{cases} \tag{1.10}$$

where

$$\alpha > -1, \ \beta > -1, \ x_0 = \frac{\beta-\alpha}{\alpha+\beta+1}$$

and x' is the maximum of x_0. Since

$$\nu - \tfrac{1}{2} > -\tfrac{1}{2}, \ \mu - \tfrac{1}{2} > -\tfrac{1}{2}$$

and

$$\max_{-1\leq x\leq 1} \left| P_n^{(\nu-\frac{1}{2}, \mu-\frac{1}{2})}(x) \right| \backsim n^q, \ q = \max(\nu-\tfrac{1}{2}, \mu-\tfrac{1}{2})$$

the series (1.9) is dominated by the series below:

$$\sum_{n=0}^{\infty} A \frac{\left| J_{\mu+\nu+2n}(kr) \ J_{\mu+\nu+2n}(k\rho) \right|}{\left[J_{\mu+\nu+2n}(kR) \right]^2} n^{2q} \tag{1.11}$$

In (1.11) A is a constant independent from n. Besides for every n we can write

$$\left| \frac{J_{\mu+\nu+2n}(kr)}{J_{\mu+\nu+2n}(kR)} \right| \sim \left| \frac{r}{R} \right|^{\mu+\nu+2n}, \left| \frac{r}{R} \right| < 1 \tag{1.12}$$

Hence (1.11) is convergent. Consequently the series (1.9) is uniformly convergent. Therefore (1.9) defines a function K (P, Q) which is differentiable.

Now let us give a definition.

Definition I.

If the function f, defined in the interval $-1 \leq x \leq 1$, can be expanded in a uniformly convergent Jacobi series, that is, if the coefficients

$$a_n = [C_{n,\alpha,\beta}]^{-1} \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x) P_n^{(\alpha,\beta)}(x)\, dx \qquad (1.13)$$

can be obtained such that

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) , \qquad (1.14)$$

then we say that f belongs to the class L.

3. *The Interior Dirichlet Problem.*

The curve C in the line integral (1.5) given for the inner product consists of Γ_x, Γ_y and C_R. Since the integrals along Γ_x and Γ_y vanishes the inner product takes the form

$$(U, V) = \int_0^{\pi/2} R^{2\mu+2\nu+1} \cos^{2\mu}\theta \sin^{2\nu}\theta\, V \frac{\partial U}{\partial r}\, d\theta$$

$$+ k^2 \int_0^R \int_0^{\pi/2} r^{2\mu+2\nu+1} \cos^{2\mu}\theta \sin^{2\nu}\theta\, VU\, dr\, d\theta. \qquad (1.15)$$

Thbs

$$(K(P, Q), U(Q)) = R^{2\mu+2\nu+1} \int_0^{\pi/2} \cos^{2\mu}\theta \sin^{2\nu}\theta \left.\frac{\partial K(P, Q)}{\partial r}\right|_{r=R} U(Q) d\theta$$

$$+ k^2 \int_0^R \int_0^{\pi/2} r^{2\mu+2\nu+1} \cos^{2\mu}\theta \sin^{2\nu}\theta\, K(P, Q)\, U(Q)\, dr\, d\theta \qquad (1.16)$$

The derivative of the K (P, Q) in the direction of the normal to the spherical boundary is

$$\left.\frac{\partial K}{\partial r}\right|_{r=R} = F(P,Q) = F(\rho, \varnothing ; R, \theta)$$

$$= \sum_{n=0}^{\infty} \frac{2^{\mu+\nu+1} \left[2nR^{-(\mu+\nu+1)} J_{\mu+\nu+2n}(kR) - kR^{-(\mu+\nu)} J_{\mu+\nu+2n+1}(kR) \right]}{C_{n,\nu-\frac{1}{2},\mu-\frac{1}{2}} \left[J_{\mu+\nu+2n}(kR) \right]^2}$$

(1.17)

$$\times\ \rho^{-(\mu+\nu)} J_{\mu+\nu+2n}(k\rho)\ P_n^{(\nu-\frac{1}{2},\ \mu-\frac{1}{2})}(\cos 2\theta)\quad P_n^{(\nu-\frac{1}{2},\ \mu-\frac{1}{2})}(\cos 2\varnothing)\ .$$

Let us substitute (1.17) in (1.16):

$$(K\ (P,Q),\ U\ (Q)\ ) = \sum_{n=0}^{\infty} \frac{2^{\mu+\nu+1}}{C_{n,\ \nu-\frac{1}{2},\ \mu-\frac{1}{2}}} R_{n,1}\ \rho^{-(\mu+\nu)} J_{\mu+\nu+2n}(k\rho)\ P_n^{(\nu-\frac{1}{2},\ \mu-\frac{1}{2})}(\cos 2\varnothing)$$

$$\times \int_0^{\pi/2} \cos^{2\mu}\theta\ \sin^{2\nu}\theta\ U\ (Q)\ \ P_n^{(\nu-\frac{1}{2},\ \mu-\frac{1}{2})}(\cos 2\ \theta)\ d\theta$$

$$+\ k^2 \sum_{n=0}^{\infty} \frac{2^{\mu+\nu+1}}{C_{n,\ \nu-\frac{1}{2},\ \mu-\frac{1}{2}}} R_{n,2}\ \rho^{-(\mu+\nu)}\ J_{\mu+\nu+2n}(k\rho)\ \ P_n^{(\nu-\frac{1}{2},\ \mu-\frac{1}{2})}(\cos 2\ \varnothing) \qquad (1.18)$$

$$\times \int_0^{\pi/2} \cos^{2\mu}\theta\ \sin^{2\nu}\theta\ U\ (Q)\ \ P_n^{(\nu-\frac{1}{2},\ \mu-\frac{1}{2})}(\cos 2\ \theta)\ \ d\theta.$$

