COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A_1 : Mathématiques

TOME 31 ANNÉE 1982

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Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences de l'Vniversité d'Ankara

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A Numerical Solution To The Radial Equation Of The Tidal Wave Propagation*

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(Received May 25. 1982: accepted December 1, 1982)

Abstract

The tidal wave function y (x) is a solution to an inhomogeneous, linear, second-order differential equation with variable coefficient. Numerical values for the height-dependence terms, in the observed tides, have been utilized in finding y (x) as a solution to an initial-value problem. Complex Fast Fourier Transform technique is also used to obtain the solution in a complex form.

Based on a realistic temperature structure, the atmospherebelow -110 km- has been divided into layers with distinct characteristics, and thus the technique of propagation in stratified media has been applied. The reduced homogeneous equation assumes the form of Helmholtz equation and with initial conditions the general solution is obtained.

1. INTRODUCTION

In the development of the tidal theory[1,2], the assumption is made such that the dependence of the tides on height, co-latitude and longitude are separable.

The latitude-dependences of the tidal fields are expressed in terms of the Hough functions, which are solutions of Laplace's tidal equation. It is an eigenvalue-eigenfunction problem and has been the subject of extensive study during the last decade[3,4,5,6].

The vertical structure of the tidal wave propagation, on the other hand, is expressed in terms of the wave function y (x), which is the solu-

*International Centre for Theoretical Physics, Preprint IC/81/145, Trieste, Italy
*Accepted for presentation at XV Biennial Fluid Dynamic Symposium, Poland, Sep. 1981.

tion of the Radial Wave equation. For a given mode of oscillation, the variable coefficient, in that equation, depends solely on the vertical temperature distribution. Previous investigatore[7,8,9,10]adopted, however, simplified vertical temperature structure in order to render the mathematical treatment more tractable.

Classical tidal theory has, therefore, limitations, arising out of the assumptions on which it is based, which need to be recognized when making close comparison with the observational data[11]. With a realistic temperature structure, however, no closed form solution-to the vertical wave equation-exists and the solution has to be approached numerically. This is the goal of the present study.

Analytical expressions for the general solution, in conformity with the observed tidal oscillations in the atmosphere, have been obtained. The numerical solutions for the oscillatory and trapped modes of tidal wave propagation have also been presented and discussed.

2. THE RADIAL WAVE EQUATION

The basic equation of the theory of atmospheric tides are given in Siebert[1]and their derivations need not, therefore be repeated. The wave function $y_n(x)$ is expressed in terms of the velocity divergence $\chi_n(x)$ and the rate of heating $J_n(x)$, per unit mass, as

$$y_n(x) = [\chi_n - kJ_n / gH] / \exp(x/2) \tag{1}$$

where x, the reduced height, is defined by

$$x = \int^z dZ' / H(Z') \tag{2}$$

$H(Z) = RT(Z)/g$ and $k = 2/7 = (\gamma - 1) / \gamma$. n refers to the mode number, and may take negative sign-depending on that of the equivalent depth of the mode, h_n; the latter is the eigenvalue, obtained through the solution of Laplace's tidal equation, which depends only on the period of the mode. We refer, thereafter, to the modes by the symbols (S, n); S is the longitudinal wavenumber. In the present study, we deal with the most important modes of the migrating solar diurnal (1,– 2), (1,1) and semidiurnal (2,2) oscillations; $h_n = -12.2287$, 0.6988 and 7.9175 km, respectively[6].

The wave function $y_n(x)$ is a solution to the Radial wave Equation, written in the form:

$$d^2y_n / dx^2 - \tfrac{1}{4} [\, 1 - 4/h_n(kH + dH/dx) \,] y_n = (k/\gamma\, gh_n) J_n(x) e^{-x/2} \quad (3)$$

Subject to the usual boundary conditions:

(i) Zero vertical motion at the surface[12], this requires that:

$$dy_n / dx - y_n / 2 \Big|_{x=0} = -(H(o) / h_n) y_n(o) \quad (4)$$

and

(ii) y_n is bounded and the flow of energy is in the upward direction, at sufficiently high levels, $Z \sim 100$ km[8].

