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S. K. VAISH and S. C. GUPTA

20

Faculté des Sciences de l'Université d'Ankara
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## A Theorem On The Geometric Mean Of An Entire Function

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1. Let $f(z) = \sum_{n=o}^{\infty} a_n z^n$, $z = re^{i\theta}$, be a power series which converges for all z, and let

$$M(r) = \max_{|z|=r} |f(z)|,$$

$$\mu(r) = \max_{o \leq n < \infty} |a_n| r^n,$$

and

$$\nu(r) = \max \{N : \mu(r) = |a_N| r^N\}.$$

The mth ($m \geq 1$) order $\varrho$ for f(z) is given by

$$(1.1) \qquad \limsup_{r \to \infty} \frac{\log \log M(r)}{\log^{[m]} r} = \rho(m) \equiv \rho,$$

where $\log^{[o]} x = x$ and $\log^{[n]} x = \log (\log^{[n-1]} x)$ for $0 < \log^{[n-1]} x < \infty$.

Polya and Szegö [2] defined the geometric mean of f(z) for $|z| = r$ as:

$$(1.2) \qquad G(r,f) = \exp \left\{ \frac{1}{2\pi} \int_o^{2\pi} \log |f(re^{i\theta})| \, d\theta \right\}.$$

The above geometric mean for the nth derivative $f^{(n)}(z)$ of f(z) is given by

$$(1.3) \qquad G(r,f^{(n)}) = \exp \left\{ \frac{1}{2\pi} \int_o^{2\pi} \log |f^{(n)}(re^{i\theta})| \, d\theta \right\}.$$

On the lines of the proof of the following two results by Lakshminarasimhan [1] and Shah and Ishaq [3], respectively,

$$\limsup_{r\to\infty} \frac{\log \nu(r)}{\log r} = \rho(1)$$

and

$$\limsup_{r\to\infty} \frac{\log \nu(r)}{\log \log r} = \rho(2)-1,$$

we can easily prove that

$$\limsup_{r\to\infty} \frac{\log \nu(r)}{\log^{[m]} r} = \Phi, \quad 0 \leq \Phi \leq \infty. \tag{1.4}$$

where

$$\begin{aligned} \rho &= \Phi && \text{if } m = 1 \\ &= 1+\Phi && \text{if } m = 2 \\ &= \infty && \text{if } m \geq 3. \end{aligned}$$

In this paper we obtain a theorem relating the geometric means $G(r,f)$ and $G(r,f^{(n)})$ with the mth order $\rho(m)$.

2. *Theorem* 1. *For every entire function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *of mth order* $\rho$, *we find*

$$\limsup_{r\to\infty} \frac{\log \left\{ r \left( \frac{G(r,f^{(1)})}{G(r,f)} \right) \right\}}{\log^{[m]} r} = \Phi \tag{2.1}$$

*in the neighbourhood of points where* $|f(z)| > M(r)\,\nu^{-1/8}$.

*Proof.* From (1.3), we have

$$G(r,f^{(1)}) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |f^{(1)}(re^{i\theta})| \, d\theta \right\} \tag{2.2}$$

$$= \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f^{(1)}(re^{i\theta})}{f(re^{i\theta})} \right| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta \right\}$$

Also, we have [4], in the neighbourhood of points, where $| f(z) | > M(r) \nu^{-1/8}$,

$$(2.3) \qquad \frac{f^{(1)}(z)}{f(z)} = \{ 1+h(z) \ \nu(R)^{-1/16}\} \ \frac{\nu(r)}{z} \ , \ |h| < k.$$

On using (1.2) and (2.3) in (2.2), we find

$$G(r,f^{(1)}) = G(r,f)\exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log \left( \left| 1+h(z)\nu(R)^{-1/16} \right| \left| \frac{\nu(r)}{z} \right| \right) d\theta \right\}.$$

This gives

$$(2.4) \qquad G(r,f^{(1)}) > G(r,f) \ \frac{\nu(r)}{r} \ (1-k \ \nu(R)^{-1/16})$$

and

$$(2.5) \qquad G(r,f^{(1)}) < G(r,f) \ \frac{\nu(r)}{r} \ (1+k \ \nu(R)^{-1/16})$$

in the neighbourhood of points where $|f(z)| > M(r) \ \nu^{-1/8}$.

Proceeding to superior limits, as $r \to \infty$, on both the sides in (2.4) and (2.5) and using (1.4), we get

$$\limsup_{r\to\infty} \frac{\log \left\{ r \left( \frac{G(r,f^{(1)})}{G(r,f)} \right) \right\}}{\log^{[m]} r} = \limsup_{r\to\infty} \frac{\log \nu(r)}{\log^{[m]} r} = \Phi$$

This proves Theorem 1.

*Corollary* 1. For the entire function f(z) of mth order $\rho$,

$$(2.6) \qquad \limsup_{r\to\infty} \frac{\log \left\{ r \left( \frac{G(r,f^{(n)})}{G(r,f)} \right)^{1/n} \right\}}{\log^{[m]} r} = \Phi$$

Writing (2.4) and (2.5) for the sth derivative of f(z), we get

$$G(r,f^{(s)}) > G(r,f^{(s-1)}) \frac{\nu(r)}{r} (1-k\ \nu(R)^{-1/16})$$

and

$$G(r,f^{(s)}) < G(r,f^{(s-1)}) \frac{\nu(r)}{r} (1+k\ \nu(R)^{-1/16}),$$

respectively. Taking $s = 1,2, \ldots, n$, multiplying all the inequalities thus obtained and proceeding to limit superior, (2.6) follows.

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**DR. S. K. VAISH**
Department of Mathematics,
University of Rookee,
Roorkee-247667 (U.P.),
INDIA.

**PROF. S. C. GUPTA**
Department of Mathematics,
Sahu Jain College,
Najibabad-246763 (U.P.),
INDIA.