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The Singular Solutions of the Generalized Poly-Axially Symmetric Helmholcz Equations

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The Singular Solutions of the Generalized Poly-Axially Symmetric Helmholtz Equations

İ. Ethem ANAR

SUMMARY

In this article the solution of the generalized poly-axially symmetric Helmholtz equation is obtained on the common intersection of the singular hyperplanes. In the case of spherical region the solution on the common intersection of the singularity hyperplanes is given by a Poisson type integral. We also have obtained solutions of the equations $h^{p}_{\Sigma}[u]=o$ and $h^{p}\Sigma[u]=f$.

I. INTRODUCTION

In this paper we shall investigate the solutions of the generalized poly–axially symmetric Helmbholtz equation

$$h_{\Sigma}[U] \equiv \sum_{i=1}^{n} U_{x_i x_i} + \sum_{j=l}^{n} \frac{2k_j}{x_j} U_{x_j} + k^2 U = 0. \qquad (1.1)$$

The equation (1.1) has the singularity hyperplanes $x_j = o$ if $k_j \neq o$. Throughout this paper we assume that k_j and k are real positive constants. Let us denote by D_0 the common intersection of the singularity hyperplanes, and consider the portion of the n–dimensional space, defined as follows

$$Q = \{(x_1, \ldots, x_{l-1}, x_l, \ldots, x_n): x_l \geq 0, \ldots, x_n \geq 0\}. \qquad (1.2)$$

Let D be a bounded region in Q, and ∂D denote the boundary of D. Let us asssume that $\partial D = D'_0$ US where $D'_0 \subset D_0$. Let n be the inward normal on ∂D.

A solution of (1.1.) is

$$U = r^{-s} N_s(kr),\ 2s = n - 2 + 2 \sum_{j=l}^{n} k_j \qquad (1.3)$$

where

$$r = [(x_1 - \xi_1)^2 + \dots + (x_{l-1} - \xi_{l-1})^2 + x_l^2 + \dots + x_n^2]^{\frac{1}{2}}.$$

In the representation (1.3)

$P = (x_1, \dots, x_{l-1}, x_l, \dots, x_n) \in D$, $P_o = (\xi_1, \dots, \xi_{l-1}, 0, \dots, 0) \in D'_o$ and N_s is a Bessel function of the second–kind of order s. Let us denote the sphere with radius ε, centred at a point on D'_0 by $D\varepsilon$ and its boundary by $\partial D\varepsilon$. Let u, v be two functions having continuous derivatives of second order in D. Then the Green's formula for the operator h_Σ can be written as follows,

$$\int_D \prod_{j=l}^{n} x_j^{2k_j} (uh_\Sigma[v] - vh_\Sigma[u]) \, d\mathcal{G} = -\int_{\partial D} \prod_{j=l}^{n} x_j^{2k_j} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) d\sigma. \quad (1.4)$$

Here $d\mathcal{G}$ and $d\sigma$ are the volume and surface elements respectively.

2. THE SOLUTION OF THE EOUATION $h_\Sigma[u] = f$ ON D'_0

By a similar reasoning used in [1] we choose the function v as a singular solution of (1.3). Let us write the Green's formula (1. 4) for the region D–Dε as follows:

$$\int_{D-D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} [u(P)h_\Sigma(r^{-s}N_s(kr)) - r^{-s}N_s(kr)h_\Sigma(u(P))] \, d\mathcal{G}$$

$$= -\int_{\partial D+\partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} \left\{ u(P) \frac{\partial}{\partial n} [r^{-s}N_s(kr)] - r^{-s}N_s(kr) \frac{\partial u(P)}{\partial n} \right\} d\sigma. \quad (2.1)$$

We already know that [3] a Bessel function of the second–kind has the property

$$\frac{d}{dx} [x^{-\nu}N_\nu(kx)] = - k\, x^{-\nu} N_{\nu+1}(kx) \quad (2.2)$$

and for very small x's,

$$N_\nu(x) \sim \begin{cases} -\dfrac{1}{\pi} 2^\nu \Gamma(\nu)\, x^{-\nu} \;;\; \nu \neq 0 \\ \qquad\qquad\qquad\qquad\qquad \nu \neq -1, -2, \dots \\ \dfrac{2}{\pi} \ln x \qquad\qquad ;\; \nu = 0 \end{cases} \quad (2.3)$$