Here $R_{n,1}$ and $R_{n,2}$ are given as;

$$R_{n,1} = \frac{2n\ R^{\mu+\nu}\ \ J_{\mu+\nu+2n}(kR)\ - k\ R^{\mu+\nu+1}\ \ J_{\mu+\nu+2n+1}(kR)}{\left[\ J_{\mu+\nu+2n}(kR)\ \right]^2}$$

$$R_{n,2} = \frac{\int_0^R r^{\mu+\nu+1}\ \ J_{\mu+\nu+2n}(kr)\ dr}{\left[\ J_{\mu+\nu+2n}(kR)\ \right]^2}$$

If U belongs to the class L, then we can write

$$(K\ (P,\ Q),\ U\ (Q)\ ) = \sum_{n=0}^{\infty}\ a_n\ (R_{n,1} + k^2\ R_{n,2})\ \rho^{-(\mu+\nu)}\ \ J_{\mu+\nu+2n}(k\rho)$$

$$\times\ P_n^{(\nu-\frac{1}{2},\ \mu-\frac{1}{2})}(\cos 2\varnothing) \qquad (1.19)$$

The infinite series in (1.19) is uniformly convergent and defines the solution of the equation (1.1) at the point P. So

$$U\ (P) = (K\ (P,\ Q),\ U\ (Q)\ ).$$

In (1.19) taking

$$b_n = a_n (R_{n,1} + k^2 R_{n,2})$$

we find

$$U(P) = (K(P,Q), U(Q)) = \rho^{-(\mu+\nu)} \sum_{n=o}^{\infty} b_n J_{\mu+\nu+2n}(k\rho) \quad P_n^{(\nu-\frac{1}{2}, \mu-\frac{1}{2})}(\cos 2\varnothing), \; P \in D \tag{1.20}$$

Now the value of U at a point on the singular boundary can be obtained by use of limit. Let $Q \in C_R$. Because of the uniform convergence of the series (1.9), we obtain

$$\lim_{P\to Q} U(P) = \lim_{\substack{P\to Q \\ (\rho,\varnothing)\to(R,\theta)}} (K(P,Q), U(Q)) = \sum_{n=o}^{\infty} b_n R^{-(\mu+\nu)} J_{\mu+\nu+2n}(kR)$$

$$\times P_n^{(\nu-\frac{1}{2}, \mu-\frac{1}{2})}(\cos 2\theta)$$

$$= \sum_{n=o}^{\infty} C_n P_n^{(\nu-\frac{1}{2}, \mu-\frac{1}{2})}(\cos 2\theta) \tag{1.21}$$

$$= f_1(\cos 2\theta)$$

where

$$C_n = b_n R^{-(\mu+\nu)} \quad J_{\mu+\nu+2n}(kR)$$

Thus if f is given then f_1 can be obtained

Theorem I.

Let f_1 be a function of the class L and C_n be the coefficients of the Fourier-Jacobi series. If f is a function belonging to the class L having the Fourier-Jacobi coefficients

$$a_n = \frac{2^{\mu+\nu+1}}{C_{n,\nu-\frac{1}{2},\mu-\frac{1}{2}}} \int_o^{\pi/2} \cos^{2\mu}\theta \sin^{2\nu}\theta \, f(\cos 2\theta) \, P_n^{(\nu-\frac{1}{2}, \mu-\frac{1}{2})}(\cos 2\theta) \, d\theta, \tag{1.22}$$

then the unique solution to the equation (1.1), taking the boundary values $f_1(Q)$ at the boundary of the region D is,

$$U(P) = (K(P,Q), U(Q)) = R^{2\mu+2\nu+1} \int_o^{\pi/2} \cos^{2\mu}\theta \sin^{2\nu}\theta$$

$$\times F(\rho, \varnothing; R, \theta) f(\cos 2\theta)\, d\theta \qquad (1.\ 23)$$

$$+ k^2 \int_0^R \int_0^{\pi/2} r^{2\mu+2\nu+1} \cos^{2\mu}\theta \sin^{2\nu}\theta\, K(\rho, \varnothing; r, \theta) f(\cos 2\theta)\, d\theta dr$$

II. THE EXTERIOR DIRICHLET PROBLEM

I. The Exterior Dirichlet Problem.

Let us consider the classical Helmholtz equation in (p+1) dimensions:

$$U_{yy} + \sum_{i=1}^{p} U_{\xi_i \xi_i} + k^2 U = 0,\ p = n - 1 + \sum_{j=1}^{n-1} k_j \qquad (2.1)$$

If the coordinate transformation

$$x_1 = \left(\xi_1^2 + \dots + \xi_{k_1+1}^2\right)^{1/2}$$

$$x_2 = \left(\xi_{k_1+2}^2 + \dots + \xi_{k_1+k_2+2}^2\right)^{1/2}$$

$$-\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -$$

$$x_{n-2} = \left(\xi_{k_1+\dots+k_{n-3}+n-2}^2 + \dots + \xi_{k_1+\dots+k_{n-2}+n-2}^2\right)^{1/2}$$

$$x_{n-1} = \left(\xi_{k_1+\dots+k_{n-2}+n-1}^2 + \dots + \xi_{k_1+\dots+k_{n-1}+n-1}^2\right)^{1/2} \qquad (2.2)$$

is applied to (2.1) we obtain the poly-symmetric Helmholtz equation:

$$h_\Sigma [U] \equiv U_{yy} + \sum_{i=1}^{n-1} \left(U_{x_i x_i} + \frac{k_i}{x_i} U_{x_i}\right) + k^2 U = 0. \qquad (2.3)$$

Let D be a bounded, simply-connected and symmetric region with respect to the hyperplanes $x_i = 0$, $(i = 1, \dots, n-1)$ in $\mathbb{R}^n$. Let us denote the boundary of the region D, which is assumed to be smooth, by ∂D and let ∂D^+ be

$$\partial D \cap \{(x_1, \ldots, x_{n-1}, y) \mid x_i \geq 0, i = 1, \ldots, n-1\}. \tag{2.4}$$

Let f be a continuous function defined on ∂D^+. Let us draw a sphere of radius r with the centre at the origin containing the region D. Also let us denote the region between D and the sphere of the radius r by B. Let $(0, \ldots, 0) \in D$.

