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An Integral Formula And Inverse Fundamental Forms On Hypersurfaces In Riemannian Manifolds

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An Integral Formula And Inverse Fundamental Forms On Hypersurfaces In Riemannian Manifolds

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ABSTRACT

In this study, the coefficients of the p-fundamental forms of a hypersurface N imbedded in n-dimensional Riemannian space M were expressed in terms of the coefficients of first and second fundamental forms. Then, by means of Cayley-Hamilton theorem, the inverse S^{-1} of the shape operator S on the hypersurface N was written as the combinations of the powers of S and the curvatures $K_1, \dots K_{n-1}$. Thus the new fundamental forms and some properties of them called the inverse fundamental forms, were defined and investigated. As a result of an application of the generalized divergence theorem of Gauss to the divergence relations of certain tensor fields over the region R of N that can be expressed in terms of polynomials involving the new defined curvatures of M an integral formula was obtained.

I. INTRODUCTION

Let N be a hypersurface imbedded in an n-dimensional Riemannian space M and S be the shape operator defined on N. The metric tensor g_{ij} of M is assumed to be of class C^3, it induces a metric on N defined by

$$g_{\alpha\beta} = \sum_{i,j=1}^{n} g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \quad (1.1)$$

Where it is supposed that M refers to local coordinates x^i and that the hypersurface N is represented parametrically by the equations

$$x^i = x^i (u^\alpha),\ i = 1, \dots, n;\ \alpha = 1, \dots, n-1 \qquad (1.2).$$

in which the variables u^α denote the parameters of N, the functions x^i are assumed to be of class C^2 in all their arguments and it is supposed that

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$$\text{rank} \begin{bmatrix} \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^1}{\partial u^{n-1}} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial x^n}{\partial u^1} & \cdots & \frac{\partial x^n}{\partial u^{n-1}} \end{bmatrix} = n - 1 \qquad (1.3)$$

That is $\det S \neq 0$. We will denote $\frac{\partial x^i}{\partial u^\alpha}$ by $X^i{}_\alpha$.

Mixed covariant derivatives of M will be denoted by a pair of vertical bars. $\Gamma^h{}_{pq}$ and $\Gamma^\lambda{}_{\alpha\beta}$ denote the Christoffel symbols of M and induced metric tensor, respectively. Mixed covariant derivative of $X^h{}_\alpha$ with respect to u^β had been given by H. Rund [2] as

$$X^h{}_{\alpha\,||\beta} = \frac{\partial X^h{}_\alpha}{\partial u^\beta} - \sum_{\lambda=1}^{n-1} \Gamma^\lambda{}_{\alpha\beta} X^h{}_\lambda + \sum_{p,q=1}^{n} \Gamma^h{}_{pq} X^p{}_\alpha X_\beta{}^q \qquad (1.4).$$

And furthermore, it is known that [2]

$$X^j{}_{\alpha\,||\beta} = \Omega_{\alpha\beta} \cdot \xi^j, \; j = 1, \ldots, n. \qquad (1.5)$$

where the $\Omega_{\alpha\beta}$ are the coefficients of the usual second fundamental form and ξ^j, $j = 1, \ldots, n$, define the unique unit normal vector field to N. If we differentiate the expressions

$$\sum_{i,j=1}^{n} g_{ij}\, \xi^i \xi^j = 1 \text{ and } \sum_{i,j=1}^{n} g_{ij}\, X^i{}_\alpha\, \xi^j = 0$$

with respect to u^β and by using the equation (1.5) [4], we obtain

$$\Omega^\varepsilon_\beta = \sum_{\alpha=1}^{n-1} g^{\varepsilon\alpha}\, \Omega_{\alpha\beta} \qquad (1.6)$$

and

$$\xi^j{}_{||\beta} = - \sum_{\varepsilon=1}^{n-1} \Omega^\varepsilon_\beta\, X^j_\varepsilon \qquad (1.7)$$

Now, let $I^{(p)}{}_{\alpha\beta}$ denote the coefficients of the p^{th} – fundamental form, then we know that [3],

$$I^{(p)}_{\alpha\beta} = \sum_{\lambda=1}^{n-1} I^{(p-1)}_{\alpha\lambda} \; \Omega^{\lambda}_{\beta} \; , 2 \leq p \leq n, \qquad (1.8).$$