Using (2.2) and (2.3) we obtain

$$-\int_{\partial D_\varepsilon} \prod_{i=l}^{n} x_j^{2k_j} \; u(P) \; \frac{\partial}{\partial n} \; [r^{-s}N_s(kr)] \; d\sigma_\varepsilon$$

$$= -\int_{\partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} \; u(P) \; \frac{\partial}{\partial r} \; [r^{-s}N_s(kr)] \; d\sigma_\varepsilon$$

$$= k \int_{C} u(P) r^{s+1} N_{s+1}(kr) \prod_{j=l}^{n-1} \sin^{2\sum_{l}^{j} k_s} \theta_j \prod_{j=l}^{n} \cos^{2kj} \theta_j \; dw$$

If ε tends to zero we get

$$-\int_{\partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} \; u(P) \; \frac{\partial}{\partial n} \; [r^{-s}N_s(kr)] d\sigma_\varepsilon = - \; u(P_0)w^*, \tag{2.4}$$

where

$$w^* = \frac{k}{\pi} \left(\frac{2}{k}\right)^{s+1} \Gamma(s+1) \int_{C} \prod_{j=l}^{n-1} \sin^{2\sum_{l}^{j} k_s} \theta_j \prod_{j=l}^{n} \cos^{2k_j}\theta_j \; dw \tag{2.5}$$

$$d\sigma = r^{n-1}\, dw, \;\; dw = \sin^{n-2}\theta_{n-1} \sin^{n-3}\theta_{n-2} \cdots \sin\theta_2 \, d\theta_{n-1} d\theta_{n-2} \cdots d\theta_1 \tag{2.6}$$

The second integral over $\partial D\varepsilon$ appearing in the equation (2.1) eaisily gives

$$\int_{\partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} \; r^{-s}N_s(kr) \; \frac{\partial u(P)}{\partial n} \; d\sigma_\varepsilon \; = \; 0. \tag{2.7}$$

Let u be a solution of the equation $h_\Sigma[u] = f(P)$ and f be a continuous function defined in D. Hence the solution of $h_\Sigma[u] = f(P)$ at the point P_0 can be written as follows:

$$u(P_0) = \frac{1}{w^*} \int_{\partial D} \prod_{j=l}^{n} x_j^{2k_j} \left\{ u(P) \frac{\partial}{\partial n} \; [r^{-s}N_s(kr)] - r^{-s}N_s(kr) \frac{\partial}{\partial n} u(P) \right\} d\sigma$$

$$+ \frac{1}{w^*} \int_{D} \prod_{j=l}^{n} x_j^{2k_j} \; f(P) \; r^{-s} \, N_s(kr) \; d\mathcal{G}. \tag{2.8}$$

Obviously the solution of $h_\Sigma[u] = 0$ in D'_0 is

$$u(P_0) = \frac{1}{w^*} \int_{\partial D} \prod_{j=l}^{n} x_j^{2k_j} \left\{ u(P) \frac{\partial}{\partial n} [r^{-s} N_s(kr)] - r^{-s} N_s(kr) \frac{\partial}{\partial n} u(P) \right\} d\sigma \tag{2.9}$$

3. A FUNDAMENTAL FUNCTION FOR THE SPHERICAL REGIONS

Let D be the spherical region with the radius R, centred at the common intersection of the singularity hyperplanes. Let us define

$$R_1 = (\rho^2 + r^2_1 - 2r_1\rho \cos\theta)^{1/2} \quad R^1_1 = (r_1^2 + \frac{R^4}{\rho^2} - 2r_1 \frac{R^2}{\rho} \cos\theta)^{1/2}$$

and

$$R_1 \Big|_{r_1 = R} = R_1^* \qquad R^1_1 \Big|_{r_1 = R} = R^1{}_1^*$$

where

$$\rho = \overline{OP_0}, \qquad \overline{OP_0} \cdot \overline{OP'_0} = R^2, \qquad r_1 = \overline{OP}$$

In the spherical region D the fundamental function γ $(P;\ P_0)$ of the equation $h_{\ddot{\Sigma}}[u] = 0$ is