We want to obtain the solution of the exterior Dirichlet problem given as follows,

$$h_\Sigma [U] = 0 \text{ outside } D, \tag{2.5}$$

$$U = f \quad \text{on} \quad \partial D^+. \tag{2.6}$$

A solution of the equation (2.3) can be written as in [3]

$$U(r,\theta_1,\ldots,\theta_{n-1}) = r^{-s^*} J_{s^*+p_0}(kr) \, P_{\frac{1}{2}P_{n-2}}^{(\frac{1}{2}(k_{n-1}-1), \frac{1}{2}(k_{n-2}-1))}(\cos 2\theta_{n-1}) \, (\sin\theta_1)^{p_1}$$

$$\times P_{\frac{1}{2}(p_0-p_1)}^{(\frac{1}{2}(2p_1+m_2^*), -\frac{1}{2})}(\cos 2\theta_1) \tag{2.7}$$

$$\times \prod_{j=1}^{n-3} (\sin\theta_{j+1})^{p_{j+1}} P_{\frac{1}{2}(P_j - p_{j+1})}^{(\frac{1}{2}(2p_{j+1}+m^*_{j+2}), \frac{1}{2}(k_{j+1}-1))}(\cos 2\theta_{j+1})$$

where

$$2s^* = n-2 + \sum_{i=1}^{n-1} k_i, \quad m_j^* = n - j - 1 + k_{j-1} + \ldots + k_{n-1}$$

Let us put

$$V(\theta_1,\ldots,\theta_{n-2}) = (\sin\theta_1)^{p_1} P_{\frac{1}{2}(p_0-p_1)}^{(\frac{1}{2}(2p_1+m_2^*), -\frac{1}{2})}(\cos 2\theta_1) \tag{2.8}$$

$$\times \prod_{j=1}^{n-3} (\sin\theta_{j+1})^{p_{j+1}} P_{\frac{1}{2}(p_j-p_{j+1})}^{(\frac{1}{2}(2p_{j+1}+m^*_{j+2}), \frac{1}{2}(k_{j+1}-1))}(\cos 2\theta_{j+1})$$

and choose $p_0 = p_{n-2} = 2m$, $(m = 0, 1, 2, \ldots)$. Hence the equation (2.7) takes the form

$$U(r, \theta_1,\ldots,\theta_{n-1}) = r^{-s^*} J_{s^*+2m}(kr) \, P_m^{(\frac{1}{2}(k_{n-1}-1), \frac{1}{2}(k_{n-2}-1))}(\cos 2\theta_{n-1})$$

$$x\ V(\theta_1, \ldots, \theta_{n-2}). \tag{2.9}$$

An other solution of (2.3) is

$$U_m(r, \theta_1, \ldots, \theta_{n-1}) = V(\theta_1, \ldots, \theta_{n-2}) \left[r^{-s^*} H^{(1)}_{s^*+2m}(kr) + r^{-s^*} H^{(2)}_{s^*+2m}(kr) \right]$$

$$x\ P_m^{(\frac{1}{2}(k_{n-1}-1), \frac{1}{2}(k_{n-2}-1))}(\cos 2\theta_{n-1}) \tag{2.10}$$

where $H^{(1)}$ and $H^{(2)}$ being Hankel functions.

As it is well known, the Jacobi polynomials are orthogonal in the interval [–1, 1] with respect to the weight function

$$w(t) = (1-t)^{\alpha}(1+t)^{\beta}.$$

Let us define the norm in L by

$$\| \varnothing \|^2 = (\varnothing, \varnothing) = \int_{-1}^{1} w(t) [\varnothing(t)]^2 dt. \tag{2.11}$$

Then we can write

$$\varnothing(t) = \sum_{m=0}^{\infty} a_m P_m^{(\alpha,\beta)}(t) = \sum_{m=0}^{\infty} \frac{(\varnothing(\eta), P_m^{(\alpha,\beta)}(\eta))}{\left\| P_m^{(\alpha,\beta)}(\eta) \right\|^2} P_m^{(\alpha,\beta)}(t) \tag{2.12}$$

Here a_m are defined as in (1.13) and

$$\left\| P_m^{(\alpha,\beta)}(\eta) \right\|^2 = C_{m,\alpha,\beta}.$$

If $U \in L$ then [4], [5]

$$U = \sum_{m=0}^{\infty} a_m(r, \theta_1, \ldots, \theta_{n-2}) P_m^{(\frac{1}{2}(k_{n-1}-1), \frac{1}{2}(n_{-2}-1))}(\cos 2\theta_{n-1}) \tag{2.13}$$

(2.13) can also be given as

$$U = \sum_{m=0}^{\infty} V(\theta_1, \ldots, \theta_{n-2}) P_m^{(\frac{1}{2}(k_{n-1}-1), \frac{1}{2}(k_{n-2}-1))}(\cos 2\theta_{n-1})$$

$$x\ r^{-s^*} \left[a'_m H^{(1)}_{s^*+2m}(kr) + b'_m H^{(2)}_{s^*+2m}(kr) \right] \tag{2.14}$$

In (2.14) U_m are defined by (2.10). Where a'_m and b'_m are arbitary constants.

