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Integrability Of Power Series With Quasi-Monotone Coefficients*

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ABSTRACT

In the present note a generalization of a theorem of Jain [1] is obtained.

1. A non-decreasing, continuous real-valued function Φ defined on the non-negative half line and vanishing only at the origin is called an Orlicz function (OF). Function $\Phi \in \text{OF}$ is said to satisfy Δ_2 -condition for large u if there are constants $C > 0$ and $u_0 \geq 0$ such that

$$\Phi(2u) \leq C \Phi(u), \text{ for } u \geq u_0.$$

A sequence $\{a_n\}$ of non-negative numbers is said to be quasi-monotone if for some $\alpha > 0$,

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha}{n}\right).$$

An equivalent definition of quasi-monotone sequence is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$.

We write

$$F(x) = \sum a_k x^k \quad (0 \leq x < 1);$$

and

$$A_n = a_0 + a_1 + \dots + a_n.$$

*This is based on Chapter III of the author's Ph. D. Thesis entitled "Integrability of function represented by Trigonometric series" submitted under the supervision of Dr. Z.U. Ahmad at Aligarh Muslim University, Aligarh in 1980.

2. We take our start from the work of Woyezynski [4] who proved the following theorem.

Theorem A. If, $a_n \geq a_{n+1} \geq 0$ ($n = 1, 2, \dots$) then

$$F(x) \in L_\Phi(0, 1),$$

if and only if

$$\{A_n\} \in L_\Phi(N, V),$$

where Φ is an Orlicz function, satisfying Δ_2 -condition, $d\mu = dx$, N stands for the set of all positive integers and V is the measure of N concentrating the mass n^{-2} at all point $n \in N$.

Recently, Jain [1] generalized Theorem A by replacing the condition $a_n \downarrow 0$ by the less stringent condition that $\{a_n\}$ is quasi-monotone in the following form.

Theorem B. Let Φ be an Orlicz function satisfying Δ_2 -condition. If $\{a_n\}$ is quasi-monotone sequence such that $0 < B_1 \leq n^\beta \leq B_2$ with some $\beta > 0$, ($n = 1, 2, \dots$) then, for $0 \leq V < 1$,

$$(1-x)^{-V} \Phi(|F(x)|) \in L(0, 1),$$

if and only if

$$\sum_{n=1}^{\infty} n^{V-2} \Phi(A_n) < \infty.$$

We prove the following theorem.

Theorem. Let Φ be an Orlicz function satisfying Δ_2 -condition, and let $\lambda(t)$ be a positive, non-increasing, integrable function on the interval $0 < t \leq 1$. If $\{a_n\}$ is quasi-monotone sequence such that $0 < B_1 \leq n^\beta a_n \leq B_2$ with some $\beta > 0$, ($n = 1, 2, \dots$), then we have

$$\lambda(1-x) \Phi(|F(x)|) \in L(0, 1),$$

if and only if

$$\sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \Phi(A_n) < \infty.$$

In the case when $\lambda(t) = t^{-V}$ ($V < 1$), our theorem reduces to Theorem B.

4. Proof of the theorem. Set $y = 1 - x$. Since, $\left(1 - \frac{1}{n}\right)^n$ is an

increasing sequence, we have, for $\frac{1}{n+1} \leq y \leq \frac{1}{n}$, $n \geq 2$:

$$\begin{aligned} F(1-y) &\geq \sum_{k=0}^n a_k (1-y)^k \\ &\geq \left(1 - \frac{1}{n}\right) \sum_{k=0}^n a_k \\ &\geq \frac{1}{4} A_n. \end{aligned}$$

Thus, we get

$$F(1-y) \geq \frac{1}{4} A_n, \text{ for } \frac{1}{n+1} \leq y \leq \frac{1}{n}, n \geq 2.$$

Now,

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda \left(\frac{1}{n}\right) n^{-2} \Phi(A_n) &\leq 2 \sum_1^{\infty} \int_{1/n+1}^{1/n} \lambda(y) \Phi(A_n) dy \\ &\leq 2 \int_{1/2}^1 \lambda(y) \Phi(A_1) dy + 2 \sum_{n=2}^{\infty} \int_{1/n+1}^{1/n} \lambda(y) \Phi(A_n) dy \\ &\leq 0(1) + B \sum_{n=2}^{\infty} \int_{1/n+1}^{1/n} \lambda(y) \Phi(F(1-y)) dy \\ &\leq 0(1) + B \int_0^1 \lambda(1-x) \Phi(F(x)) dx \\ &< \infty, \end{aligned}$$

by hypothesis.

Conversely, we have

$$\begin{aligned}
\int_0^1 \lambda(1-x) \Phi(F(x)) dx &= \sum_{n=2}^{\infty} \int_{1-\frac{1}{n+1}}^{1-\frac{1}{n}} \lambda(1-x) \Phi(F(x)) dx. \\
&= \sum_{n=2}^{\infty} \int_{1/n}^{1/n+1} \lambda(x) \Phi(F(1-x)) dx \\
&= \sum_{n=2}^{\infty} \int_{1/n}^{1/n-1} \lambda(x) \Phi\left(\sum_{k=0}^{\infty} a_k(1-x)^k\right) dx \\
&\leq \sum_{n=2}^{\infty} \int_{1/n}^{1/n-1} \lambda(x) \Phi\left(\sum_{k=0}^{\infty} a_k \left(1-\frac{1}{n}\right)^k\right) dx \\
&= 0(1) \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \Phi\left(\sum_{k=0}^{\infty} a_k \left(1-\frac{1}{n}\right)^k\right) \\
&= 0(1) \sum_{n=2}^{\infty} \lambda(1/n) n^{-2} \Phi\left(\sum_{k=0}^{\infty} \sum_{j=nk}^{n(k+1)} a_j \left(1-\frac{1}{n}\right)^j\right) \\
&= 0(1) \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \Phi\left(\sum_{k=0}^{\infty} e^{-k} \left(\sum_{j=0}^n a_j + \sum_{nk}^{n(k+1)} a_j\right)\right) \\
&= 0(1) \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \Phi\left(\sum_{k=0}^{\infty} e^{-k} (A_n + Bn^{1-\beta} (k+1))\right) \\
&= 0(1) \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \Phi\left(\sum_{k=0}^{\infty} B(k+2) e^{-k} A_n\right) \\
&= 0(1) \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \Phi(A_n), \\
&< \infty,
\end{aligned}$$

by hypothesis.

This completes the proof of the theorem.

REFERENCES

- [1] **Jain. P.**, Integrability of power series, Proc. Amer. Math. Soc., 42 (1974), 569-594.
- [2] **Krasnoselskii, M.A. and Rutickii, ya. B.**. Convex functions and Orlicz space, Groningen, 1961.
- [3] **Szasz, O.**. Quasi-monotone series, Amer. Jour. Math., 70 (1948), 203-206.
- [4] **Woyczynski. W.A.** Positive coefficients element of Hardy-Orlicz space, Colloq. Math., 21 (1970), 103-110.