In order to evaluate the reduced height (Eq. 2) realistic temperature data are utilized. [13,14] The scale height H (x) has then been obtained, by using quadratic interpolating formula, at x = 0 (0.1) 17. By least squares approximation H (x) is found to be respresented fairly accurately by a fourth degree polynomial in the region $0 < x < 15.6$. At $x > 15.6$, a constant temperature gradient (8.5°k/km, i.e. $dH/dZ = 0.25$) is assumed. The approximated scale height H (x), thus obtained, is given in table la.

The variable coefficient of y_n(Eq. 3):

$$\mu^2{}_n(x) = -\tfrac{1}{4} [\, 1 - 4/h_n(kH + dH/dx) \,] \quad (5)$$

has been evaluated and presented in table 1b, for the modes under consideration. (The subscript n will be dropped, there-after, for simplicity in writing, and we deal with one mode at a time).

The region below x = 15.6 has been, further, divided into 13 equally spaced subregions, for each of which:

$$\mu^2(x) = Px - q \quad (6)$$

The results for P are also given in table 1b, from which it can be inferred that the atmosphere is divided into four distinct regions, with regard to the characteristics of its refractive index, $\mu^2(x)$, to the different modes. These regions are separated by the levels x = 4.8, 9.6 and 12.0. At levels $x \geq 15.6$. the temperature model assumes the form $dH/dZ = \varepsilon$, and the corresponding expression for $\mu^2$is, therefore;

$$\mu^2(x) = Ee^{\varepsilon x} - \varepsilon \quad (7)$$

3. THE AUXILIARY FIRST-ORDER DIFFERENTIAL EQUATION

The vertical dependence-terms in the tidal fields are usually expressed in terms of the solution y (x), to Eq.(3), as:

$$(dy/dx - y/x)e^{x/2} = \xi(x)\, e^{i\delta}(x) \qquad (8)$$

These terms had been previously obtained at Z = 2.5 (2.5) 60 km in analysing 6 years of horizontal wind date[15], and at Z = 0 using the best available data of surface pressure amplitudes and phases [16,17]. By Lagrangian quadratic interpolation, the vertical dependence terms (Eq. 8) are therefore evaluated at x = 0 (0.1) 8.4.

With the inhomogeneous terms being given in tabulated forms, at $\Delta x = 0.1$, the linear first-order differential equation, (Eq. 8), has been integrated numerically[18], using Simpson's rule, and its solution is:

$$y(x) = e^{x/2}\{y(o) - {}_0\!\int^x \xi(x') \exp(i\delta(x') - x')\, dx'\} \qquad (9)$$

The initial values y (o) satisfy the lower boundary condition (4). The numerical values for the solution at x = 0 (1.2) 8.4 are presented in table 2.

On the other hand, if the inhomogeneous term (Eq. 8) attains a reasonable analytical form, the integral in (9) can also be evaluated. By inspection, it was found that this term can be approximated fairly accurately - by Complex Fourier Series expansion - in the form:

$$(dy/dx - y/2)\, e^{x/2} = \xi(x)\, e^{i\delta(x)} = \sum_{v=o}^{3} C_v e^{i\alpha_v x} \qquad (10)$$

The complex coefficients $C_v (= a_v + ib_v)$ are obtained for the two regions $0 < x < 4.8$ and $4.8 < x < 9.6$, independently, using the Complex Fast Fourier Transform Technique, usually used in Computational Complex analysis[19]. The two regions are chosen this way as they are characterized by the different temperature structure (Eq. 6) and consequently different wavelengths, for each mode.

With this analytical expression (10), the corresponding solution y (x) is, therefore, given by:

$$y(x) = y(x_0) \exp(x - x_0)/2 - e^{x/2}\{e^{-x} \sum_v C_v / (i\alpha_v - 1)\, e^{i\alpha_v x} - e^{-x_0} \sum_v C_v / (i\alpha_v - 1)\} \qquad (11)$$

The initial values $y(x_0)$ are those obtained by the integration method (9) at $x_0 = 4.8$. The numerical values for the solution (11) are also presented in table 2, and they are found to be comparable, within differences $\pm$ 0.5 % – on the average, with those obtained by direct integration (9). This reflects the validity of the approximation (10) and also the numerical stability of the solution.

