COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A_1: Mathématiques

TOME : 32 ANNÉE : 1983

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19

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences de l'Université d'Ankara

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Properties of 2-Dimensional Ruled Surfaces In The Euclidean n-Space E^n And Massey's Theorem

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(Received June 6, 1983, and accepteed August 9. 1983)

ABSTRACT

In this paper we find new characteristic properties for 2-dimensional ruled surfaces M in E^n and we give the sufficient and necessary conditions for which the ruled surface M is to be total geodesic. In addition, the Massey's theorem which is well-known for the ruled surfaces in the Euclidean 3-space, [3], was generalized for the ruled surfaces in E .

I. INTRODUCTION

We will assume throughout this paper that all manifolds, maps, vector fields, etc. ... are differentiable of class C^∞. Consider a general submanifold M of the Euclidean n-space E^n. Suppose that $\bar{D}$ is the Riemann connection of E^n, while D is the Riemann connection of M. Then, if X and Y are the vector fields of M and if V is the second fundamental form of M, we have by decomposing $\bar{D}_X Y$ in a tangential and a normal component

$$(I.1) \qquad \bar{D}_X Y = D_X Y + V(X,Y).$$

The equation (I.1) is called *Gauss equation.*

If ξ is any normal vector field on M, we find the Weingarten equation by decomposing $\bar{D}_X \xi$ in a tangential component and a normal component

$$(I.2) \qquad \bar{D}_X \xi = - A_\xi(X) + \overset{\perp}{D}_X \xi.$$

A_ξ determines at each point a self- adjoint linear map and $\overset{\perp}{D}$ is a metric connection in the normal bundle $\overset{\perp}{\chi}(M)$. We use the same notation A_ξ for the linear map and the matrix of the linear map. A normal vector field ξ is called *paralel* in the normal bundle $\overset{\perp}{\chi}(M)$ if we have $\overset{\perp}{D}_X\xi = 0$ for each $X \in \chi(M)$. If η is a normal unit vector at the point $p \in M$, then

$$\text{(I.3)} \qquad G(p,\eta) = \det A_\eta$$

is the Lipschitz-Killing curvature of M at p in the direction η.

Suppose that X and Y are vector fields on M, while ξ is a normal vector field, then, if the standard metric tensor of E^n is denoted by $<\ ,\ >$

$$\text{(I.4)} \qquad X<Y,\xi> = <\bar{D}_X Y,\xi> + <Y,\bar{D}_X\xi> = 0$$

or

$$<V(X,Y),\xi> = <Y,A_\xi(X)> .$$

If $\xi_1, \xi_2, \ldots, \xi_{n-2}$ constitute an orthonormal base field of the normal bundle $\overset{\perp}{\chi}(M)$, then we set

$$\text{(I.5)} \qquad <V(X,Y),\ \xi_i> = V_i(X,Y)$$

or

$$V(X,Y) = \sum_{i=1}^{n-2} V_i(X,Y)\xi_i.$$

The mean curvature vector H of M at the point p is given by

$$\text{(I.6)} \qquad H = \sum_{i=1}^{n-2} \operatorname{tr}A_{\xi_i}/2.\xi_i.$$

$\|H\|$ is the mean curvature. If $H=0$ at each point p of M, then M is said to be *minimal.*

II. 2-DIMENSIONAL RULED SURFACES IN THE EUCLIDEAN n-SPACE E^n

Suppose that the base curve r(s) of the 2-dimensional ruled surface M in E^n is an orthogonal trajectory of the generators, which have the direction of the unit vector e(s); then M can locally be represented by

$\varphi(s,l) = r(s) + le(s)$.

DEFINITION II.1: Let M be a 2-ruled surface in E^n and V be the second fundamental form of M. If $V(X,X) = 0$ for all $X \in \chi(M)$, then X is called an *asymptotic vector field* on M.

THEOREM II.1: Let M be a 2-dimensional ruled surface in E^n. Then the generators of M are asymptotics and geodesics of M.

Proof: Since the generators are the geodesics of E^n, we have

$$\bar{D}_e e = 0.$$

If we set this in the Gauss equation, we get

$$D_e e + V(e,e) = 0 \text{ or } D_e e = - V(e,e).$$

Since $D_e e \in \chi(M)$ and $V(e,e) \in \chi^{\perp}(M)$ we find $D_e e = 0$ and $V(e,e) = 0$

Therefore the generators of M are the asymptotics and geodesics of M.

