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**On A Generalization Of The Dual Summability Methods**

by

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10

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## On A Generalization Of The Dual Summability Methods

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### SUMMARY

In this paper, the concept of Dual Summability Methods has been introduced and later generalized. Furthermore various inclusion theorems have been given connected with defined new methods.

### I. INTRODUCTION

A sequence method of summation with the matrix  $A = (a_{nk})$  is based on the transformation

$$\sigma_n = A_n s = \sum_{k=0}^{\infty} a_{nk} s_k, \quad n = 0, 1, 2, \dots \quad (1)$$

of the sequence  $s = (s_n)$  into the sequence  $As = (\sigma_n)$ ; and a series method with the matrix  $A' = (a'_{nk})$  corresponds to the transformation

$$\tau_n = \sum_{k=0}^{\infty} a'_{nk} u_k, \quad n = 0, 1, 2, \dots \quad (2)$$

of the series  $\sum u_n$ ,  $u_n = s_n - s_{n-1}$ ,  $s_{-1} = 0$ , into the sequence  $(\tau_n)$ .

It is natural that the matrix  $A'$  of (2) is called dual to the matrix  $A$  of (1) (or  $A$  dual  $A'$ ) if  $\sigma_n$  becomes  $\tau_n$  (or  $\tau_n$  becomes  $\sigma_n$ ) by the suitable way: This is equivalent to the relation.

$$a'_{nk} = \sum_{i=k}^{\infty} a_{ni} \quad \text{or to the relation} \quad a_{nk} = a'_{nk} - a'_{n, k+1}, \quad [2].$$

It is well known that  $A$  is regular if and only if  $A'$  is regular [1].

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## 2. Generalization of the Dual Summability Methods by the Euler Mean of first order.

In a recent paper [3], the concept of dual summability methods was generalized by the Cesaro Mean of first order.

We shall now make the generality by the Euler Mean of first order First we give some definitions.

**Definition 1.** Let  $A = (a_{nk})$  be the matrix transforming the sequence  $s = (s_n)$  into the sequence  $(\sigma_n)$ , that is to say that,

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk} s_k, \quad n = 0, 1, 2, \dots \quad (3)$$

If,

$$t_n = \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} \sigma_v, \quad t_n \rightarrow \sigma \quad (n \rightarrow \infty) \quad (4)$$

we shall say that the sequence  $(s_n)$  is  $(A, E_1)$ -limitable to the value " $\sigma$ ".

This expression can obviously be written in the form,

$$t_n = \sum_{k=0}^{\infty} b_{nk} s_k; \quad b_{nk} = \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} a_{vk}, \quad n = 0, 1, 2, \dots$$

This new method we have just defined is a sequence-to-sequence transforming the sequence  $(s_n)$  to the sequence  $(t_n)$ , i.e. it is a sequence method. The necessary and sufficient condition for this method to be regular is that

$$(b_{nk}) = \left( \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} a_{vk} \right); \quad n, k = 0, 1, 2, \dots \quad (5)$$

should be a T-matrix [1].

**Theorem 2. 1.** If the matrix  $A = (a_{nk})$  is a T-matrix, then the matrix  $B = (b_{nk})$  is also a T-matrix. But not conversely in general.

**Proof.** The proof of the first part of the theorem can be easily shown. The second part of the theorem will be proven by giving a counter-example. Let us define the matrix  $B = (b_{nk})$  as,

$$b_{nk} = \begin{cases} \frac{2^{k-1}}{2^n}, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

Because of the relation (5) we can determine the matrix  $A = (a_{nk})$  with the aid of the inverse transformation matrix of the method  $E_1$  as,

$$(E_1^{-1})_{nk} = \begin{cases} (-1)^{n-k} \binom{n}{k} 2^k, & 0 \leq k \leq n \\ 0 & , k > n . \end{cases}$$

According to this,

$$a_{nk} = \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} 2^v b_{vk}; \quad n, k = 0, 1, 2, \dots \quad (6)$$