For the very large values of the independent variable

Hankel functions have the following expansions:

$$H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{i(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} + O(x^{-\frac{3}{2}});\ x \to \infty$$

$$H_\nu^{(2)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{-i(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} + O(x^{-\frac{3}{2}});\ x \to \infty \tag{2.15}$$

Now let us examine the radiation condition

$$\lim_{r\to\infty} r^{s^*+\frac{1}{2}} \left(\frac{\partial U}{\partial r} - ikU \right) = o \tag{2.16}$$

If the radiation condition is imposed on (2.14) then we get $b'_m = o$. Since $U \in L$ we obtain

$$a_m(r, \theta_1, \ldots, \theta_{n-2}) = \frac{2^{\frac{1}{2}}(k_{n-1} + k_{n-2} + 2)}{C_{m, \frac{1}{2}(k_{n-1}-1), \frac{1}{2}(k_{n-2}-1)}} \int_o^{\pi/2} \sin^{k_{n-1}} \theta_{n-1}$$

$$\times \cos^{k_{n-2}} \theta_{n-1}\, U\, P_m^{(\frac{1}{2}(k_{n-1}-1), \frac{1}{2}(k_{n-2}-1))}(\cos 2\theta_{n-1})\, d\theta_{n-1}$$

$$= a'_m V(\theta_1, \ldots, \theta_{n-2})\, r^{-s^*} H^{(1)}_{s^*+2m}(kr) \tag{2.17}$$

Thus the solution of the Dirichlet problem (2.5), (2.6) satisfying the radiation condition is

$$U(r, \theta_1, \ldots, \theta_{n-1}) = r^{-s^*} V(\theta_1, \ldots, \theta_{n-2}) \sum_{m=o}^{\infty} a'_m H^{(1)}_{s^*+2m}(kr)$$

$$\times P_m^{(\frac{1}{2}(k_{n-1}-1), \frac{1}{2}(k_{n-2}-1))}(\cos 2\theta_{n-1}) \,. \tag{2.18}$$

2. *The Uniqueness of the Solution.*

In this section, we will show that the unique solution for the exterior Dirichlet problem defined by (2.5), (2.6) is of the type (2.18). For this purpose when $f = o$ on ∂D^+ it is sufficient to find also that $u \equiv o$ exterior to D.

Let us write the Green identity for the region B:

$$\int_B \prod_{i=1}^{n-1} x_i^{k_i} \left(U h_\Sigma [\bar{U}] - \bar{U} h_\Sigma [U] \right) d\tau = \int_{\partial B} \prod_{i=1}^{n-1} x_i^{k_i}$$

$$x \left(U \frac{\partial \bar{U}}{\partial n} - \bar{U} \frac{\partial U}{\partial n} \right) d\sigma \qquad (2.19)$$

Here $\bar{U}$ is the complex conjugate of U and n is the outward normal of the region B. Obviously if

$$h_\Sigma [U] = o \text{ then } h_\Sigma [\bar{U}] = o.$$

The surface elements is

$$d\sigma = r^{n-1} \prod_{i=1}^{n-2} \sin^{n-1-i} \theta_i \, d\theta_1 \ldots d\theta_{n-1}.$$

Thus, (2.19) can be written in the form

$$o = \int_{\partial B} \prod_{i=1}^{n-1} x_i^{k_i} \left(U \frac{\partial \bar{U}}{\partial n} - \bar{U} \frac{\partial U}{\partial n} \right) d\sigma$$

$$= \int_o^\pi d\theta_1 \int_o^{\pi/2} d\theta_2 \ldots \int_o^{\pi/2} \Phi r^{n-1+\sum_1^{n-1} k_i} \sin^{k_{n-1}} \theta_{n-1} \cos^{k_{n-2}} \theta_{n-1}$$

$$x \left(U \frac{\partial \bar{U}}{\partial r} - \bar{U} \frac{\partial U}{\partial r} \right) d\theta_{n-1}, \qquad (2.20)$$

where

$$\Phi = \prod_{i=1}^{n-2} \sin^{\sum_i^{n-1} k_s} \theta_i \prod_{j=1}^{n-3} \cos^{k_j} \theta_{j+1} \prod_{i=1}^{n-2} \sin^{n-1-i} \theta_i, \qquad (2.21)$$

$$U \frac{\partial \bar{U}}{\partial r} - \bar{U} \frac{\partial U}{\partial r} = |a'_m|^2 \, [V(\theta_1, \ldots, \theta_{n-2})]^2$$

$$\times \left[P_m^{(\frac{1}{2}(k_{n-1}-1),\ \frac{1}{2}(k_{n-2}-1))}(\cos 2\theta_{n-1}) \right]^2$$

$$\times r^{-2s^*} \left(H^{(1)}_{s^*+2m}(kr) \frac{d}{dr} H^{(2)}_{s^*+2m}(kr) - H^{(2)}_{s^*+2m}(kr) \frac{d}{dr} H^{(1)}_{s^*+2m}(kr) \right) \tag{2.22}$$

and

$$W\left[H^{(1)}_{s^*+2m}(kr),\ H^{(2)}_{s^*+2m}(kr) \right]$$

$$= H^{(1)}_{s^*+2m}(kr) \frac{d}{dr} H^{(2)}_{s^*+2m}(kr) - H^{(2)}_{s^*+2m}(kr) \frac{d}{dr} H^{(1)}_{s^*+2m}(kr) = -\frac{4i}{\pi kr} \tag{2.23}$$

Thus, from (2.19) we have

$$o = \frac{-4i}{\pi k} |a'_m|^2 \int_o^{\pi} d\theta_1 \int_o^{\pi/2} d\theta_2 \ldots \int_o^{\pi/2} \Phi V^2 d\theta_{n-2}$$

$$\times \int_o^{\pi/2} \sin^{k_{n-1}} \theta_{n-1} \cos^{k_{n-2}} \theta_{n-1} \left[P_m^{(\frac{1}{2}(k_{n-1}-1),\ \frac{1}{2}(k_{n-2}-1))}(\cos 2\theta_{n-1}) \right]^2 d\theta_{n-1}$$

$$= \frac{-4i}{\pi k} |a'_m|^2 \frac{C_{m, \frac{1}{2}(k_{n-1}-1), \frac{1}{2}(k_{n-2}-1)}}{\frac{1}{2}(k_{n-1}+k_{n-2}+2)} \tag{2.24}$$

$$2$$

$$\times \int_o^{\pi} \int_o^{\pi/2} \ldots \int_o^{\pi/2} \Phi V^2 d\theta_1 \ldots d\theta_{n-2}$$

Then $a'_m = o$ and we obtain $U \equiv o$. So we have

Theorem 2.