If we write this expression (p–1) times for $2 \leq p \leq n$ and substitude the coefficients of each fundamental form in those which come later, then we have

$$I^{(p)}{}_{\alpha\beta} = \sum_{\lambda_1,..,\lambda_{p-2}=1}^{n-1} \Omega_{\alpha\lambda_{p-2}} \; \Omega^{\lambda_{p-2}}_{\lambda_{p-3}} \; \Omega^{\lambda_{p-3}}_{\lambda_{p-4}} \ldots \Omega^{\lambda_3}_{\lambda_2} \; \Omega^{\lambda_2}_{\lambda_1} \; \Omega^{\lambda_1}_{\beta} \quad (1.9)$$

or using Eq (1.6)

$$I^{(p)}{}_{\alpha\beta} = \sum_{\substack{\lambda_1,\ldots,\lambda_{p-2}=1 \\ \alpha_1,\ldots,\alpha_{p-2}=1}}^{n-1} g^{\lambda_1\alpha_1} g^{\lambda_2\alpha_2} \ldots g^{\lambda_{p-2}\alpha_{p-2}} \Omega_{\alpha_1\beta} \Omega_{\alpha_2\lambda_1} \Omega_{\alpha_3\lambda_2} \ldots \Omega_{\alpha\lambda_{p-2}} \quad (1.10)$$

Thus, we have obtained the coefficients of p^{th} – fundamental form in terms of the coefficients of the first and second fundamental forms.

II. INVERSE FUNDAMENTAL FORMS

We will denote the roots of the characteristic equation of S by $\lambda_1,\ldots,\lambda_{n-1}$. In addition p^{th} curvature K_p of N is defined to be the p^{th} elementary symmetric functions of $\lambda_1,\ldots,\lambda_{n-1}$ by

$$K_p = \sum_{\nu_1 \leq \ldots \leq \nu_p} \lambda_{\nu_1} \ldots \lambda_{\nu_p} \qquad (11.1).$$

By means of Cayley-Hamilton theorem and the characteristic polynomial of S, we may write

$$(-1)^{n-1} S^{n-1} + \sum_{p=1}^{n-1} (-1)^{n-p-1} K_p S^{n-p-1} = 0 \qquad (11.2)$$

or

$$S[S^{n-2} + (-1)^1 K_1 S^{n-3} + (-1)^2 K_2 S^{n-4} + \ldots$$
$$+ (-1)^{n-2} K_{n-2} I_{n-1} + (-1)^{n-1} K_{n-1} S^{-1}] = 0. \qquad (11.3)$$

where I_{n-1} is the identity matrix having the order $(n-1)$ and S^{-1} is the inverse of S with respect to matrix multiplication. Hence we have

$$S^{-1} = \frac{1}{K_{n-1}} [(-1)^n S^{n-2} + (-1)^{n-1} K_1 S^{n-3} + \dots + (-1)^2 K_{n-2} I_{n-1}] \quad (11.4)$$

since it is supposed that det $S \neq 0$. This expression have been telling us that the inverse of a regular linear map S is expressed by means of the curvatures $K_1, \dots, K_{n-1}$ and the powers of S. Furthermore, S^{-1} is a symmetric, regular and linear map. Thus, we may define new forms, so-called inverse fundamental forms by using S^{-1} in place of S.

Let $I_{(p)}$ denote the p^{th} inverse fundamental form, particularly we can take $I^{(1)} = I_{(1)}$.
We have similarly, from (11.4)

$$(S^{-1})^2 = \frac{1}{K_{n-1}} [(-1)^n S^{n-3} + (-1)^{n-1} K_1 S^{n-4} + \dots + (-1)^3 K_{n-3} I_{n-1} + (-1)^2 K_{n-2} S^{-1}]. \quad (11.5)$$

Finally, if we denote $(S^{-1})^{n-1} = S^{-(n-1)}$, we obtain

$$S^{-(n-1)} = \frac{1}{K_{n-1}} [\sum_{p=1}^{n-1} (-1)^{n-p+1} K_{p-1} S^{-(p-1)}]. \quad (11.6)$$

Because of the definition of the fundamental forms, we have a linear relation as

$$\sum_{p=1}^{n} (-1)^{n-p+1} K_{p-1} I_{(p)} = 0, \quad (11.7)$$

where it is assumed that $Ko = 1$.

