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by

V. KUMAR and S.L. SHUKLA

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On A Libera Integral Operator For Certain Univalent Functions

V. KUMAR and S.L. SHUKLA

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In this paper we study the Libera integral operator $F(z) = \frac{2}{z}\int_0^z f(t)\,dt$ for certain univalent functions. The results obtained are sharp and improve some known results of Goel and Sohi, and Livingston for the univalent functions having negative coefficients.

1. INTRODUCTION

Let P (α,β) denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are regular in the unit disc $U = \{ z : |z| < 1 \}$ and satisfy

$$|\{f'(z)-1\} / \{f'(z) + (1-2\alpha)\}| < \beta, \; z \in U$$

where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. Equivalently, $f(z) \in P(\alpha,\beta)$ if and only if there exists a function $w(z)$ regular in U and satisfying $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$ such that

$$f'(z) = \frac{1 + \beta(1-2\alpha)w(z)}{1-\beta w(z)}, \; z \in U.$$

It is well known that the functions in P (α,β) are univalent in U. Clearly $P(\alpha_2,\beta) \subset P(\alpha_1,\beta)$ if $\alpha_1 < \alpha_2$ and $P(\alpha, \beta_1) \subset P(\alpha, \beta_2)$ if $\beta_1 < \beta_2$. Also, $f(z) \in P(\alpha, 1)$ if and only if $\text{Re}\, \{f'(z)\} > \alpha$, $z \in U$. Let us identify $P(\alpha, 1) \equiv P(\alpha)$ and $P(0) \equiv P$.

Libera [3] showed that, if $f(z) \in P$, then so does the function F(z) defined by

$$F(z) = \frac{2}{z} \int_0^z f(t)\ dt. \tag{1.1}$$

Subsequently Livingston [4] considered the converse problem and proved that, if $F(z) \in P$, then $f(z) \in P$ in $|z| < (\sqrt{5}-1)/2$. Recently Goel and Sohi [1] have improved the result of Libera by showing that $F(z) \in P(1/5)$. In this note, our aim is to show that, if $f(z) \in P(\alpha,\beta)$ and the coefficients from the second on in the Taylor expansion of $f(z)$ are negative, then $F(z)$ belongs to a certain subclass of $P(\alpha,\beta)$. We also consider the converse problem. In particular our results improve the above mentioned results of Goel and Sohi, and Livingston for the functions in P having negative coefficients.

We require the following lemma which is due to Gupta and Jain [2].

LEMMA A. Let $f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n$. Then $f(z) \in P(\alpha, \beta)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n\ (1+\beta)}{2\beta(1-\alpha)}\ |a_n| \leq 1. \tag{1.2}$$

The result is sharp.

2. MAIN RESULTS

THEOREM 2.1. If $f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n \in P(\alpha, \beta)$, then the function $F(z)$ defined by (1.1) belongs to $P(\gamma, \beta)$ where $\gamma = (1+2\alpha)/3$. Further, the result is sharp.

Proof. Since $f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n$, by (1.1), we have $F(z) = z - \sum_{n=2}^{\infty} |b_n|\, z^n$, where $|b_n| = \left(\frac{2}{n+1}\right) |a_n|$. Let $F(z) \in P\ (\sigma, \beta)$, then, by Lemma A, it holds if and only if

$$\sum_{n=2}^{\infty} \frac{n\ (1+\beta)}{2\ \beta(1-\sigma)}\ |b_n| \leq 1. \tag{2.1}$$

In order to show that $F(z) \in P(\gamma, \beta)$ we want to find the maximum value of σ provided (2.1) is satisfied. Now, in view of (1.2), the inequality (2.1) holds if

$$\frac{n(1+\beta)}{2\beta(1-\sigma)} |b_n| \leq \frac{n(1+\beta)}{2\beta(1-\alpha)} |a_n|, \text{ for each } n = 2, 3, \ldots \text{ or if}$$

$$\sigma \leq \frac{n-1+2\alpha}{n+1}$$

$$= \gamma_n, \text{ say, for each } n = 2, 3, \ldots .$$

Clearly, $\gamma = \inf_{n \geq 2} \gamma_n$. It is easy to verify that γ_n is an increasing function of n. Therefore $\Upsilon = \Upsilon_2 = (1 + 2\alpha)/3$. Hence $F(z) \in P(\gamma, \beta)$.

In order to establish the sharpness we take

$f(z) = z - [\beta(1-\alpha)/(1+\beta)] z^2$. Clearly, $f(z) \in P(\alpha, \beta)$ and

$F(z) = z - [2\beta(1-\alpha)/(3(1+\beta))] z^2$. Now

$$|\{F'(z) - 1\}/\{F'(z) + (1-2\gamma)\}| = \beta, \text{ for } z = 1.$$

Hence the result is sharp.