Let D be a bounded and symmetric region with respect to the hyperplanes $x_i = o$. The problem defined by

$$h_\Sigma [u] = o \text{ outside } D$$

$$u = o \quad \text{on } \partial D^+ \tag{2.25}$$

has a unique solution which satisfies the radiation condition

$$\lim_{r\to\infty} r^{s^*+\frac{1}{2}}\left(\frac{\partial u}{\partial r} - iku\right) = 0. \tag{2.26}$$

III. THE EXTERIOR NEUMANN PROBLEM

I. *Statement of the problem.*

Let S be a closed and smooth surface in $\mathbb{R}^3$ containing the origin. Let us denote the region exterior to S by V_e, and let the interior region of the S by V_i. Let P (x, y, z) be an arbitary point and denote the distance between the points P (x, y, z) and P′ (o, o, z) by $R = \overline{PP'}$. Let n be the unit normal of the surface S in the direction of V_e.

Now consider the following equations,

$$\Delta_\Sigma\,[U] \equiv U_{xx} + U_{yy} + U_{zz} + \frac{2\mu}{x} U_x + \frac{2\nu}{y} U_y = o. \tag{3.1}$$

$$h_\Sigma\,[U] \equiv \Delta_\Sigma\,[U] + k^2 U = o,\ \mu, \nu, k > o. \tag{3.2}$$

We will investigate the exterior Neumann problems given for the equations (3.1) and (3.2) together. x = o, y = o are the singular planes for both of the equations. If we choose k = o in (3.2) we obtain (3.1) The fundamental solutions of these equations are

$$u_o(P) = R^{-(2\mu+2\nu+1)} \tag{3.3}$$

$$u(P) = R^{-(\mu+\nu+\frac{1}{2})} N_{\mu+\nu+\frac{1}{2}}(kR) \tag{3.4}$$

respectively, where N is the Bessel function of the second kind.

Definition 2.

Let the function f: $V_e \to \mathbb{R}$ be given. If $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ and the conditions

$$\lim_{r\to\infty} \left| r^{2\mu+2\nu+1} f(P) \right| < \infty \text{ and } \lim_{r\to\infty} \left| r^{2\mu+2\nu+2} \frac{\partial}{\partial r} f(P) \right| < \infty$$

are satisfied f is called regular at the infinity.

Now let us define an exterior Neumann problem for the equation (3.2):

(i) For each u_k (P), let there be the functions $u^i{}_k$ (P) and $u^s{}_k$ (P) such that

$$u_k(P) = \overset{i}{\underset{k}{u}}(P) + \overset{s}{\underset{k}{u}}(P),$$

(ii) For every $P \in V_e$, $h_\Sigma \left[\overset{s}{\underset{k}{u}}(P) \right] = o.$

(iii) For every $P \in S$, $\dfrac{\partial}{\partial n} u_k(P) = o,$

and

(iv) the radiation conditions

$$\lim_{r \to \infty} \left| r^{\mu+\nu+1} \overset{s}{\underset{k}{u}}(P) \right| < \infty, \quad \lim_{r \to \infty} \left| r^{\mu+\nu+1} \frac{\partial}{\partial r} \overset{s}{\underset{k}{u}}(P) \right| < \infty$$

be satisfied.

For the equation (3.1) we can define the exterior Neumann problem as in the following:

(i) For each u_o (P) let there be the functions $u^i{}_o$ (P) and $u_o{}^s$ (P) such that

$$u_o(P) = \overset{i}{\underset{o}{u}}(P) + \overset{s}{\underset{o}{u}}(P),$$

(ii) For every $P \in V_e$, $\Delta_\Sigma [u^s{}_o(P)] = o$

(iii) For every $P \in S$, $\dfrac{\partial}{\partial n} u_o(P) = o$

and

(iv) $\overset{s}{\underset{o}{u}}(P)$ be regular at the infinity.

2. *Derivation of the integral equations.*

Let us denote the solutions of the equation $\Delta_\Sigma[u] = o$ in V_e by $u^s{}_0$, and in V_i by $u^i{}_0$. Let $u_0 = u^i{}_0 + u^s{}_0$ be satisfied. Let the functions $u^s{}_0$ and $u^i{}_0$ be continuous when P is approached to the singularity planes from the two different directions.

Denoting the sphere centered at P' by B_ε, let us write the Green identity for the operator Δ_Σ in the region V_e:

$$\int_{V_e - B_\varepsilon} x^{2\mu} y^{2\nu} \left[\overset{s}{\underset{o}{u}} \, \Delta_\Sigma R^{-(2\mu+2\nu+1)} - R^{-(2\mu+2\nu+1)} \Delta_\Sigma \overset{s}{\underset{o}{u}} \right] d\tau \qquad (3.5)$$

$$= - \int_{S+\partial B_\varepsilon} x^{2\mu} y^{2\nu} \left[\overset{s}{\underset{o}{u}} \frac{\partial}{\partial n} R^{-(2\mu+2\nu+1)} - R^{-(2\mu+2\nu+1)} \frac{\partial}{\partial n} \overset{s}{\underset{o}{u}} \right] d\sigma.$$