Let $v_1, \dots, v_{n-1}$ be this basis vectors of the characteristic spaces corresponding to characteristic values $\lambda_1, \dots, \lambda_{n-1}$, then, we have

$$S(v_\alpha) = \lambda_\alpha v_\alpha, \quad \alpha = 1, \dots, n-1. \quad (11.8)$$

or

$$S^{-1}(\lambda_\alpha v_\alpha) = v_\alpha, \ \alpha = 1, \dots, n-1.$$

For S^{-1} is a linear map, we have

$$S^{-1}(v_\alpha) = \frac{1}{\lambda_\alpha} v_\alpha, \ \alpha = 1,\ldots, n-1. \qquad (11.9)$$

Therefore $\frac{1}{\lambda_\alpha}$ scalars are the characteristic values of S^{-1} and the basis vectors of the characteristic subspaces corresponding them are the same vectors $Y_1,\ldots, Y_{n-1}$. The matrix of S^{-1} with respect to this basis is

$$S^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \ldots & 0 \\ 0 & \frac{1}{\lambda_2} & \ldots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \ldots & \frac{1}{\lambda_{n-1}} \end{bmatrix} . \qquad (11.10)$$

Thus, we can define the new curvatures under the auspices of S^{-1} as

$$IK_p = \sum_{\nu_1 \leq \ldots \leq \nu_p} \frac{1}{\lambda_{\nu 1}} \cdots \frac{1}{\lambda_{\nu p}}, \ p = 1,\ldots, n—1. \qquad (11.11)$$

Consequently there exists a relation between K_p and IK_p represented by

$$IK_p = (K_{n-1})^{-1} \ K_{n-(p+1)} \qquad (11.2)$$

where $(K_{n-1})^{-1}$ denotes the inverse function of K_{n-1}.

In that case, we can write the relation (11.7), being $K_{n-1} \neq 0$, as

$$\sum_{p=1}^{n} (-1)^{n-p} \ IK_{n-p} \ I_{(p)} = 0. \qquad (11.3)$$

III. A GENERALIZED INTEGRAL FORMULA

Let $I_{(p)}$ denote the coefficients of the p^{th} inverse fundamental form. Let us define the coefficients of the associated inverse fundamental form by the relations

$$(-1)^{\varepsilon+1} Q_{(\varepsilon)\lambda'\varepsilon'} = \sum_{\gamma=1}^{\varepsilon} (-1)^{\gamma} IK_{n-\gamma-1} I_{(\gamma)\lambda'\varepsilon'}, \quad \varepsilon = 1,\ldots, n-1. \qquad (111.1)$$

By the aid of those, we will define an integral formula over an (n–1)-dimensional region R of M and its boundry ∂R, which is an (n-2) dimensional manifold. It is supposed that R refers to local coordinates u^{α}, and that the boundry ∂R is represented parametrically by the equations

$$u^{\alpha} = u^{\alpha} (v^{\beta'}), (\alpha = 1,\ldots, n-1; \beta' = 1,\ldots, n-2), \qquad (111.2)$$

where the variables $v^{\beta'}$ denote the parameters of ∂R. The functions u^{α} in (III.2) are assumed to be of class C^2 in all their arguments. Metric tensors of R and ∂R, $g_{\alpha\beta}$ and $g_{\alpha'\beta'}$ respectively. are assumed to be of class C^3, $g_{\alpha\beta}$ induces the metric tensor $g_{\alpha'\beta'}$ on ∂R defined as, [2],

$$g_{\alpha'\beta'} = \sum_{\alpha,\beta=1}^{n-1} g_{\alpha\beta} \frac{\partial u^{\alpha}}{\partial v^{\alpha'}} \frac{\partial u^{\beta}}{\partial v^{\beta'}}. \qquad (111.3)$$

The volume elements of R and ∂R are also, defined by

$$ds = \det (g_{\alpha\beta})\, du^1 \ldots du^{n-1},$$
$$ds' = \det (g_{\alpha'\beta'})\, dv^1 \ldots dv^{n-2}, \qquad (111.4)$$

respectively. These can be written as

$$\sum_{\alpha,\beta=1}^{n-1} g_{\alpha\beta}\, n^{\alpha} \frac{\partial u^{\beta}}{\partial v^{\beta'}} = 0 \text{ and } \sum_{\alpha,\beta=1}^{n-1} g_{\alpha\beta}\, n^{\alpha} n^{\beta} = 1, \qquad (III.5)$$

where n^{α} ($\alpha = 1,\ldots, n-1$) are the components of unit normal vector field over R and the restriction of this vector field over ∂R has the component $n^{\alpha'}$.