$\gamma(P;P_0) = (R_1R^1{}_1^*)^{-s}N_s(kR_1)\,N_s(kR^1{}_1^*) - (R^1{}_1R_1^*)^{-s}N_s(kR^1{}_1)N_s(kR_1^*)$
where γ $(P;P_0)$ vanishes on $\partial D - D_0'$. If we substitute $\gamma(P;P_0)$ for the solution (1.3) in the formula (2.1) we obtain

$$-\int_{\partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} u(P) \frac{\partial \gamma(P;P_0)}{\partial n} d\sigma_\varepsilon$$

$$= -\int_{\partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} u(P) \frac{\partial}{\partial R_1} \gamma(P;P_0)\, d\sigma_\varepsilon$$

$$= -\int_{\partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} u(P) \frac{\partial}{\partial R_1} \{(R_1R^1{}_1^*)^{-s}N_s(kR_1)N_s(kR^1{}_1^*)\}\, d\sigma_\varepsilon \tag{3.1}$$

$$= -\, w^* \,(R^1{}_1^*)^{-s}N_s(kR^1{}_1^*)\; u(P_0)$$

as in the previous section. On the other hand we can easily show that

$$\int_{\partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} \gamma(P;P_0) \frac{\partial u(P)}{\partial n} d\sigma_\varepsilon = 0. \tag{3.2}$$

Making use of (3.1) and (3.2) in (2.1) we get

$$\int_D \prod_{j=l}^{n} x_j^{2k_j} \{u(P)\ h_\Sigma[\gamma(P;P_o)] - \gamma(P;P_o)\ h_\Sigma[u(P)]\}\ d\mathcal{T}$$

$$= -\ w^*\ (R^1{}_1{}^*)^{-s} N_s(kR^1{}_1{}^*)\ u(P_o) \tag{3.3}$$

$$-\int_{\partial D} \prod_{j=l}^{n} x_j^{2k_j} \left\{ u(P) \frac{\partial}{\partial n} [\gamma(P;P_o)] - \gamma(P;P_o) \frac{\partial u(P)}{\partial n} \right\} d\sigma$$

or

$$u(P_o) = \frac{(R^1{}_1{}^*)^s}{w^*\ N_s(kR^1{}_1{}^*)} \left\{ \int_D \prod_{j=l}^{n} x_j^{2k_j}\ \gamma(P;P_o)\ h_\Sigma[u(P)]\ d\mathcal{T} \right.$$

$$\left. - \int_{\partial D} \prod_{j=l}^{n} x_j^{2k_j}\ u(P) \frac{\partial \gamma(P;P_o)}{\partial n}\ d\sigma \right\}. \tag{3.4}$$

In the case of spherical region (3.4) is the solution of the equation $h_\Sigma [u] = f$ at the singularity point $P_o \in D'_o$. If $h_\Sigma[u] = o$ then (3.4) reduces to the form

$$u(P_o) = \frac{-(R^1{}_1{}^*)^s}{w^*\ N_s(kR^1{}_1{}^*)} \int_{\partial D} \prod_{j=l}^{n} x_j^{2k_j}\ u(P) \left[\frac{\partial}{\partial r_1} \gamma(P;P_o)\right]\Bigg|_{r_1 = R} d\sigma. \tag{3.5}$$

The value of the normal derivative of the fundamental solution on the spherical surface is

$$\frac{\partial \gamma}{\partial r_1}\Bigg|_{r_1 = R} = k(R^*)^{-(s+1)} (R^1{}_1{}^*)^{-s}\ \{(\rho - R\ \cos\theta)\ N_s(kR^*{}_1)\ N_{s+1}(kR^1{}_1{}^*)$$

$$-\ (R - \rho\ \cos\theta)\ N_s(kR^1{}_1{}^*)\ N_{s+1}\ (kR^*{}_1)\}$$

Thus the solution (3.5) becomes

$$u(P_o) = \frac{k(R^1{}_1{}^*)^s}{w^*\ N_s(kR^1{}_1{}^*)} \int_{\partial D} \prod_{j=l}^{n} x_j^{2k_j}\ u(P)(R_1{}^*)^{-(s+1)}\ (R^1{}_1{}^*)^{-s} \tag{3.6}$$

$$x\ \{(R - \rho\cos\theta)\ N_s(kR^1{}_1{}^*)\ N_{s+1}(kR_1{}^*) - (\rho - R\cos\theta)\ N_s(kR_1{}^*)\ N_{s+1}(kR^1{}_1{}^*)\}\ d\sigma$$