Suppose that $\{e_1, e\}$ is an orthonormal base field of the tangential bundle $\chi(M)$ and $\{\xi_1, \xi_1, \ldots, \xi_{n-2}\}$ is an orthonormal base field of the normal bundle $\chi^{\perp}(M)$. Then we have the following equations.

$$\bar{D}_e \xi_j = a^j{}_{11} e + a^j{}_{12} e_1 + \sum_{i=1}^{n-2} b^j{}_{1i} \xi_i$$

(II.1)

$$\bar{D}_{e_1} \xi_j = a^j{}_{12} e + a^j{}_{22} e_1 + \sum_{i=1}^{n-2} b^j{}_{2i} \xi_i, \quad 1 \leq j \leq n-2.$$

From these equations we observe that

$$A\xi_j = - \begin{bmatrix} a^j{}_{12} & a^j{}_{12} \\ a^j{}_{12} & a^j{}_{22} \end{bmatrix}, \quad 1 \leq j \leq n\text{-}2.$$

Since $\bar{D}_e \xi_j$ and $\bar{D}_{e_1} \xi_j$ are orthogonal to ξ_j, we have $b^j{}_{1j} = b^j{}_{2j} = 0$

On the other hand, $a^j{}_{11} = \langle \bar{D}_e \xi_j, e \rangle = - \langle \xi_j, \bar{D}_e e \rangle$ and $\bar{D}_e e$ 0, thus we find $a^j{}_{11} = 0$, $1 \leq j \leq n-2$. We also have

(II.2) $$a^j_{12} = <\bar{D}_e \xi_j, e_1> = - <\xi_j, \bar{D}_e e_1>$$

and

(II.3) $$<\bar{D}_e e_1, e> = - <e_1, \bar{D}_e e> = 0$$

while

(II.4) $$<\bar{D}_e e_1, e_1> = - <e_1, \bar{D}_e e_1> = 0.$$

From (II.3) and (II.4) we observe that $\bar{D}_e e_1 \in \overset{\perp}{\chi}(M)$ or $\bar{D}_e e_1 = V(e,e_1)$, because of (II.2) we have

(II.5) $$\bar{D}_e e_1 = V(e,e_1) = \sum_{j=1}^{n-2} <\xi_j, \bar{D}_e e_1> = - \sum_{j=1}^{n-2} a^j_{12} \xi_j \ .$$

Because of (I.4) and (II.1) we find

(II.6) $$a^j_{22} = <\bar{D}_e \xi_j, e_1> = - <A_{\xi_j}(e_1), e_1> = - <V(e_1,e_1), \xi_j>$$

and

(II.7) $$\mathrm{tr} A_{\xi_j} = - a^j_{22} = <V(e_1,e_1), \xi_j>, \ 1 \leq j \leq n-2.$$

THEOREM II.2: Let M be a 2-ruled surface in E^n and $\{e_1, e\}$ be the orthonormal base field of M. Then the Gauss curvature G is given by

$$G = - <\bar{D}_e e_1, \bar{D}_e e_1>$$

where $\bar{D}$ denotes the Riemann connection of E^n, [4].

By using Theorem II.2 and (II.5) we find

(II.8) $$G = - \sum_{j=1}^{n-2} (a^j_{12})^2 \ .$$

On the other hand, because of the expressions stated in (I.6) and (II.7) we have

(II.9) $$H = \sum_{j=1}^{n-2} \frac{<V(e_1,e_1), \xi_j> \xi_j}{2} = 1/2 \ V(e_1,e_1) \ .$$

DEFINITION II.2: Let M be a 2-ruled surface in E^n. If the tangent planes of M are constant along the generators of M, M is called *developable*, [2].

DEFINITION II.3: Let M be a 2-dimensional ruled surface in E^n and V be a second fundamental form of M. If

$$V(X,Y) = 0$$

for all $X,Y \in \chi(M)$, then M is called *totally geodesic*, [1].

THEOREM II.3: A 2-ruled surface M in E^n is developable and minimal iff M is total geodesic.