Now, let $Y_{\alpha} = Y_{\alpha} (u^{\beta})$ be a covariant vector field defined over the region R. The components of Y_{α} vector field are assumed to be of class C^1 throughout R. The projection of Y_{α} vector field on ∂R boundary has the components

$$Y_{\beta'} = \sum_{\alpha=1}^{n-1} Z^{\alpha}{}_{\beta'} Y_{\alpha}, \quad \beta' = 1,\ldots, n-2 \qquad (III.6)$$

where $Z^{\alpha}{}_{\beta'} = \dfrac{\partial u^{\alpha}}{\partial v^{\beta'}}$.

The length of the projection on the unit normal vector field of Y_α vector field is

$$q = \sum_{\alpha=1}^{n-1} Y_\alpha n^\alpha. \tag{III.7}$$

Replacing this in Eq (III.6) we have the mixed covariant derivative of $Y_{\beta'}$ with respect to $v^{\alpha'}$

$$Y_{\beta'||\alpha'} = q\, I^{(2)}{}_{\beta'\alpha'} + \sum_{\alpha,\beta=1}^{n-1} Y_{\alpha|\beta} Z^\alpha{}_{\beta'} Z^\beta{}_{\alpha'} \tag{III.8}$$

where $Y_{\alpha|\beta}$ denotes the covariant derivative of Y_α with respect to v^β. Let assume that

$$P_{\alpha'\beta'} = \sum_{\alpha,\beta=1}^{n-1} Y_{\alpha|\beta} Z^\alpha{}_{\beta'} Z^\beta{}_{\alpha'} \quad , \tag{III.9}$$

then we have

$$Y_{\beta'||\alpha'} = q\, I^{(2)}{}_{\beta'\alpha'} + P_{\alpha'\beta'} \quad . \tag{III.10}$$

We may define the new functions for $1 \leq \alpha \leq n-2$ by

$$\begin{aligned} H^2{}_\alpha &= H^2{}_\alpha (|K_{n-(\alpha+2)}), \\ H^3{}_\alpha &= H^3{}_\alpha (|K_{n-(\alpha+2)}, |K_{n-(\alpha+3)}) \\ &\;\;\vdots \\ H_\alpha{}^{n-1} &= H_\alpha{}^{n-1} (|K_{n-(\alpha+2)}, |K_{n-(\alpha+3)}, \ldots, |K_{n-(\alpha+n-2)}). \end{aligned} \tag{III.11}$$

It is easy to express these functions more explicitly. Hence, we can write the inverse fundamental forms as a linear combination of the ordinary fundamental forms in terms of the functions

$$I_{(\varepsilon)\alpha'\beta'} = \sum_{\alpha=1}^{n-1} \overset{\varepsilon}{\underset{\alpha}{H}}\, I^{(\alpha)}{}_{\alpha'\beta'} \quad , \quad \varepsilon = 1, \ldots, n-1. \tag{III.12}$$

So we can obtain from (III.1) the following:

$$(-1)^{\varepsilon+1} Q_{(\varepsilon)\lambda'\varepsilon'} = \sum_{\gamma=1}^{\varepsilon} \sum_{\alpha=1}^{n-1} (-1)^\gamma |K_{n-\gamma-1} H^\gamma{}_\alpha I^{(\alpha)}{}_{\lambda'\varepsilon'}. \tag{III.13}$$

On the other hand it is written from elementary tensor calculus [4]

$$Q_{(\varepsilon)}{}^{\alpha'\beta'} = \sum_{\lambda',\varepsilon'=1}^{n-2} g^{\alpha'\lambda'} g^{\beta'\varepsilon'} Q_{(\varepsilon)\lambda'\varepsilon'} \tag{III.14}$$

or from (III.13),

$$Q_{(\varepsilon)}{}^{\alpha'\beta'} = \sum_{\lambda',\varepsilon'=1}^{n-2} \sum_{\gamma=1}^{\varepsilon} \sum_{\alpha=1}^{n-1} (-1)^{\gamma-\varepsilon-1} g^{\alpha'\lambda'} g^{\beta'\varepsilon'} \mathbb{K}_{n-\gamma-1} H^{\gamma}{}_{\alpha} I^{(\alpha)}{}_{\lambda'\varepsilon'} . \tag{III.15}$$

