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ON THE LIMITING DISTRIBUTION FOR BERNOULLI TRIALS WITHIN A MARKOV CHAIN CONTEXT

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ABSTRACT

Let X_1, X_1,... be a sequence of Bernoulli trials governed by a homogeneous third-order two-state Markov chain. The probabilirty function of S_n, the number of occurrences in n successive trials, is obtained. In addition to this, assuming that steady state is already attained the limiting function of S_n is obtained under the condition that $nP\ (X_i = 1) = u$ as $n \to \infty$. Finally we can note that this limiting probability function can be generalized in terms of Leguerre polynomials as already shown in the relevant literature.

KEY WORDS: Third-Order Markov Chain, Markov Bernoulli Sequence

INTRODUCTION

It is well known that for the independent Bernoulli sequence, the limit of the probability function of S_n is Poisson with parameter u. In 1960, Edwards [3] formulated the problem as a Markov chain for the Bernoulli sequence with a correlation between trials and called such a sequence of dependent random variables "Markov Bernoulli sequence". The unconditional probabilities of this sequence are $P(X_i = 1) = p$ and $P(X_i = 0) = q = 1 - p$ for all $i = 1, 2 \ldots$.

Wang [4] obtained the limiting probability function of S_n for the Bernoulli sequence governed by a first–order two–state Markov chain. Brainerd and Chang [1] derived the probability function of S_n in the case of the second–order Markov chain and Brainerd [2] obtained the limit of this probability function.

THE DISTRIBUTION OF S_n IN THE THIRD–ORDER CASE

Let $X_1, X_2, \ldots$ be a Markov Bernoulli sequence governed by a third–order Markov chain. Denote the unconditional probabilities of X_i by $P\ (X_i = 1) = p$, $P\ (X_i = 0) = q = (1–p)$ and the conditional

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probabilities by $P(X_i = 0 / X_{i-1} = 1) = \alpha$, $P(X_i = 0 / X_{i-2} = 1, X_{i-1} = 0) = \beta$, $P(X_i = 0 / X_{i-3} = 1, X_{i-2} = 0, X_{i-1} = 0) = \S$, $P(X_i = 0 / X_{i-1} = 0) = w$, $P(X_i = 0 / X_{i-2} = 0, X_{i-1} = 0) = \lambda$, $P(X_i = 0 / X_{i-3} = 0, X_{i-2} = 0, X_{i-1} = 0) = \delta$ for all $i = 1, 2, \ldots$ These probabilities are independent of i. In the third-order Markov chain the following identities can be written immediately:

$$P(X_{i-1} = 0, X_i = 1) = P(X_{i-1} = 1, X_i = 0), \tag{1}$$

$$P(X_{i-2} = 0, X_{i-1} = 0, X_i = 1) = P(X_{i-2}=1, X_{i-1}=0, X_i=0) \tag{2}$$

$$P(X_{i-3} = 0, X_{i-2} = 0, X_{i-1} = 0, X_i = 1) = P(X_{i-3} = 1, X_{i-2} = 0, X_{i-1} = 0, X_i = 0) \tag{3}$$

Let

Y = The number of trials to observe the first occurrence of 1, after i. The conditional probabilities of Y are

$$P(Y = 0 / X_i = 1) = 0,$$
$$P(Y = 1 / X_i = 1) = 1 - \alpha,$$
$$P(Y = 2 / X_i = 1) = \alpha (1-\beta),$$
$$\vdots$$
$$P(Y = k / X_i = 1) = \alpha \beta \S \delta^{k-4} (1-\delta)$$

and

$$P(Y = 0 / X_i = 0) = 0,$$
$$P(Y = 1 / X_i = 0) = 1-w,$$
$$P(Y = 2 / X_i = 0) = w (1 - \lambda),$$
$$\vdots$$
$$P(Y = k / X_i = 0) = w \lambda \delta^{k-3} (1-\delta).$$

The probability generating functions of Y are for $P(Y = k / X_i = 1)$

$$g_1(t) = \sum_{k=0}^{\infty} P(Y = k / X_i = 1) t^k$$

$$= \frac{(1-\alpha)t + [\alpha(1-\beta) - \delta(1-\alpha)]t^2 + [\alpha\beta(1-\S) - \alpha\delta(1-\beta)]t^3}{1 - \delta t} + \frac{[\alpha\beta\S(1-\delta) - \alpha\beta(1-\S)\delta]t^4}{1 - \delta t}, \tag{4}$$

for $P(Y = k / X_i = 0)$

$$g_0(t) = \sum_{k=0}^{\infty} P(Y = k / X_i = 0)\ t^k$$

$$= \frac{(1-w)t + [w(1-\lambda) - \delta(1-w)]t^2 + [w\lambda(1-\delta) - w\delta(1-\lambda)]\, t^3}{1 - \delta t},$$

and for $P(Y = k)$

$$g(t) = \sum_{k=0}^{\infty} P(Y = k)t^k = P(X_i = 1)\, g_1(t) + P(X_i = 0)\, g_0(t)$$

$$= \frac{pt + p(\alpha - \delta)t^2 + p(\alpha\beta - \alpha\delta)t^3 + p\,[\alpha\beta(\S - \delta)]t^4}{1 - \delta t} \qquad (5)$$

where the following identities obtained from (1), (2) and (3) have been used:

$$\alpha p = q\,(1-w), \qquad \alpha\beta p = (1-\lambda)\, wp, \qquad \alpha\beta\S p = \lambda wq\,(1-\delta). \qquad (6)$$

