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Some Characterizations For the Natural Lift Curves and The Geodesic Sprays

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Some Characterizations For The Natural Lift Curves and The Geodesic Sprays

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ABSTRACT

In this paper, we dealt with the natural lift curves of the spherical indicatries of tangent, principal normal, binormal vectors and the fixed centrode of a curve.

Furthermore, some interesting results about the original curve were obtained, depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the bundle [2.]

1. THE NATURAL LIFT CURVES AND GEODESIC

Definition 1.1:

Let M be a hypersurface in E^{n+1} and let $\alpha: I \to M$ be a parametrized curve. X being a smooth tangent vector field on M, α is called an integral curve of X if

$$(1) \qquad \frac{d}{dt}(\alpha(t)) = X(\alpha(t) \qquad \text{(for all } t \in I).$$

T_pM being the tangent space of M at p, we have

$$TM = \bigcup_{p \in M} T_pM = \chi(M),$$

where $\chi(M)$ is the space of vector fields of M [1].

Definition 1.2:

For any parametrized curve $\alpha: I \to M$, the parametrized curve $\bar{\alpha}: I \to TM$
given by

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(2) $\bar{\alpha}(t) = (\alpha(t), \dot{\alpha}(t)) = \dot{\alpha}(t)|_{\alpha(t)}$

is called the naturel lift of α on TM [2].

Thus, we can write

$$(3) \qquad \frac{d\bar{\alpha}}{dt} = \frac{d}{dt}(\dot{\alpha}(t))|_{\alpha(t)} = D_{\dot{\alpha}(t)}\dot{\alpha}(t),$$

where D is the connection on E^{n+1}.

Definition 1.3:

For $v \in TM$, the smooth vector field $X \in \chi(TM)$ defined by

$$(4) \qquad X(v) = - \langle v, S(v) \rangle N|_p$$

is called the geodesic spray on the manifold TM [2], where N is the unit normal vector field of M.

THEOREM 1.1:

The natural lift $\bar{\alpha}$ of the curve α is an integral curve of the geodesic spray X if and only if α is a geodesic on M.

Proof ($\Rightarrow$): Let α be an integral curve of the geodesic spray X. Then we have

$$(5) \qquad X(\bar{\alpha}(t)) = \frac{d}{dt}(\bar{\alpha}(t))\Big|_{\bar{\alpha}(t)}.$$

Since X is a geodesic spray on TM, we have

$$(6) \qquad X(\bar{\alpha}(t)) = - \langle \bar{\alpha}(t), S(\bar{\alpha}(t)) \rangle N\Big|_{\alpha(t)}.$$

Form (2), (5) and (6) we get:

$$(7) \qquad \frac{d}{dt}(\dot{\alpha}(t)|_{\alpha(t)}) = - \langle \dot{\alpha}(t)|_{\alpha(t)}, S(\dot{\alpha}(t)|_{\alpha(t)}) \rangle N|_{\alpha(t)}.$$

Since the last equation is true for all $\alpha(t)$, using (3) we find that

$$(8) \qquad D_{\dot{\alpha}(t)}\dot{\alpha}(t) = - \langle \dot{\alpha}(t), S(\dot{\alpha}(t)) \rangle N.$$

Thus, from the last equation and Gauss Equation we have

$$(9) \qquad \bar{D}_{\dot{\alpha}(t)}\dot{\alpha}(t) = 0,$$

where $\bar{D}$ is the Gauss-Connection on M. Hence, we have seen that α is a geodesic on M.

($\Leftarrow$): Now, assume that α be a geodesic on M. Then

$$\bar{D}_{\dot{\alpha}(t)} \dot{\alpha}(t) = 0 .$$

Hence, by the Gauss-Equation we have

$$D_{\dot{\alpha}(t)} \dot{\alpha}(t) \Big|_{\alpha(t)} + < \dot{\alpha}(t), S(\dot{\alpha}(t)) > N|_{\alpha(t)} = 0 .$$

Since X is the geodesic spray, we can write:

$$\frac{d}{dt} (\dot{\alpha}(t)|_{\alpha(t)}) - X \left(\dot{\alpha}(t) \right) |_{\alpha(t)} = 0$$

$$\Rightarrow \frac{d}{dt} (\dot{\alpha}(t)|_{\alpha(t)}) = X \left(\dot{\alpha}(t)|_{\alpha(t)} \right).$$

From the definition (1.2) we find that

$$\frac{d}{dt} (\bar{\alpha}(t)) = X (\bar{\alpha}(t))$$

2. THE NATURAL LIFT OF THE SPHERICAL INDICATRIX OF TANGENT VECTORS OF A CURVE

We will investigate how α must be a curve satisfying the condition that $\bar{\alpha}_T$ is an integral curve of the geodesic spray, where α_T being the spherical indicatrix of tangent vectors of α, $\bar{\alpha}_T$ is the natural lift of the curve α_T.