We easily obtain

$$\frac{1}{(2\mu+2\nu+1)W} \int_S x^{2\mu} y^{2\nu} \left[\overset{s}{\underset{o}{u}} \frac{\partial}{\partial n} R^{-(2\mu+2\nu+1)} - R^{-(2\mu+2\nu+1)} \frac{\partial}{\partial n} \times \overset{s}{\underset{o}{u}} \right] ds$$

$$= \begin{cases} \overset{s}{\underset{o}{u}} (P'), & P' \in V_e \\ \frac{1}{2} \overset{s}{\underset{o}{u}} (P'), & P' \in S \\ o, & P' \in V_i \end{cases} \qquad (3.6)$$

where

$$W = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^{2\mu+2\nu+1}\theta \cos^{2\mu}\varphi \sin^{2\nu}\varphi \, d\theta.$$

Similarly, we write the Green identity for the operator Δ_Σ in the region V_i:

$$\int_{V_i - B_\varepsilon} x^{2\mu} y^{2\nu} \left[\overset{i}{\underset{o}{u}} \, \Delta_\Sigma R^{-(2\mu+2\nu+1)} - R^{-(2\mu+2\nu+1)} \Delta_\Sigma \overset{i}{\underset{o}{u}} \right] d\tau \qquad (3.7)$$

$$= \int_{S+\partial B_\varepsilon} x^{2\mu} y^{2\nu} \left[\overset{i}{\underset{o}{u}} \frac{\partial}{\partial n} R^{-(2\mu+2\nu+1)} - R^{-(2\mu+2\nu+1)} \frac{\partial}{\partial n} \overset{i}{\underset{o}{u}} \right] d\sigma$$

and obtain

$$\frac{1}{(2\mu+2\nu+1)W} \int_S x^{2\mu} y^{2\nu} \left[\overset{i}{\underset{o}{u}} \frac{\partial}{\partial n} R^{-(2\mu+2\nu+1)} - R^{-(2\mu+2\nu+1)} \frac{\partial}{\partial n} \right.$$

$$\left. \times \overset{i}{\underset{o}{u}} \right] ds$$

$$= \begin{cases} o & , P' \in V_e \\ -\frac{1}{2} \overset{i}{\underset{o}{u}} (P'), & P' \in S \\ -\overset{i}{\underset{o}{u}} (P') & , P' \in V_i \, . \end{cases} \quad (3.8)$$

Using the operator h_Σ we obtain the following corresponding equations:

$$\frac{1}{W} \int_S x^{2\mu} y^{2\nu} \left[\overset{s}{\underset{k}{u}} \frac{\partial}{\partial n} R^{(\mu+\nu+\frac{1}{2})} N_{\mu+\nu+\frac{1}{2}} (kR) - R^{-(\mu+\nu+\frac{1}{2})} N_{\mu+\nu+\frac{1}{2}} (kR) \right.$$

$$\left. \times \frac{\partial}{\partial n} \overset{s}{\underset{k}{u}} \right] ds$$

$$= \begin{cases} -M \overset{s}{\underset{k}{u}} (P') & , P' \in V_e \\ -\frac{1}{2} \underset{o}{M} \overset{s}{\underset{k}{u}} (P') & , P' \in S \\ & , P' \in V_i \end{cases} \quad (3.9)$$

and

$$\frac{1}{W} \int_S x^{2\mu} y^{2\nu} \left[\overset{i}{\underset{k}{u}} \frac{\partial}{\partial n} R^{-(\mu+\nu+\frac{1}{2})} N_{\mu+\nu+\frac{1}{2}} (kR) - R^{-(\mu+\nu+\frac{1}{2})} N_{\mu+\nu+\frac{1}{2}} (kR) \right.$$

$$\left. \times \frac{\partial}{\partial n} \overset{i}{\underset{k}{u}} \right] ds$$

$$= \begin{cases} o & , P' \in V_e \\ \frac{1}{2} M u^i_k (P') & , P' \in S \\ M u^i_k (P') & , P' \in V_i \end{cases} \quad (3.10)$$

where

$$M = \frac{k}{\pi} \left(\frac{2}{k} \right)^{\mu+\nu+\frac{3}{2}} \Gamma(\mu+\nu+\tfrac{3}{2}).$$

Here u^s_k is a solution of $h_\Sigma\,[u] = o$ in V_e and, u^i_k is a solution in V_i; besides $u_k = u^s_k + u^i_k$. If we choose $u^i_0 = 1$ we get

$$\frac{1}{(2\mu+2\nu+1)W} \int_S x^{2\mu} y^{2\nu} \frac{\partial}{\partial n} R^{-(2\mu+2\nu+1)}\, ds = \begin{cases} o & , P' \in V_e \\ -\frac{1}{2} & , P' \in S \\ -1 & , P' \in V_i \end{cases} \quad (3.11)$$

from (3.8). From (3.6) and (3.8) we obtain

$$u^i_o(P') + \frac{1}{(2\mu+2\nu+1)W} \int_S x^{2\mu}\, y^{2\nu}\, u_o\,(P) \frac{\partial}{\partial n} R^{-(2\mu+2\nu+1)}\, ds$$

$$= \begin{cases} U_o(P) & , \quad P' \in V_e \\ \frac{1}{2} u_o(P) & , \quad P' \in S \\ 0 & ; \quad P' \in V_i \end{cases} \quad (3.12)$$

Now let us consider the case $P' \in S$:

$$2\, u^i_o\,(P') + \frac{2}{(2\mu+2\nu+1)W} \int_S x^{2\mu} y^{2\nu}\, u_o\,(P) \frac{\partial}{\partial n} R^{-(2\mu+2\nu+1)}$$

$$x\ ds = u_o\,(P'),\ P' \in S \quad (3.13)$$

This integral equation can be written

$$u_o - T\,u_o = 2\,u^i_o$$

where

$$T\ u_o = \frac{2}{(2\mu+2\nu+1)W} \int_S x^{2\mu} y^{2\nu}\, u_o\,(P) \frac{\partial}{\partial n} R^{-(2\mu+2\nu+1)}\, dS.$$