On the other hand, the inhomogeneous term (Eq. 10) corresponds to the applied tidal force. Therefore, in differentiating both sides of (10) and substituting into the radial wave equation (3), we can express the general solution $y_G(x)$ as:

$$y_G(x) = Q(x) \,/\, (\mu^2(x) + \tfrac{1}{4}) - e^{-x/2} \Sigma\, i\alpha_v C_v e^{i\alpha_v x} \,/\, (\mu^2(x) + \tfrac{1}{4}) \qquad (12)$$

where

$$Q(x) = (k/\gamma\, gh)\, J(x)\, e^{-x/2} \qquad (13)$$

At high levels (70-80 km), it is generally accepted[9,10,11] that the applied tidal force J (x) vanishes. Assume that this occurs at $x^*(= 12$ say, corresponding to $Z \sim 82$ km). In the intervening region we may, therefore, write:

$$(dy/dx - y/2)\, e^{x/2} = C\,(x - x^*)\, e^{i\alpha} \qquad (14)$$

$$9.6 \leq x \leq x^*$$

Therefore, the solution in this region, is:

$$y_G(x) = Q(x) \,/\, (\mu^2(x) + \tfrac{1}{4}) - e^{-x/2} C e^{i\alpha} \,/\, (\mu^2(x) + \tfrac{1}{4}) \qquad (15)$$

With given values of Q (x) the general solution can be obtained numerically. Nevertheless, such function is given in the literature – in rather simplified analytical forms which do not conform with the observations. Consequently, another approach is given to obtain the general solution of the inhomogeneous equation (3).

4. THE REDUCED HOMOGENEOUS EQUATION:

Basically, the general solution of Eq. (3) may be expressed as the sum of the complementary function and a particular integral. The complementary function is the general solution of the reduced homogeneous equation:

$$d^2y/dx^2 + \mu^2(x)\, y = 0 \qquad (16)$$

An examination of this reduced equation (16) shows that there is a simple analogy with propagation of plane waves in a medium of varying refractive index $\mu^2(x)$[12]. This state of affairs can then be described by the Helmholtz equation, which in general, is a typical one for oscillating, systems[20].

In order to obtain exact solutions to the Helmholtz equation, the variable coefficient $\mu^2(x)$ has been replaced by a collection of continuous functions (Eqs. 6 and 7), whick look like acceptable approximation to $\mu^2(x)$. It has been shown [21] that the general solutions of the reduced homogeneous equation (16) are expressed in terms of:

(i) Bessel functions $J_{\pm 1/3}$ and $I_{\pm 1/3}$ in case $\mu^2 > 0$ and $\mu^2 < 0$, respectively, for $0 \leq x \leq 15.6$.

and

(ii) Bessel functions J_4, y_4 (or modified Bessel functions I_4, k_4) of the first and second kinds, in case of $\mu^2 > 0$ (or $\mu^2 < 0$), for $15.6 \leq x \leq 16.8$. In the former case ($\mu^2 > 0$), the solution is oscillatory and the requirement that $y(x)$ is bounded for large x is met. But in the latter case ($\mu^2 < 0$), the modified Bessel function of the first kind grows indefinitely with x, and its coefficient is set to zero to conform with the requirement of the bounded solution.

To conclude, the general solution to the homogeneous equation (16) has been represented in the form:

$$y_c(x) = ay_1(x) + by_2(x) \qquad (17)$$

where y_1 and y_2 are the two linearly independent solutions, expressed in terms of the appropriate Bessel functions, and $a\ (= Ae^{i\alpha})$, $b\ (= Be^{i\beta})$ are two complex integration constants.

Imposing the lower boundary conditions (4) on the solution (17), the two constants a and b are related and we may write:

$$y_c(x) = Bv(x)\, e^{i\beta} \qquad (18)$$

A second alternative technique to solve the homogeneous equation with arbitrary $\mu^2(x)$, is the WKB approximation method[20]. It consists in assuming that the changes in $\mu(x)$ became small enough over a wavelength in the vertical, i.e.

$$| 1/\mu \, d/dx \ln \mu | \ll 1 \qquad (19)$$

It has been found that this is the case for the modes under consideration, except for the mode (2,2) at $x = 2.4$, 7.2 & 14.4, where $\mu^2 = 0$, and and WKB approximation breaks down.

In the WKB approximation, the general solution of the homogeneous equation can then be written in the form:

$$y_c(x) = W/\mu^{1/2} e^{-i\mu x}; \ \mu^2 > 0 \qquad (20)$$

or

$$y_C(x) = W/\lambda^{1/2} e^{-\lambda x} \quad ; \mu^2 < 0\ (= -\lambda^2) \tag{21}$$

Signs in (20) and (21) are chosen to conform with the boundness of y (x) and upward flow of energy at great altitudes; W is a constant of integration. Comparisons of the exact solution (18) with the approximate solution (20), or (21), enable us to relate the constants W and b.