If $\bar{\alpha}_T$ is an integral curve of the geodesic spray, then by means of Theorem 1.1

$$\bar{D}_{\dot{\alpha}_T} \dot{\alpha}_T = 0,$$

that is,

$$D_{\dot{\alpha}_T} \dot{\alpha}_T + < \dot{\alpha}_T, S \left(\dot{\alpha}_T \right) > T(s) = 0,$$

where s is the arc-length of α.

Since $S = I_2$ for the unit sphere, we have

$$D_{\dot{\alpha}_T} \dot{\alpha}_T + \| \dot{\alpha}_T \|^2 T(s) = 0,$$

$$\Rightarrow D_{\dot{\alpha}_T} k_1 N + k_1^2 T(s) = 0,$$

$$\Rightarrow \frac{d}{ds_T}(k_1 N) + k_1^2 T(s) = 0,$$

where s_T is the arc-length of α_T . After some algebraic calculation we find that

$$(k_1^2 - k_1)\, T + \frac{\dot{k}_1}{k_1} N - k_2 B = 0\,.$$

Because of T, N,B are linear independent, we have

$k_1^2 - k_1 = 0$, $(k_1 = 0{,}1)$;

$(\dot{k}_1 / k_1) = 0$, (k_1 = cons. and $k_1 \neq 0$);

$k_2 = 0$.

COROLLARY 1:

If the curve α is a unit circle, then its spherical indicatrix α_T is a great circle on the unit sphere. In this case, the natural lift $\bar{\alpha}_T$ of α_T is an integral curve of the geodesic spray on the tangent bundle T (S^2), where S^2 is an unit 2– sphere in E^3.

3. THE NATURAL LIFT OF THE SPHERICAL INDICARTIX OF PRINCIPAL NORMAL VECTORS OF α

We will investigate in this section, how α must be a curve satisfying the condition that $\bar{\alpha}_N$ is an integral curve of the geodesic spray, where α_N being the spherical indicatrix of principal normal vectors of α, $\bar{\alpha}_N$ is the natural lift of the curve α_N.

If $\bar{\alpha}_N$ is an integral curve of the geodesic spray, then by means of Theorem 1.1 we have

$$\bar{D}_{\dot{\alpha}_N} \dot{\alpha}_N = 0,$$

that is,

$$D_{\dot{\alpha}_N} \dot{\alpha}_N + < \dot{\alpha}_N, S(\dot{\alpha}_N) > N(s) = 0$$

$$\Rightarrow D_{\dot{\alpha}_N} \dot{\alpha}_N + \| \dot{\alpha}_N \|^2 N(s) = 0,$$

$$\Rightarrow D_{\dot{\alpha}_N} (-k_1 T + k_2 B) + (k_1^2 + k_2^2) N = 0$$

$$\Rightarrow \frac{d}{ds_N}(-k_1 T + k_2 B) + (k_1^2 + k_2^2) N = 0,$$

where s_N is the arc-length of α_N. After some algebraic calculation we find that

$$-\dot{k}_1 T + (\| w \|^3 - \| w \|^2) N + \dot{k}_2 B = 0,$$

where $\| w \|^2$ is equal to $k_1^2 + k_2^2$, that is, w is the Darboux vector. Since T,N,B are linear independent,

$$\dot{k}_1 = 0, (k_1 = \text{cons.});$$

$$\dot{k}_2 = 0, (k_2 = \text{cons.});$$

$$k_1 = k_2 = 0 \text{ or } k_1^2 + k_2^2 = 1.$$

COROLLARY 2:

If the curve α is a circular helix, then its spherical indicatrix α_N is a great circle on the unit sphere. In this case, the natural lift $\bar{\alpha}_N$ of α_N is an integral curve of the geodesic spray on the tangent bundle T (S^2).

4. THE NATURAL LIFT OF THE SPHERICAL INDICATRIX OF THE BINORMAL VECTORS OF α

We will investigate how α must be a curve satisfying the condition that α_B is an intergral curve of the geodesic spray, where α_B being the spherical indicatrix of binormal vectors of α, $\bar{\alpha}_B$ is the natural lift of the curve α_B.