Since

$$|\, Tu \,| = \left| \frac{2}{(2\mu+2\nu+1)W} \int_S x^{2\mu} y^{2\nu}\, u \frac{\partial}{\partial n} R^{-(2\mu+2\nu+1)}\, ds \right|$$

$$\leq \max |\, u \,|$$

T is not necessarily a contraction mapping. Therefore it is not possible to obtain a solution using the interative method defined as

$$u^{(0)}_o\,(P') = 2\,u^i_o\,(P'),$$

$$u^{(1)}_o\,(P') = 2\,u^i_o\,(P') + \frac{1}{(2\mu+2\nu+1)W} \int_S x^{2\mu} y^{2\nu}\, u^{(0)}_o\,(P) \frac{\partial}{\partial n}$$

$$\vdots \qquad x\ R^{-(2\mu+2\nu+1)}\, ds,$$

$$\overset{(n+1)}{\underset{o}{u}}(P') = 2\, u^i_o(P') + \frac{1}{(2\mu+2\nu+1)W} \int_S x^{2\mu} y^{2\nu} \overset{(n)}{\underset{o}{u}}(P) \frac{\partial}{\partial n} \times R^{-(2\mu+2\nu+1)}\, ds.$$

But the integral equation

$$u^i_o(P') + \frac{1}{(2\mu+2\nu+1)W} \int_S x^{2\mu} y^{2\nu} [u_o(P) - u_o(P')] \frac{\partial}{\partial n} \times R^{-(2\mu+2\nu+1)}\, dSp = u_o(P'),\ P' \in V_e \cup S. \qquad (3.14)$$

obtained from (3.11) and (3.12) is more convenient than (3.13).

In a similar way

$$-M u^i_k(P') + \frac{1}{W} \int_S x^{2\mu} y^{2\nu} u_k(P) \frac{\partial}{\partial n} R^{-(\mu+\nu+\frac{1}{2})} N_{\mu+\nu+\frac{1}{2}}(kR)\, ds = \begin{cases} -M u_k(P') & , P' \in V_e \\ -\frac{1}{2} M u_k(P') & , P' \in S \\ o & , P' \in V_i \end{cases} \qquad (3.15)$$

can be obtained for the Helmholtz equation (3.2) by the addition of (3.9) and (3.10). Now if (3.11) is multiplied by $Mu_k(P')$ and is added to (3. 15)

$$u^i_k(P') + \frac{1}{(2\mu+2\nu+1)W} \int_S x^{2\mu} y^{2\nu} [u_k(P) - u_k(P')] \frac{\partial}{\partial n_p} \times R^{-(2\mu+2\nu+1)}\, dSp$$

$$- \frac{1}{W} \int_S x^{2\mu} y^{2\nu} u_k(P) \frac{\partial}{\partial n_p} \left[\frac{1}{M} R^{-(\mu+\nu+\frac{1}{2})} N_{\mu+\nu+\frac{1}{2}}(kR) + \frac{1}{(2\mu+2\nu+1)} R^{-(2\mu+2\nu+1)} \right] dSp$$

$$= u_k(P'),\quad P' \in V_e \cup S \qquad (3.16)$$

is obtained. Now let us introduce the representations

$$L_o u = \frac{1}{(2\mu+2\nu+1)W} \int_S x^{2\mu} y^{2\nu} [u(P) - u(P')] \frac{\partial}{\partial n_p}$$

$$xR^{-(2\mu+2\nu+1)}\, dSp \tag{3.17}$$

and

$$L\,u = L_0\, u - \frac{1}{W}\int_S x^{2\mu} y^{2\nu}\, u(P)\, \frac{1}{\partial n_p}\left[\frac{1}{M} R^{-(\mu+\nu+\frac{1}{2})} N_{\mu+\nu+\frac{1}{2}}(kR) + \frac{R^{-(2\mu+2\nu+1)}}{(2\mu+2\nu+1)}\right] dSp. \tag{3.18}$$

So we can write the integral equations (3.14) and (3.16) as:

$$(I - L_0)\, u_0 = u^i_{\,0}, \tag{3.19}$$

$$(I - L)\, u_k = u^i_{\,k} \tag{3.20}$$

3. *Derivation of Solutions*

In this section, we wish to obtain the solutions of the operator equations (3.19) and (3.20) using these Neumann series [1], [2].

$$u_0 = \sum_{n=0}^{\infty} L_0^n\, u^i_{\,0} \tag{3.21}$$

$$u_k = \sum_{\nu=0}^{\infty} L^n\, u^i_{\,k}. \tag{3.22}$$

Let us denote the space of the real-valued functions on S by $C_{\mathbb{R}}$ (S). Let $f \in C_{\mathbb{R}}$ (S). Define the norm of the function f as

$$\| f \| = \sup_{p \in S} | f(P) | \tag{3.23}$$

It is easy to show that $C_{\mathbb{R}}$ (S) is a Banach space with the norm (3.23) [6]. Let K be a linear and real operator defined on S, that is

$$K: C_{\mathbb{R}}(S) \rightarrow C_{\mathbb{R}}(S).$$

Let us define the norm of K as

$$\| K \|_{\mathbb{R}} = \sup_{\substack{u \in C_{\mathbb{R}}(S) \\ u \not\equiv_{\mathbb{R}} 0}} \frac{\| Ku \|}{\| u \|} \tag{3.24}$$

Then

$$L_o \;:\; C_{|R}(S) \to C_{|R}(S)$$
$$L \;:\; C_{|R}(S) \to C_{|R}(S)$$

and $u \in C_{|R}(S)$.