4. A SOLUTION OF THE EOUATION $h^{p}_{\Sigma}[u] = o$

If we want to obtain a solution of (1.1), which depends only on

$r = [(x_1 - \xi_1)^2 + \dots + (x_{l-1} - \xi_{l-1})^2 + x_l^2 + \dots + x^2_n]^{1/2}$

then the equation becomes

$$\frac{d^2u}{dr^2} + \frac{2s+1}{r} \frac{du}{dr} + k^2u = o. \tag{4.1}$$

We shall use the method of induction to find a solution of the equation $h^p_\Sigma [u] = o$, where p is a natural number. First let us consider the equation $h^2_\Sigma[u] = o$: the system of equation

$$h_\Sigma[\varphi_1] = o$$

$$h_\Sigma[u] = \varphi_1 \tag{4.2}$$

corresponds to the equation

$$h^2_\Sigma[u] \equiv h_\Sigma(h_\Sigma[u]) = o. \tag{4.3}$$

By (1.3) a solution of the equation $h_\Sigma[\varphi_1] = o$ is,

$$\varphi_1 = r^{-s}J_s(kr) \tag{4.4}$$

Here $J_s(kr)$ is the Bessel function of the first–kind. Thus the equation (4.2) takes the form

$$r^2 \frac{d^2u}{dr} + (2s+1)r \frac{du}{dr} + k^2r^2u = r^{2-s}J_s(kr). \tag{4.5}$$

Now let us look for a solution of (4.5) in the form

$$u = A\, r^{-s+1}J_{s-1}(kr) \tag{4.6}$$

After the necessary calculations we obtain

$$\{Ak[2s - n - 2 \sum_{j=l}^{n} k_j] - 1\}\, r^{-s+2} J_s(kr) = o.$$

Thus in order to get a solution of the form (4.6) we must choose

$$A = -\frac{1}{2k}.$$

Hence a solution of the equation $h^2_\Sigma[u] = o$ is

$$u_2 = - \frac{1}{2k} r^{-s+1} J_{s-1}(kr). \tag{4.7}$$

Now, let us suppose that a solution of the equation

$$h_\Sigma h_\Sigma \ldots h_\Sigma[u] \equiv h_\Sigma^{p-1}[u] = o \tag{4.8}$$

is

$$u_{p-1} = \frac{(-1)^{p-2}}{2^{p-2}(p-2)!\, k^{p-2}} r^{-s+p-2} J_{s-(p-2)}(kr) . \tag{4.9}$$

Since

$$h_\Sigma^p [u] \equiv h_\Sigma^{p-1} (h_\Sigma[u]) = o. \tag{4.10}$$

we get the system

$$h_\Sigma^{p-1}[\varphi_{p-1}] = o$$

$$h_\Sigma[u] = \varphi_{p-1} . \tag{4.11}$$

Substituting (4.9) into (4.11), we get

$$r^2 \frac{d^2u}{dr^2} + (2s+1)\, r \frac{du}{dr} + k^2r^2u = \frac{(-1)^{p-2}}{2^{p-2}(p-2)!\, k^{p-2}} r^{-s+p} J_{s-(p-2)}(kr) \tag{4.12}$$

If we try to find a solution of (4.12) in the form

$$u = A\, r^{-s+p-1} J_{s-(p-1)}(kr) \tag{4.13}$$

we obtain

$$A = \frac{(-1)^{p-1}}{(2k)^{p-1}(p-1)!}$$

Hence a solution of the equation $h_\Sigma^p[u] = o$ is

$$u_p = \frac{(-1)^{p-1}}{(2k)^{p-1}(p-1)!} r^{-s+p-1} J_{s-(p-1)}(kr). \tag{4.14}$$