Let

Y_k = The number of trials to observe the k th occurrence of 1 after i th trial. At the initial trial X_i is 1 or 0. Thus we can write

$$Y_k = Y + (k-1)\, Y.$$

Y_k is equal to the sum of k independent random variables. The probability generating function of Y_k is for $k \geq 1$

$$f_k(t) = \sum_{n=0}^{\infty} P(Y_k = n)\ t^n$$

$$= g(t)\,[g_1(t)]^{k-1}.$$

Since

$$P(S_n = k) = P(S_n \geq k) - P(S_n \geq k+1)$$

$$= P(Y_k \leq n) - P(Y_{k+1} \leq n)$$

the probability generating function of S_n is of the form

$$G_k(t) = \sum_{n=0}^{\infty} P(S_n = k)t^n = \sum_{n=0}^{\infty} P(Y_k \leq n)t^n - \sum_{n=0}^{\infty} P(Y_{k+1} \leq n)t^n \qquad (7)$$

$$= \frac{f_k(t)}{1-t} - \frac{f_{k+1}(t)}{1-t}$$

$$= \frac{g(t)\,[g_1(t)]^{k-1} - g(t)\,[g_1(t)]^{k}}{1-t}$$

$$= \frac{g(t)\,[g_1(t)]^{k-1}\,[1-g_1(t)]}{1-t}\,.$$

From (4) and (5) we can obtain $pt[1-g_1(t)] = (1-t)\,g(t)$ and according to this equation, $G_k(t)$ can be written

$$G_k(t) = \frac{[g(t)]^2\,[g_1(t)]^{k-1}}{pt}$$

$$= pt^k\,(a + bt + ct^2 + dt^3)^2\,(f + ht + lt^2 + dt^3)^{k-1}\,\frac{(1-\delta)^{k+1}}{(1-\delta t)^{k+1}} \tag{8}$$

where

$$a = \frac{1}{1-\delta}\,, \quad b = \frac{\alpha-\delta}{1-\delta}\,, \quad c = \frac{\alpha\beta-\alpha\delta}{1-\delta}\,, \quad d = \frac{\alpha\beta\,(\S-\delta)}{1-\delta}\,,$$

$$f = \frac{1-\alpha}{1-\delta}\,, \quad h = \frac{\alpha\,(1-\beta)-\delta\,(1-\alpha)}{1-\delta}\,,$$

$$l = \frac{\alpha\beta\,(1-\S)-\alpha\beta\,(1-\beta)}{1-\delta}\,.$$

In (8) $(f + ht + lt^2 + dt^3)^{k-1}$ is the probability generating function of the multinomial distribution and $(1-\delta)^{k+1}/(1-\delta t)^{k+1}$ is the probability generating function of the negative binomial distribution. From (7) it can be shown that $P(S_n = k)$ is the coefficient of t^n.

The expansion of the $G_k(t)$ allows us to write for $k \geq 1$

$$P(S_n = k) = p[a^2 C_{n-k} + 2ab\,C_{n-k-1} + (b^2 + ac)\,C_{n-k-2} + (2ad + 2bc)\,C_{n-k-3} + (c^2 + 2bd)\,C_{n-k-4} + 2cd\,C_{n-k-5} + d^2\,C_{n-k-6}], \tag{9}$$

and for $k = 0$

$$P(S_n = 0) = P(X_i = 0,\ X_{i+1} = 0,\ \ldots,\ X_{i+n} = 0) = qw\lambda\delta^{n-2} \tag{10}$$

where

$$C_{n-k-r} = \sum_{m=0}^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{i}\binom{k-1}{j,\,i-j,\,m-i}\binom{k+n-k-m-i-j-r}{n-k-m-i-j-r}\,d^{j}\,l^{i-j}\,h^{m-i}\,f^{k-1-m}\,\delta^{n-k-m-i-j-r} \tag{11}$$

LIMITING PROBABILITY FUNCTION

If np = u is held fixed as $n \to \infty$ from (6) we can write

$$1 - w = \frac{\alpha u}{n - u}, \quad 1 - \lambda = \frac{\alpha\beta\, u}{w(n-u)}, \quad 1 - \delta = \frac{\alpha\beta\S u}{\lambda w\,(n - u)}. \quad (12)$$