Proof: We assume that M is developable and minimal. If $X,Y \in \chi(M)$, we have $X=ae+be_1$ and $Y=ce+de_1$. Therefore we get

$$V(X,Y) = acV(e,e)+(ad+bc)V(e,e_1)+bdV(e_1,e_1). \tag{II.10}$$

Because of Theorem II.1 and minimality of M we have $V(e,e) = 0$ and $V(e_1,e_1) = 0$. Moreover, since M is developable $\bar{D}_e e_1 = 0$. Thus we can write $V(e,e_1) = 0$ and $V(X,Y) = 0$ for all $X,Y \in \chi(M)$.

Now, suppose that $V(X,Y) = 0, \forall X,Y \in \chi(M)$. Then we have $V(e,e) = 0$, $V(e,e_1) = 0$ and $V(e_1,e_1) = 0$. Because of Theorem II.1 we have

$$\langle \bar{D}_e e_1, e \rangle = 0 \text{ and } \langle \bar{D}_e e_1, e_1 \rangle = 0.$$

This means that $\bar{D}_e e_1$ is a normal vector field or $\bar{D}_e e_1 = V(e,e_1)$.

Therefore we have $\bar{D}_e e_1 = 0$. This implies that M is developable and $V(e_1,e_1) = 0$ implies that M is minimal.

That completes the proof of the theorem.

III. THE MASSEY'S THEOREM FOR 2-DIMENSIONAL RULED SURFACES IN THE EUCLIDEAN n-SPACE E^n

Consider a 2-dimensional ruled surface M in E^n and the unit vector field e of the generator, then the orthonormal base field $\{e_1,e\}$ of the tangential bundle of M at each point p of M and the orthonormal base field $\{\xi_1,\xi_2, \ldots, \xi_{n-2}\}$ of the normal bundle of M at each point p of M constitute an orthonormal base field of E^n at each point p of E^n.

On the other hand, we have the equations of covariant derivative of the orthonormal base field $\{e_1,e,\xi_1,\xi_2, \ldots, \xi_{n-2}\}$ of E^n, in matrix form, as follows:

$$\text{(III.1)} \quad \begin{bmatrix} \bar{D}_{e_1} e_1 \\ \bar{D}_{e_1} e \\ \bar{D}_{e_1} \xi_1 \\ \cdots\cdots \\ \bar{D}_{e_1} \xi_{n-2} \end{bmatrix} = \begin{bmatrix} o & c_{12} & c_{13} & \cdots & c_{1n} \\ -c_{12} & o & c_{23} & \cdots & c_{2n} \\ -c_{13} & -c_{23} & o & \cdots & c_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -c_{1n} & -c_{2n} & -c_{3n} & \cdots & o \end{bmatrix} \begin{bmatrix} e_1 \\ e \\ \xi_1 \\ \cdots \\ \xi_{n-2} \end{bmatrix}$$

Now, we would like to generalize the Massey's theorem, which is well-known for the ruled surfaces in E^3, [3], to the ruled surfaces in the Euclidean n-space E^n.

THEOREM III.1: Let, M be a 2-dimensional ruled surface in E^n, $\{e_1,e\}$ be an orthonormal base field of the tangential bundle $\underset{\lambda}{\overset{\gamma}{-}}$ (M) and r(s) be an orthogonal trajectory of the generators of M. Then the following propositions are equivalent.

(i) M is developable.

(ii) The Lipschitz-Killing curvature

$G(p,\xi_j) = 0, \; 1 \leq j \leq n-2.$

(iii) The Gauss curvature $G = 0$.

(iv) In the equation (III.1), $c_{2k} = 0, \; 3 \leq k \leq n$.

(v) $A_{\xi_j}(e) = 0$.

(vi) $\bar{D}_{e_1} e \in \underset{\lambda}{\overset{\gamma}{-}}(M)$.

Proof: (i) ⇒ (ii): We assume that M is developable. Since $a^j{}_{11} = 0$, in (II.1), $1 \leq j \leq n-2$, the Lipschitz-Killing curvature at point p in the direction of ξ_j is given by

$$G(p,\xi_j) = -(a^j{}_{12}(p))^2 = 0, \; 1 \leq j \leq n-2.$$

Because of (II.5) and since M is developable we have

$$\bar{D}_e e_1 = -\sum_{j=1}^{n-2} (a^j{}_{12}) \, \xi_j = 0.$$

So we find $G(p,\xi_j) = 0, \; 1 \leq j \leq n-2$.