5. THE GENERAL SOLUTION:

Turning now to the inhomogeneous equation (3), we find that solutions in the form (12), or (15) can be written in the form:

$$y_C(x) = y_C(x).\ u\ (x) \tag{22}$$

where $y_C(x)$ is any solution of the homogeneous equation (16) and u (x) is a particular solution. Substitution into (3) yields:

$$y_C(d^2u/dx^2) - 2\ dy_C/dx)\ (du/dx) - (d^2y_C/dx^2 + \mu^2 y_C)\ u = Q\ (x) \tag{23}$$

The coefficient of u being zero since y_C is the complementary function; and we obtain a linear equation for (du/dx) with an integrating factor

$$\exp\ \{\int\ 2\ (dy_C/dx)\ /\ y_C dx\} = y^2{}_C$$

Therefore, Eq. (23) can be integrated outright, without separate calculations of a particular integral and the complementary function. Upon integration we get

$$du/dx = 1/y^2{}_C \int^x y_C Q\ dx + K/y^2{}_C$$

and $u\ (x) = \int^x 1/y_C \{ \int^x y_C Q\ dx\}\ dx + K \int^x (1/y^2{}_C)\ dx + A$

Thus:

$$y_G(x) = y_C \int^x 1/y^2{}_C\ \{ \int^x y_C Q\ dx\}\ dx + Ky_C \int^x (1/y^2{}_C) dx + Ay_C \tag{24}$$

The integrals in equation (24) can be evaluated numerically, provided Q (x) is given.

Let us consider the approximate solution to the homogeneous equation y_C(Eqs. (20) or (21)), obtained through the WKB approximation of constant μ, over a short range of x. Thus we may approximate Q (x) (13), within this range in the form:

$$Q\ (x) = qe^{-x/2} \tag{25}$$

On substitution, Eq. (24) takes, therefore, the form:

$$y_G(x) = Q\,(x) \,/\, (\mu^2 + \tfrac{1}{4}) + iK / 2W \mu^{1/2} e^{-i\mu x} + AW / \mu^{1/2} e^{i\mu x};\ \mu^2 > 0 \quad (26)$$

or

$$y_G(x) = Q\,(x) \,/\, (-\lambda^2 + \tfrac{1}{4}) + K / 2W\, \lambda^{1/2} e^{\lambda x} + AW / \lambda^{1/2}\, e^{-\lambda x};\ \mu^2 < 0\ (= -\lambda^2) \quad (27)$$

The middle term in (26) (or 27) vanishes in order to conform with the upper boundary condition of finite solution and upward energy at great altitudes.

Furthermore the general solution to the inhomogeneus equation (3) is identical to the general solution of the reduced homogeneous equation (16), where Q = 0, i.e. A = 1.

Consequently, we obtain for $y_G(x)$ similar expressions to those obtained in (12) and (15). Through that comparison, the constant W is evaluated, and hence the homogeneous solution can be evaluated numerically.

This completes the method of numerical solution to the differential equation (3).

6. RESULTS AND DISCUSSION:

In this paper we presented a numerical method of solving linear inhomogeneous second-order differential equation with a variable coefficient, as applied to the vertical tidal-wave propagation. Two solutions are obtained, a particular solution, that is derived from the observed tidal motion, and a homogeneous solution, using the characteristics of the atmospheric structure.

As an illustration to the vertical behaviour of the solutions, the results for two modes of the diurnal oscillation (1,-2) and (1,1) and for the main semidiurnal mode (2,2) are presented in table 2.

It can be inferred from table 2, that the solution grows exponenttially with x and has greater values for the trapped mode (1,-2) than the oscillating one, reflecting the effectiveness of the former mode to the excitation. Although the (1,1) mode is a propagating one, the constancy in phase is due to its large wavelength which is about 43 km in the troposphere and 25 km in the stratosphere. For the main semidiurnal mode (2,2), $\mu^2_n(x)$ is almost zero through most of the stratasphere; i.e. extremely long vertical wavelength ($\sim$ 150 km). Thus not only does this mode receive the bulk of semidiurnal oscillation, but it must also respond to the excitation with particular efficiency.