5. THE SOLUTION OF THE EQUATION $h_\Sigma^p[u] = f$ IN D'_o

Following the method used in obtaining (4.14) we can find the solution

$$u = r^{-q} N_q(kr), \qquad q = s - p + 1 \tag{5.1}$$

of the equation $h_\Sigma^p[u] = 0$. On the other hand the Green's formula for the operator h_Σ^p, [2] is,

$$\int_D \prod_{j=l}^{n} x_j^{2k_j} (u h_\Sigma^p[v] - v h_\Sigma^p[u])\, d\sigma$$

$$= -\sum_{i=o}^{p-1} \int_{\partial D} \prod_{j=l}^{n} x_j^{2k_j} \left(h^i_\Sigma[u] \frac{\partial h_\Sigma^{p-i-1}[v]}{\partial n} - h^i_\Sigma[v] \frac{\partial h_\Sigma^{p-i-1}[u]}{\partial n} \right) d\sigma \tag{5.2}$$

here $u, v \in C^{2p}(D)$ and $u, v \in C^{2p-2}(\partial D)$. In (5.2) we define $h_\Sigma^o[u] = u$. We want to use the singular solution (5.1) instead of v. Let us subtract a neighborhood D_ε of the point from the region D, v is singular. Therefore we can apply the formula (5.2) to the functions u and v in the region $D-D_\varepsilon$. From (5.2) we can write

$$\int_{D-D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} \{u\, h_\Sigma^p[r^{-q}N_q(kr)] - r^{-q}N_q(kr)\, h_\Sigma^p[u]\}\, d\tau$$

$$= -\int_{\partial D + \partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} \left\{ u \frac{\partial}{\partial n} h_\Sigma^{p-1}[r^{-q}N_q(kr)] - r^{-q}N_q(kr) \frac{\partial}{\partial n} h_\Sigma^{p-1}[u] \right\} d\sigma$$

$$- \sum_{i=1}^{p-1} \int_{\partial D + \partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} \left\{ h_\Sigma^i[u] \frac{\partial}{\partial n} h_\Sigma^{p-i-1}[r^{-q}N_q(kr)] - h_\Sigma^i[r^{-q}N_q(kr)] \right.$$

$$\left. \times \frac{\partial}{\partial n} h_\Sigma^{p-i-1}[u] \right\} d\sigma \tag{5.3}$$

By induction

$$h_\Sigma^{p-1}[r^{-q}N_q(kr)] = (2k)^{p-1} \prod_{m=1}^{p-1} (m-p)\, r^{-(q+p-1)} N_{p+q-1}(kr) \tag{5.4}$$

and

$$\frac{\partial}{\partial r} h_\Sigma^{p-1}[r^{-q}N_q(kr)] = -2^{p-1}k^p \prod_{m=1}^{p-1} (m-p)\, r^{-(q+p-1)} N_{p+q}(kr) \tag{5.5}$$

are easily obtained. Thus

$$-\int_{\partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} u(P) \frac{\partial}{\partial r} h_\Sigma^{p-1}[r^{-q}N_q(kr)] \, d\sigma_\varepsilon$$

$$= 2^{p-1}k^p \prod_{m=1}^{p-1} (m-p) \int_C r^{2\sum_l^n k_j} u(P)\, r^{-(p+q-1)} N_{p+q}(kr) \tag{5.6}$$

$$x \prod_{j=l}^{n} \sin^{2\sum_l^j k_s} \theta_j \prod_{j=l}^{n} \cos^{2k_j} \theta_j \, r^{n-1} \, dw$$

$$= - W_1 \, u(P_o),$$

where

$$W_1 = \frac{1}{\pi}\left(\frac{2}{k}\right)^{q+p} \Gamma(p+q)2^{p-1}k^p \prod_{m=1}^{p-1} (m-p) \int_C \prod_{j=l}^{n} \sin^{2\sum_l^j k_s} \theta_j \prod_{j=l}^{n} \cos$$

$$x\ ^{2k_j}\theta_j \, r^{n-1} \, dw. \tag{5.7}$$

On the other hand we can eaisily verify that

$$\int_{\partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} r^{-q}N_q(kr) \frac{\partial}{\partial r} h_\Sigma^{p-1} [u] \, d\sigma_\varepsilon = o. \tag{5.8}$$

To complete the solution we are left with the calculation of the following:

$$-\sum_{i=1}^{p-1} \int_{\partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} \left\{ h_\Sigma^{i}[u] \frac{\partial}{\partial r} h_\Sigma^{p-i-1}[r^{-q}N_q(kr)] - h_\Sigma^{i} [r^{-q}N_q(kr)] \right.$$

$$\left. x \frac{\partial}{\partial r} h_\Sigma^{p-i-1}[u] \right\} d\sigma_\varepsilon \tag{5.9}$$