If $\bar{\alpha}_B$ is an integral curve of the geodesic spray, by means of Theorem 1.1

$$\bar{D}_{\dot{\alpha}_B} \dot{\alpha}_B = 0$$

that is,

$$D_{\dot{\alpha}_B} \dot{\alpha}_B + \langle \dot{\alpha}_B, S(\dot{\alpha}_B) \rangle B(s) = 0,$$

$$\Rightarrow \frac{d}{ds_B}(\dot{\alpha}_B) + \| \dot{\alpha}_B \|^2 B = 0,$$

where s_B is the arc-length of the curve α_B. Thus we find that

$$k_1 T + (\dot{k}_2 / k_2) N + (k_2^2 - k_2) B = 0.$$

Since T,N,B are linear independent, we have

$$k_1 = 0, \dot{k}_2 / k_2 = 0, k_2^2 - k_2 = 0.$$

Therefore we get $k_1 = 0$ and $k_2 = 1$. Since we don't find a curve whose its curvature is equal to 0 torsion is equal to 1, in the same time. We may give the following corollary:

COROLLARY 3:

There is no curve α whose the spherical indicatrix α_B is a great circle on the unit sphere. Therefore, the natural lift $\bar{\alpha}_B$ of the curve α_B can never be an integral curve of the geodesic spray on the tangent bundle T (S^2).

5. ON THE NATURAL LIFT OF THE FIXED CENTRODE

Let α_C be the fixed centrode of the motion described by the curve α, [3]. Then the curve is given by

$$\alpha_C = C(s), \text{ and } C = w / \|w\|.$$

where w being the Darboux vector,

If $\Phi = \Phi(s)$ denotes the angle between B and C, then we have

$$k_1 = \|w\| \cos\Phi,$$

$$k_2 = \|w\| \sin\Phi.$$

Now we will investigate, how α must be a curve satisfying the condition that $\bar{\alpha}_C$ is an integral curve of the geodesic spray. Let the natural lift $\bar{\alpha}_C$ of α_C be an integral curve of the geodesic spray. According to the Theorem 1.1, we write

$$\bar{D}_{\dot{\alpha}_C} \dot{\alpha}_C = 0,$$

$$\Rightarrow D_{\dot{\alpha}_C} \dot{\alpha}_C + \langle \dot{\alpha}_C, S(\dot{\alpha}_C) \rangle C = 0;$$

$$\Rightarrow \frac{d}{ds_C} (\dot{\alpha}_C) + \| \alpha_C \|^2 C = 0,$$

where s_C is the arc-length of the curve α_C and $C = \sin\Phi\, T + \cos\Phi\, B$, [4]. After some algebraic calculation we find that

$$(\ddot{\phi}\cos\Phi - \dot{\phi}^2 \sin\Phi + \dot{\phi}^3 \sin\Phi)\, T + (k_1\, \dot{\phi} \cos\Phi + k_2\, \dot{\phi}\sin\Phi)\, N$$

$$+ (-\ddot{\phi}\sin\Phi - \dot{\phi}^2 \cos\Phi + \dot{\phi}^3 \cos\Phi)\, B = 0.$$

Since the Serret-Frenet vectors, T,N,B are linear independent,

$$\ddot{\phi}\cos\Phi - \dot{\phi}^2 \sin\Phi + \dot{\phi}^3 \sin\Phi = 0,$$

$$k_1 \dot{\phi}\cos\Phi + k_2\, \dot{\phi}\sin\Phi = 0,$$

$$-\dot{\phi}\sin\Phi - \dot{\phi}^2 \cos\Phi + \dot{\phi}^3 \cos\Phi = 0.$$

The last equations imply that $\dot{\phi} = 0$ or $k_1 = k_2 = 0$. Since $\dot{\phi} = 0$, k_1/k_2 is constant. This implies that α is an helix. The condition $k_1 = k_2 = 0$ implies that α is a line. In the second case, we don't have a solution. Then we have the following result.

COROLLARY 4:

If the curve α is an helix, then, its fixed centrode α_C is a great circle on the unit sphere. In this case, the natural lift $\bar{\alpha}_C$ of α_C is an integral curve of the geodesic spray on the tangent bundle $T(S^2)$.

Form Corollary 2 and Corollary 4, we obtain the following result:

COROLLARY 5:

If the natural lift $\bar{\alpha}_N$ of α_N is an integral curve of the geodesic spray on $T(S^2)$, then the natural lift $\bar{\alpha}_C$ of the fixed centrode α_C is an integral curve of the geodesic spray on $T(S^2)$.

ÖZET:

Bu çalışmada, bir eğrinin tegetler, aslinormaller ve binormallerinin küresel göstergelerine, ve sabit pol eğrisine ait tabii lift eğrileri üzerinde durulmuştur.

Bundan başka, bu lift eğrilerinin T(S2) demeti üzerindeki geodezik spraylerin birer intergral eğrisi olması için esas eğriye ait bazı ilginç neticeler elde edilmiştir.

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