Since  $b_{vk} = 0$ , for  $k > v$ , we have

$$a_{nk} = \sum_{v=k}^n (-1)^{n-v} \binom{n}{v} 2^v \frac{2^{k-1}}{2^v} = (-1)^{n+1} (-2)^{k-1} \binom{n-1}{n-k}.$$

Hence,

$$a_{nk} = \begin{cases} (-1)^{n+1} (-2)^{k-1} \binom{n-1}{n-k}, & 0 \leq k \leq n \\ 0 & , k > n \end{cases}$$

It is evident that the matrix  $B = (B_{nk})$  is a T-matrix. But, for instance, if,  $k = 1$ ,  $a_{n1} = (-1)^{n+1}$  and since  $\lim_{n \rightarrow \infty} a_{n1} = \lim_{n \rightarrow \infty} (-1)^{n+1}$  does

not exist, and so the matrix  $A = (a_{nk})$  is not a T-matrix.

**Definition 2.** Let the matrix  $A' = (a'_{nk})$  transform the series  $\sum u_n$ ,  $u_n = s_n - s_{n-1}$ ,  $s_{-1} = 0$ , to the sequence  $(\tau_n)$ ; that is to say that,

$$\tau_n = \sum_{k=0}^{\infty} a'_{nk} u_k; \quad n = 0, 1, 2, \dots \quad (7)$$

If,

$$w_n = \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} \tau_v, \quad w_n \rightarrow \tau \quad (n \rightarrow \infty) \quad (8)$$

then we shall say that the series  $\sum u_n$  is  $(A', E_1)$  - summable to the value " $\tau$ ".

Formally, this can be expressed as,

$$w_n = \sum_{k=0}^{\infty} b'_{nk} u_k; \quad b'_{nk} = \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} a'_{vk}, \quad n = 0, 1, \dots \quad (9).$$

The new method we have just defined is a series-to-sequence transformation. In other words, it is a series method. The regularity of this method is known [1].

**Theorem 2.2.** The methods B and B' are dual if and only if the methods A and A' are dual.

**Proof.** Sufficiency. Let the methods A and A' be dual. In this case, according to the definition of dual methods we have

$$\begin{aligned} b_{nk} &= \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} a_{vk} = \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} (a'_{vk} - a'_{v, k+1}) \\ &= \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} a'_{vk} - \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} a'_{v, k+1} \\ &= b'_{nk} - b'_{n, k+1} \end{aligned}$$

Necessity. Let the methods B and B' be dual. Then the relation  $b_{nk} = b'_{nk} - b'_{n, k+1}$  exists. (10)

If (6) and (10) are considered together, then

$$\begin{aligned} a_{nk} &= \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} 2^v b_{vk} \\ &= \sum_{v=0}^n (-1)^{n-v} 2^v b'_{vk} - \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} 2^v b'_{v, k+1} \\ &= a'_{nk} - a'_{n, k+1} \end{aligned}$$

is obtained. Thus, the proof of the theorem is completed.

**Theorem 2.3.** If the method A' is regular then the method B' is also regular. But not conversely in general.

**Proof.** The proof of the first part of the theorem is easy.

We showed that in the second part of theorem 2.1; There is a regular method B such that the method A is regular. Which corresponding to this. If we consider the fact that A is regular if and only if A' is regular, the dual B' of B is regular while the dual A' of A is not regular. This means, there is a non-regular method A' which corresponds the method B' is regular. This completes the proof of the theorem.

**Theorem 2.4.** Let the matrix  $A = (a_{nk})$  be positive for  $n, k = 0, 1, \dots$  and furthermore the methods  $A$  and  $A'$  be dual. If the sequence  $(s_n)$  is positive and nondecreasing then the method  $B'$  is stronger than the method  $A$ .