From (5.4), we have

$$h_\Sigma^{j}[r^{-q}N_q(kr)] = (2k)^j \prod_{m=1}^{j} (m-j-1) r^{-(q+j)} N_{q+j}(kr), \ 1 \leq j \leq p-1 \tag{5.10}$$

Let us choose the term

$$I_{\mu-1} = \int_{\partial D_\varepsilon} \prod_{j=l}^{n} x_j^{2k_j} \{h_\Sigma^{\mu-1}[u] \frac{\partial}{\partial r} h_\Sigma^{p-\mu}[r^{-q}N_q(kr)] - h_\Sigma^{\mu-1}[r^{-q}N_q(kr)] \frac{\partial}{\partial r} \times h_\Sigma^{p-\mu}[u]\} \, d\sigma_\varepsilon \tag{5.11}$$

$$(\mu = 2,3,\ldots, p)$$

in the sum (5.9). In order to obtain the value of (5.9) it suffices to calculate the integral $I_{\mu-1}$. Putting $j = p - \mu$ in (5.10) ve get

$$\frac{\partial}{\partial r} h_\Sigma^{p-\mu}[r^{-q}N_q(kr)] = -k\,(2k)^{p-\mu} \prod_{m=1}^{p-\mu} (m-p+\mu-1)\, r^{(-p+q-\mu)} N_{p+q-\mu+1}(kr) \tag{5.12}$$

and from (5.10) we find

$$h_\Sigma^{\mu-1}\,[r^{-q}N_q(kr)] = (2k)^{\mu-1} \prod_{m=1}^{\mu-1} (m-\mu) r^{-(q+\mu-1)} N_{q+\mu-1}(kr) \tag{5.13}$$

for $j = \mu-1$. So (5.11) takes the

$$I_{\mu-1} = \int_C r^{2\sum_{l}^{n} k_j} \prod_{j=l}^{n-1} \sin^{2\sum_{l}^{j} k_s} \theta_j \prod_{j=l}^{n} \cos^{2k_j} \theta_j \Big\{ -k(2k)^{p-\mu} \prod_{m=1}^{p-\mu} (m-p+\mu-1) r^{-(p+q-\mu)} \tag{5.14}$$

$$\times N_{q+p-\mu+1}(kr)\, h_\Sigma^{\mu-1}[u] - (2k)^{\mu-1} \prod_{m=1}^{\mu-1} (m-\mu)\, r^{-(q+\mu-1)} N_{q+\mu-1}(kr) \frac{\partial}{\partial r} h_\Sigma^{p-\mu}$$

$$\times [u]\Big\}\, r^{n-1}\, dw.$$

Taking the limit when $r \longrightarrow o$ we obtain $I_{\mu-1} = 0$. Hence the solution of the equation

$$h_\Sigma^p[u] = f$$

at the singularity point $P_0 \in D_0'$ is

$$u\,(P_0) = \frac{1}{W_1} \Big\{ \int_D \prod_{j=l}^{n} x_j^{2k_j} r^{-q}N_q(kr)\, f(P)\, d\mathfrak{G}$$

$$- \sum_{i=o}^{p-1} \int_{\partial D} \prod_{i=l}^{n} x_j^{2k_j} (h^i_\Sigma[u(P)] \frac{\partial}{\partial n} h_\Sigma^{p-i-1} [r^{-q}N_q(kr)] - h^i_\Sigma[r^{-q}N_q$$

$$(kr)] \frac{\partial}{\partial n} h_\Sigma^{p-i-1} [u(P)])\, d\sigma \Big\}$$

ÖZET

Bu makalede, genelleştirilmiş çok simetrili Helmholtz denkleminin çözümü, tekil hiperdüzlemlerin arakesiti üzerindeki noktalarda elde edilmiştir. Bölgenin küresel bir bölge olması halinde, bir temel fonksiyon elde ederek tekil hiperdüzlemlerin arakesiti üzerindeki noktalarda çözüm, Poisson integraline benzer biçimde verilmştir. Ayrıca $h^p\Sigma[u]=o$ denkleminin bir çözümü elde edilerek$h^p\Sigma[u]=f$ denkleminin tekil çözümü incelenmiştir.

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