The equations in (12) show that w, λ and δ approach 1 as $n \to \infty$ and we can also obtain

$$\lim_{n\to\infty} p = \lim_{n\to\infty} \frac{u}{n} = 0, \quad \lim_{n\to\infty} q = 1,$$

$$\lim_{n\to\infty} \delta^n = \lim_{n\to\infty} \left[1 - \frac{\alpha\beta\S u}{\lambda w\,(n - u)}\right]^n = e^{-\alpha\beta\S u}.$$

If we rearrange the expression (11) and let $n \to \infty$ we obtain

$$C_k = \lim_{n\to\infty} \frac{C_{n-k-r}}{1-\delta} = \alpha\beta\S u e^{-\alpha\beta\S u} \sum_{m=o}^{k-1} \binom{k-1}{m} \frac{(1 - \alpha\beta\S)^m\,(\alpha^2\beta^2\S^2 u)^{k-1-m}}{(k - m)!}$$

which is independent of r. From (9) and (10) it can be written for $k \geq 1$

$$\lim_{n\to\infty} P\,(S_n = k) = \lim_{n\to\infty} \frac{\frac{u}{n}\,[1 + (\alpha-\delta) + (\alpha\beta-\alpha\delta) + \alpha\beta\,(\S-\delta)]^2}{\frac{\alpha\beta\S}{\lambda w\,(n-u)}} \cdot \frac{C_{n-k-r}}{1 - \delta} = \alpha\beta\S C_k$$

$$= \alpha^2\beta^2\S^2 u e^{-\alpha\beta u\S} \sum_{m=o}^{k-1} \frac{(k-1)!}{m!(k-1-m)!} \frac{(1-\alpha\beta\S)^m\,(\alpha^2\beta^2\S^2 u)^{k-1-m}}{(k - m)!}, \quad (13)$$

and for $k = 0$

$$\lim_{n\to\infty} P\,(S_n = 0) = \lim_{n\to\infty} \left(1 - \frac{u}{n}\right) w\lambda \left[1 - \frac{\alpha\beta\S u}{\lambda w\,(n-u)}\right]^{n-2} = e^{-\alpha\beta\S u}.$$

In (13) by taking $k-1-m = j$ it is obtained

$$\lim_{n\to\infty} P(S_n = k) = \frac{\alpha^2\beta^2\S^2 u\,(1-\alpha\beta\S)^{k-1}}{k}\, e^{-\alpha\beta\S u} \sum_{j=o}^{k-1} \frac{k!}{j!\,(j+1)!\,(k-1-m)!} \left(\frac{\alpha^2\beta^2\S^2 u}{1-\alpha\beta}\right)^j. \quad (14)$$

Since

$$\overset{(1)}{L_r}(y) = \sum_{i=0}^{r} \frac{(r+1)!\,(-y)^i}{(r-i)!\,i!\,(1+i)!}, \quad r = 0, 1, 2, \ldots$$

is the first–order Leguerre polynomials (see [2]), it can be shown that (14) is of the form

$$\lim_{n\to\infty} P(S_n = k) = \frac{\alpha^2\beta^2§^2u\,(1-\alpha\beta§)^{k-1}}{k} e^{-\alpha\beta§u} \overset{(1)}{L_{k-1}} \left(-\frac{\alpha^2\beta^2§^2u}{1-\alpha\beta§}\right). \quad (15)$$

CONCLUDING REMARKS

In the second–order Markov chain $§ = \delta = \lambda$. For this case, from (15) we can obtain for $k \geq 1$

$$\lim_{n\to\infty} P(S_n = k) = \frac{\alpha^2\beta^2u\,(1-\alpha\beta)^{k-1}}{k} e^{-\alpha\beta u} L_{k-1} \left(-\frac{\alpha^2\beta^2u}{1-\alpha\beta}\right), \quad (16)$$

and for $k = 0$

$$\lim_{n\to\infty} P(S_n = k) = e^{-\alpha\beta u}$$

Expression (16) is the limiting probability function in [2] for the second order case.

Comparing (16) and (15) shows that we can write the following expression in the case of the v th–order Markov chain for $k \geq 1$

$$\lim_{n\to\infty} P(S_n = k) = \frac{\alpha_1^2 \ldots \alpha_v^2 u\,(1-\alpha_1 \ldots \alpha_v)^{k-1}}{k} e^{-\alpha_1 \ldots \alpha_v u} \overset{(1)}{L_{k-1}}$$

$$\left(-\frac{\alpha_1^2 \ldots \alpha_v^2 u}{1-\alpha_1 \ldots \alpha_v}\right)$$

and for $k = 0$

$$\lim_{n\to\infty} P(S_n = k) = e^{-\alpha_1 \ldots \alpha_v u}$$

where for $j = 1, 2, \ldots, v$

$$\alpha_j = P(X_i = 0 \,/\, X_{i-j} = 1, \quad X_{i-j-1} = 0, \quad X_{i-1} = 0).$$

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