**Proof.** Let the summability field of the method  $A$  be  $U$ .

To prove the theorem, it is necessary to show that, for every fixed  $n$ , the existence of the limit

$$\lim_{p \rightarrow \infty} \sum_{k=0}^p b'_{nk} u_k = \lim_{p \rightarrow \infty} \left\{ \sum_{k=0}^{p-1} (b'_{nk} - b'_{n, k+1}) s_k + b'_{np} s_p \right\}$$

for every  $s = (s_n) \in U$ . Considering the relations (5) and (9) we have,

$$\lim_{p \rightarrow \infty} \sum_{k=0}^p b'_{nk} u_k = \lim_{p \rightarrow \infty} \left\{ \sum_{k=0}^{p-1} \frac{1}{2^n} \binom{n}{v} a_{vk} s_k + \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} \left( \sum_{i=p}^{\infty} a_{vi} \right) s_p \right\} \quad (11)$$

Since  $s \in U$ , the series  $\sum_{k=0}^{\infty} a_{nk} s_k$  converges to a definite value  $\sigma_n$  for

every  $n$ . Therefore, for  $v = 0, 1, 2, \dots$

$$\lim_{p \rightarrow \infty} \sum_{i=p}^{\infty} a_{vi} s_i = 0. \quad (12)$$

In (11), the first term of the statement within the paranthesis converges to  $\sigma_n$  as  $p \rightarrow \infty$  and  $\sigma_n$  also converges to  $\sigma$  as  $n \rightarrow \infty$ . Hence, it remains to show that the second term converges to zero as  $p \rightarrow \infty$ . According to (12),

$$\lim_{p \rightarrow \infty} \frac{1}{2^n} \sum_{v=0}^n \left( \sum_{i=p}^{\infty} a_{vi} \right) s_p \leq \frac{1}{2^n} \sum_{v=0}^n \lim_{p \rightarrow \infty} \sum_{i=p}^{\infty} a_{vi} s_i = 0$$

is found. Since the method  $B'$  sums every sequence  $s = (s_n)$  which belongs to  $U$ , the method  $B'$  is stronger than the method  $A$ . This completes the proof.

**Theorem 2.5.** Let us suppose that the methods  $A$  and  $A'$  be dual and  $\lim_{k \rightarrow \infty} a'_{nk} = 0$ , for  $n = 0, 1, 2, \dots$ . Then for every series  $\sum u_k$  who-

se the sequence of partial sums forms a bounded sequence, the method B is stronger than the method A'.

**Proof.** Let the summability field of the method A' be F. To prove the theorem we must show existence of the limit

$$t_n = \lim_{p \rightarrow \infty} \sum_{k=0}^p b_{nk} s_k$$

for every fixed n and every  $u = (u_k) \in F$ . Since the methods A and A' are dual, the methods B and B' are dual because of theorem 2.2. Furthermore using the fact that B and B' are dual and (9)

$$\begin{aligned} \lim_{p \rightarrow \infty} \sum_{k=0}^p b_{nk} s_k &= \lim_{p \rightarrow \infty} \sum_{k=0}^p (b'_{nk} - b'_{n,k+1}) s_k \\ &= \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} \lim_{p \rightarrow \infty} \sum_{k=0}^p a'_{vk} u_k - \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} \lim_{p \rightarrow \infty} a'_{v,p+1} s_p \quad (13) \end{aligned}$$

is found. Since  $u = (u_k) \in F$ , the series  $\sum u_k$  is summable by the method A'. If the series  $\sum_{k=0}^{\infty} u_k = c$  (A') the first term of the statement in

(13) converges to  $\tau_n$  as  $p \rightarrow \infty$ ,  $\tau_n$  also converges to  $c$  as  $n \rightarrow \infty$ . Hence, for the proof of the theorem it is sufficient to show the second term of the statement (13) tends to zero. The limit of  $a'_{vp}$  is equal to zero as  $p \rightarrow \infty$  in the hypothesis of the theorem. This implies  $\lim_{p \rightarrow \infty} a'_{v,p+1} s_p = 0$ .

Therefore the proof of the theorem is completed.

#### ÖZET

Bu çalışmada, Dual Toplama Metodları kavramı tanımlandı ve daha sonra geliştirildi. Ayrıca tarif edilen yeni metodlarla ilgili çeşitli içerirlik teoremleri verildi.

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