M_ε is called the associated curvatures on ∂R that had been defined by H. Rund [1] in the following way:

$$M_\varepsilon = \sum_{\alpha',\beta'=1}^{n-2} g^{\alpha'\beta'} I_{\alpha'\beta'}{}^{(\varepsilon+1)}, \ \varepsilon = 1,\dots, n-3. \tag{III.16}$$

Thus, it can be written

$$I^{(\varepsilon)}{}_{\alpha'\beta'} = g_{\varepsilon'\lambda'} M_{\varepsilon-1} \tag{III.17}$$

then, we have

$$Q_{(\varepsilon)}{}^{\alpha'\beta'} = \sum_{\gamma=1}^{\varepsilon} \sum_{\alpha=1}^{n-1} (-1)^{\gamma-\varepsilon-1} g^{\alpha'\beta'} M_{\varepsilon-1} \mathbb{K}_{n-\gamma-1} H^{\gamma}{}_{\alpha}. \tag{III.18}$$

Now, let us define the contravariant vector fields on ∂R as follows,

$$Q^{\alpha'}{}_{(\varepsilon)} = \sum_{\beta'=1}^{n-2} (-1)^{\varepsilon+1} Q_{(\varepsilon)}{}^{\alpha'\beta'} Y_{\beta'}. \tag{III.19}$$

The divergence of the $Q^{\alpha'}{}_{(\varepsilon)}$, with respect to $v^{\alpha'}$, is given by

$$Q^{\alpha'}{}_{(\varepsilon)||\alpha'} = \sum_{\beta'=1}^{n-2} (-1)^{\varepsilon+1} (Q^{\alpha'\beta'}{}_{(\varepsilon)||\alpha'} Y_{\beta'} + Q^{\alpha'\beta'}{}_{(\varepsilon)} Y_{\beta'||\alpha'}), \tag{III.20}$$

Assuming

$$\sum_{\gamma=1}^{\varepsilon} (-1)^{\gamma} M_{\varepsilon-1} \mathbb{K}_{n-\gamma-1} H^{\gamma}{}_{\alpha} = \mathbb{R}^{\gamma}_{\varepsilon\alpha}$$

from (III.10) and (III.18) it follows that

$$Q^{\alpha}{}_{(\varepsilon)||\alpha'} = \sum_{\beta'=1}^{n-2} \sum_{\alpha=1}^{n-1} g^{\alpha'\beta'} \mathbb{R}^{\gamma}{}_{\varepsilon\alpha||\alpha'} Y_{\beta'} + \mathbb{R}^{\gamma}_{\varepsilon\alpha} (qM_1 + g^{\alpha'\beta'} P_{\alpha'\beta'}). \tag{III.21}$$

Finally by using the divergence theorem of Gauss and for $n^{\alpha'} = \sum_{\beta'=1}^{n-2} g^{\alpha'\beta'} n_{\beta'}$, we obtain an integral formula as follows,

$$\int_{\partial R} n_{\alpha'} Q^{\alpha'}(\varepsilon)\, ds' = \int_{R} \sum_{\alpha=1}^{n-1} \sum_{\beta'=1}^{n-2} \Big[g^{\alpha'\beta'} \frac{\partial R^{\gamma}{}_{\varepsilon\alpha}}{\partial v^{\alpha'}} Y_{\beta'} + R^{\gamma}_{\varepsilon\alpha} \left(q M_1 + g^{\alpha'\beta'} P_{\alpha'\beta'} \right) \Big] ds. \qquad \text{(III.22)}$$

ÖZET

Rıemann Manifoldlarının Hiperyüzeyleri Üzerinde İnvers Temel Formlar Ve Bir İntegral Formülü

Bu çalışmada n-boyutlu Riemann uzayı M nin bir N hiperyüzeyinin p tane formu, birinci ve ikinci temel formların katsayıları sayesinde ifade edildi. Sonra N hiperyüzeyi üstünde tanımlanan S şekil operatörünün inversi S^{-1} Cayley-Hamilton teoremi sayesinde S nin kuvvetleri ve yüksek mertebeden $K_1, \dots K_{n-1}$ Gauss eğrilikleri cinsinden yazıldı. Böylece invers temel formlar dediğimiz yeni temel formlar ve onların bazı özellikleri tanımlanıp incelendi. Daha sonra N nin birR bölgesi üzerinde bazı tensör alanlarının divergensinin M nin yeni tanımlanan eğriliklerini kapsayan polinomlar cinsinden olan ifadesine Gauss'un genelleştirilmiş divergens teoremini uygulamak suretiyle genel bir integral formülü elde edildi.

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