With respect to the homogeneous solution, the function y (x) has been found to be non-oscillatory and the amplitude is exponentially increasing for $0 < x < 4.8$ and decreasing above; in the latter region the energy is trapped a feature associated with the negative mode. For the oscillating mode the function y (x) is in general oscillatory and thus it is very sensitive to the variation in temperature.

To conclude, with such a method of solution, a general solution to the radial equation can be expressed analytically. It will be interesting to show how this solution can be used to evaluate the thermal drive for the atmospheric tides. This will be the subject of the forthcoming work.

ACKNOWLEDGEMENT

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the

TABLE 1. THE MODEL ATMOSPHERE

1. a The Reduced Height			1. b The Refractive Index Characteristic					
x	Z (km)	H (km)	$\mu^2_{(1,-2)}$	$P^*_{(1,-2)}$	$\mu^2_{(1,1)}$	$P_{(1,1)}$	$\mu^2_{(2,2)}$	$P_{(2,2)}$
0.0	0.0	8.4742	−0.2887		0.4280		−0.1896	
				−0.050		0.870		0.077
1.2	9.0027	6.8966	−0.3484		1.4721		−0.0967	
				−0.040		0.707		0.063
2.4	16.9021	6.4497	−0.3969		2.3205		−0.0212	
				−0.027		0.475		0.042
3.6	24.7208	6.6345	−0.4295		2.8909		0.0296	
				−0.012		0.210		0.019
4.8	32.9472	7.0523	−0.4439		3.1430		0.0520	
				0.003		−0.054		−0.005
6.0	41.6587	7.4077	−0.4402		3.0785		0.0463	
				0.016		−0.281		−0.025
7.2	50.6448	7.5080	−0.4209		2.7411		0.0163	
				0.025		−0.437		−0.039
8.4	59.5307	7.2630	−0.3909		2.2166		−0.0304	
				0.028		−0.487		−0.043
9.6	67.9003	6.6857	−0.3576		1.6324		−0.0824	
				0.023		−0.395		−0.035
10.8	75.4192	5.8913	−0.3305		1.1585		−0.1246	
				0.007		−0.127		−0.011
12.0	81.9589	5.0979	−0.3218		1.0063		−0.1382	
				−0.020		0.353		0.031
13.2	87.7188	4.6265	−0.3460		1.4296		−0.1005	
				−0.062		1.079		0.096
14.4	93.3506	4.9006	−0.4200		2.7241		0.0147	
				−0.119		2.086		0.186
15.6	100.0805	6.4465	−0.5630		5.2273		0.2376	
16.8	109.2152	8.7018	−0.6312		6.4210		0.3438	

*P as obtained using Eq. (6)

International Centre for Theoretical Physics, Trieste. Thanks are due to the Centre's members of staff for their cooperation and for use of the computer facilities. He would also like to thank Prof. A.H. Cook of Cavendish Laboratory and Prof. D. Tritton of Newcastle University for fruitful discussions concerning the problem.

TABLE 2. THE NUMERICAL SOLUTION TO AUXILIARY WAVE EQUATION (8)

	Diurnal Oscillation (1,-2)			Diurnal Oscillation (1,1)			Semidiurnal Oscillation (2,2)		
x	y**	y*	Φ*	y**	y*	Φ*	y**	y*	Φ*
0	0.016	0.016	0.53	0.006	0.006	0.53	0.123	0.123	0.94
1.2	0.159	0.178	0.90	0.074	0.177	0.08	0.580	0.576	0.74
2.4	0.506	0.517	0.88	0.352	0.326	0.20	1.507	1.404	0.73
3.6	0.964	0.969	0.88	0.582	0.582	0.20	2.847	2.847	0.73
4.8	1.771	1.771	0.88	1.082	1.082	0.20	5.213	5.213	0.73
6.0	3.239	3.239	0.88	1.969	1.939	0.21	9.432	9.503	0.73
7.2	5.912	5.925	0.88	3.582	3.520	0.21	17.190	17.305	0.73
8.4	10.758	10.795	0.88	6.519	6.411	0.12	31.314	31.541	0.73
9.6	—	19.662	0.88	—	11.679	0.21	—	57.453	0.73

*Values obtained from the integration, using the approximation given by Eq. (10).

**Values obtained from the integration (9), as obtained from the observations.

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