If

$$\| L_o \|_{|R} < 1, \quad \| L \|_{|R} < 1 \tag{3.25}$$

the solution of the operator equations (3.19) and (3.20) are respectively (3.21) and (3.22). Now let us show the existence of the inequalities (3.25). If we take the absolute value of (3.17) we get

$$| L_o u | \leq \frac{1}{(2\mu+2\nu+1)\ |W|} \int_S | u(P) - u(P') | \ | x^{2\mu} y^{2\nu} \frac{\partial}{\partial n_p}$$
$$\times R^{-(2\mu+2\nu+1)} | \, dSp. \tag{3.26}$$

Since u is continuous, $| u(P) - u(P') |$ is also continuous for $P \in S$. Let us assume that

$$| u(P) | \neq | u(P') |$$

if $P \neq P'$. Consequently we can write

$$| u(P) - u(P') | < 2 \| u \|, \ \forall \ P, P' \in S. \tag{3.27}$$

If we consider $P' \in S$ in (3.11)

$$| L_o u | < \| u \|$$

is obtained. Now taking the supremum over S, we get

$$\| L_o u \| < \| u \|.$$

Therefore

$$\| L_o \|_{|R} = \sup_{\substack{u \in C_{|R}(S) \\ u \neq o}} \frac{\| L_o u \|}{\| u \|} < 1 \tag{3.28}$$

is obtained. Since $u \in C_{|R}(S)$, the supremum of u (P) is the maximum value of u (P).

Now let us show that

$$\| L \|_{|R} < 1$$

Define the operator L_k as,

$$L_k u = \frac{1}{W} \int_S x^{2\mu} y^{2\nu} u(P) \frac{\partial}{\partial n_p} \left[\frac{1}{M} R^{-(\mu+\nu+\frac{1}{2})} N_{\mu+\nu+\frac{1}{2}}(kR) + \frac{R^{-(2\mu_2+2\nu+1)}}{(2\mu+2\nu+1)} \right] dSp \quad (3.29)$$

The equation (3.18) takes the form

$$Lu = L_0 u - L_k u.$$

So, we have

$$| Lu | \leq | L_0 u | + | L_k u |,$$

hence

$$|| L ||_{\mathcal{R}} \leq || L_0 ||_{\mathcal{R}} + || L_k ||_{\mathcal{R}}. \quad (3.30)$$

On the other hand we can obtain

$$| L_k u | \leq \frac{|| u ||}{| W |} \int_S | x^{2\mu} y^{2\nu} \frac{\partial}{\partial n_p} \left[\frac{1}{M} R^{-(\mu+\nu_2+\frac{1}{2})} N_{\mu+\nu+\frac{1}{2}}(kR) + \frac{R^{-(2\mu+2\nu+1)}}{(2\mu+2\nu+1)} \right] | dSp$$

$$\leq \frac{|| u ||}{| W |} \int_S | x^{2\mu} y^{2\nu} \left[\frac{\pi (kR)^{\mu+\nu+\frac{3}{2}} N_{\mu+\nu+\frac{3}{2}}(kR)}{2^{\mu+\nu+\frac{3}{2}} \Gamma(\mu+\nu+\frac{3}{2})} \frac{1}{R^{2\mu+2\nu+1}} + \frac{1}{R^{2\mu+2\nu+1}} \right] | dSp \quad (3.31)$$

In order to estimate the value of the integral at the right side we need the following property of the Bessel function of the second kind. For the sufficiently small values of k we can write

$$(kR)^{\mu+\nu+\frac{3}{2}} N_{\mu+\nu+\frac{3}{2}}(kR) = - \frac{1}{\pi} 2^{\mu+\nu+\frac{3}{2}} \Gamma(\mu+\nu+\tfrac{3}{2}) + \frac{O[(kR)^2]}{2^{-(\mu+\nu-\frac{1}{2})} \Gamma(\frac{1}{2}-\mu-\nu) \sin[(\mu+\nu+\frac{3}{2})\pi]}$$

In this manner (3.31) takes the form below

$$| L_k u \ | \leq \frac{|| u ||}{| W |} \int_S | x^{2\mu} y^{2\nu}$$

$$\times \frac{O\ [(kR)^2]}{4\ \Gamma(\frac{1}{2}-\mu-\nu)\ \Gamma(\mu+\nu+\frac{3}{2})\ R^{2\mu+2\nu+2} \sin\ [(\mu+\nu+\frac{3}{2})\pi\ ]} \ | dS.$$

$$\leq C\ || u ||\ k^2.$$

Thus we obtain

$$|| L_k u || \leq C\ || u ||\ k^2$$

and consequently

$$|| L_k ||_{\mathcal{R}} = \sup_{\substack{u \in C_{\mathcal{R}}(S) \\ u \not\equiv o}} \frac{|| L_k u ||}{|| u ||} \leq Ck^2 \qquad (3.32$$

So, (3.30) gives us

$$|| L ||_{\mathcal{R}} \leq || L_o ||_{\mathcal{R}} + ck^2 \qquad (3.33)$$

By (3.33) we find that

$$|| L ||_{\mathcal{R}} < 1$$

for the sufficiently small values of k. Therefore the solutions of the operator equations (3.19) and (3.20) can be given by the Neumann series

$$u_o = \sum_{n=o}^{\infty} L_o^n \ u^i{}_o, \ u_k = \sum_{n=o}^{\infty} L^n \ u^i{}_k$$

respectively.

As a result of the above discussion we obtained

Theorem 3.

Let, S be a smooth closed surface and

$$L: C_{\mathcal{R}}(S) \rightarrow C_{\mathcal{R}}(S)$$

be a linear operator defined by (3.18). Then there exists a k such that

$$|| L ||_{\mathcal{R}} < 1.$$

ÖZET

Bu makalede Genelleştirilmiş Çok Simetrili Helmholtz denklemi için, iç ve dış problem incelenmiştir. Çalışma üç bölümden oluşmuştur. Birinci bölümde iç Dirichlet problemi, bir iç çarpım ve bir çekirdek fonksiyon tanımlanarak incelenmiştir. İkinci bölümde ise dış Dirichlet probleminin çözümü elde edilmiş, ve bu çözümün tekliği kanıtlanmıştır. Bir integral denklem sistemi yardımıyla dış Neumann probleminin incelenmesi son bölümü oluşturmaktadır.

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