This completes the proof of theorem.

NOTE. If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, then, by Lemma A, it is easy to verify that $f(z) \in P(\gamma, \beta)$ if and only if $f(z) \in P(\alpha, \delta)$, where $\gamma = (1+2\alpha)/3$ and $\delta = 2\beta/(3+\beta)$. Consequently Theorem 2.1 can also be stated in the following equivalent form.

THEOREM 2.2. If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in P(\alpha, \beta)$, then the function F(z) defined by (1.1) belongs to $P(\alpha, \delta)$, where $\delta = 2\beta/(3+\beta)$. The result is sharp.

COROLLARY 2.1. If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in P$, then the funct.on F(z) defined by (1.1) belongs to $P(1/3)$. The result is sharp.

The above corollary improves the result of Goel and Sohi [1] mentioned in the introduction for the functions in P having negative coefficients.

THEOREM 2.3. If $F(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in P(\alpha, \beta)$, then the function f(z) defined in (1.1.) belongs to $P(\rho)$ in $|z| < R(\alpha, \beta, \rho)$ where

$$R(\alpha, \beta, \rho) = \inf_{n \geq 2} \left[\frac{(1-\rho)(1+\beta)}{\beta(1-\alpha)(n+1)} \right]^{1/(n-1)}.$$

The result is sharp.

Proof. Since $F(z) \in P(\alpha, \beta)$, we have

$$\sum_{n=2}^{\infty} \frac{n(1+\beta)}{2\beta(1-\alpha)} |a_n| \leq 1. \tag{2.2}$$

Also, by the representation of f(z) we have $f(z) = z - \sum_{n=2}^{\infty} \left(\frac{n+1}{2}\right) |a_n| z^n$. Since $|f'(z)-1| < (1-\rho)$ implies $\operatorname{Re}\{f'(z)\} > \rho$, it suffices to show that $|f'(z)-1| < (1-\rho)$ holds in $|z| < R(\alpha, \beta, \rho)$. Now

$$|f'(z)-1| = \left| - \sum_{n=2}^{\infty} \frac{n(n+1)}{2} |a_n| z^{n-1} \right|$$
$$\leq \sum_{n=2}^{\infty} \frac{n(n+1)}{2} |a_n| |z|^{n-1}.$$

The right hand side of this inequality is less then $(1-\rho)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{2(1-\rho)} |a_n| |z|^{n-1} < 1. \tag{2.3}$$

But, in view of (2.2), the inequality (2.3) holds if

$$\frac{n(n+1)}{2(1-\rho)} |a_n| |z|^{n-1} < \frac{n(1+\beta)}{2\beta(1-\alpha)} |a_n|, \text{ for each}$$

$n = 2, 3, \ldots$, or if

$$|z| < \left[\frac{(1-\rho)(1+\beta)}{\beta\,(1-\alpha)(n+1)}\right]^{1/(n-1)}, \quad \text{for each } n = 2, 3, \ldots .$$

Hence $f(z) \in P(\rho)$ in $|z| < R(\alpha, \beta, \rho)$.

To show the sharpness we take $F(z) = z - [2\,\beta\,(1-\alpha)/(n(1+\beta))]\,z^n$. Then $f(z) = z - [\beta\,(1-\alpha)(n+1)/(n(1+\beta))]\,z^n$ and, therefore

$$f'(z) = 1 - [\beta(1-\alpha)(n+1)/(1+\beta)]\,z^{n-1}$$

$$= \rho, \quad \text{for } z = [(1-\rho)(1+\beta)/(\beta(1-\alpha)(n+1))]^{1/(n-1)}.$$

Hence the result is sharp.

This completes the proof of theorem.

Since $R(0, 1, 0) = 2/3$, we obtain the following corollary which improves the result of Livingston [4] mentioned in the introduction for the functions in P having negative coefficients.

COROLLARY 2.2. If $F(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n \in P$, then the function $f(z)$ defined in (1.1) belongs to P in $|z| < 2/3$. The result is sharp.

REFERENCES

1. **R.M. Goel and N.S. Sohi,** Subclasses of univalent functions, Tamkang J. Math. 11 (1980), 77–81.

2. **V.P. Gupta and P.K. Jain,** Certain classes of univalent functions with negative coefficients II. Bull. Austral. Math. Soc. 15 (1976), 467–473.

3. **R.J. Libera,** Some classes of regular univalent functions, Proc. Amer. Math. Soc. 16 (1965), 755–758.

4. **A.E. Livingston,** On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 17 (1966), 352–357.

Department of Mathematics
Janta College
Bakewar 206124
Etawah (U.P.)
India.