(ii) ⇒ (iii): Let $G(p,\xi_j) = 0, \; 1 \leq j \leq n-2$. Since we have, [4],

$$G(p) = \sum_{j=1}^{n-2} G(p,\xi_j), \ \forall p \in M$$

we observe that $G = 0, \forall p \in M$.

(iii) $\Rightarrow$ (iv): Suppose that $G = 0, \forall p \in M$. Then, because of (II.8) we have $a^j_{12} = 0, 1 \leq j \leq n-2$. So $\bar{D}_{e_1} \xi_j$ has no component in the direction e. Hence we observe that $c_{2k} = 0, 3 \leq k \leq n$, in the equation (III.1).

(iv) $\Rightarrow$ (v): Suppose $c_{2k} = 0, 3 \leq k \leq n$, in the equation (III.1). This shows that $\bar{D}_{e_1}\xi_j$ has no component in the direction e. Thus we have, in the equation (II.1), $a^j_{12} = 0, 1 \leq j \leq n-2$.

Moreover, since $a^j_{11} = \langle \bar{D}_e\xi_j, e \rangle = - \langle \xi_j, \bar{D}_e e \rangle = 0$ and because of the Weingarten equation we find

$$A_{\xi_j}(e) = 0, \ 1 \leq j \leq n-2.$$

(v) $\Rightarrow$ (vi): Let $A_\xi(e) = 0$. Then, from the Weingarten equation, we have $a^j_{11} = 0, a^j_{12} = 0, 1 \leq j \leq n-2$. Moreover, since $\langle e, \xi_j \rangle = 0$ implies $\langle \bar{D}_{e_1} e, \xi_j \rangle = - \langle e, \bar{D}_{e_1}\xi_j \rangle = - a^j_{12}$, we find

$$\langle \bar{D}_{e_1} e, \xi_j \rangle = 0.$$

So we get

$$\bar{D}_{e_1} e \in \frac{\gamma}{\lambda}(M).$$

(vi) $\Rightarrow$ (i): Let $\bar{D}_{e_1} e \in \frac{\gamma}{\lambda}(M)$. Then $\langle \bar{D}_{e_1} e, \xi_j \rangle = a^j_{12} = 0, 1 \leq j \leq n-2$. On the other hand, $\langle e_1, e_1 \rangle = 1$ implies that $\langle \bar{D}_e e_1, e_1 \rangle = 0$ and $\langle e_1, e \rangle = 0$ implies that $\langle \bar{D}_e e_1, e \rangle = 0$. Thus $\bar{D}_e e_1 \in \frac{\gamma}{\lambda}(M)$.

Because of (II.5) and since $a^j_{12} = 0, 1 \leq j \leq n-2$, we write that $\bar{D}_e e_1 = 0$.

This means the tangent planes of M are constant along the generator e of M, i.e. M is developale.

COROLLARY III.2: Let M be a 2-dimensional ruled surface in E^n with a Gauss curvature being zero. If M is minimal, then $c_{sk} = 0$, $1 \leq s \leq 2, 3 \leq k \leq n$.

Proof: Let M be minimal. Then from the equation (II.9), we have $V(e_1,e_1) = 0$. If this result is set in the Gauss equation, we find

$$\bar{D}_{e_1}e_1 = D_{e_1}e_1.$$

This means that $\bar{D}_{e_1}e_1$ has no component in $\overset{\perp}{\chi}(M)$. Therefore we have

(III.1) $\qquad c_{1k} = 0, \; 3 \leq k \leq n,$

in the equation (III.1). On the other hand, since $G = 0$, by hypothesis, and from the Theorem III.1, we know that $c_{2k} = 0$, $3 \leq k \leq n$. If we consider this together with (III.1), we observe that $c_{sk} = 0$, $1 \leq s \leq 2$, $3 \leq k \leq n$.

ÖZET

E^n, n-boyutlu Öklid uzayında tanımlı 2-boyutlu regle yüzeylerinin minimal ve açılabilir olması için gerek ve yeter şartın total geodezik olması gösterildi ve M ile gösterilen bu yüzeyler için yeni karakteristik özellikler bulundu. Ayrıca, 3-boyutlu Öklid uzayında tanımlı regle yüzeyler için iyi bilinen Massey teoremi, [3], E^n deki 2-regle yüzeyler için genelleştirildi.

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