Commun. Fac. Sci. Univ. Ank., Series A_1
V. 35, pp. 35 - 44 (1986)

CURVATURE MATRICES AND DARBOUX MATRICES OF MOTIONS ALONG A CURVE

ERDOĞAN ESİN and H. HİLMİ HACISALİHOĞLU

Department of Mathematics, Gazi University, Ankara, Turkey

(Received: September 25, 1984)

ABSTRACT

In this paper we giwe some relations between the Darboux matrices of the frame motions along a curve and the matrices of higher curvatures of the curve. First we describe entries of the Darboux matrix of the Serret-Frenet motion in E^n and hence we obtain some results. Furthermore we introduce motion of the natural frame field for the pair curve-hypersurface in E^n and so we obtain some relations between the Darboux matrix of this motion and the curvature matrix for this pair.

BASIC CONCEPTS

In E^n, the Euclidean n-space, a curve is a C^∞ map α from an open subset of IR into E^n. Let α be a curve in E^n with the unit tangent Vector field U_1, D be natural connection on E^n and the system $\{U_1,\ldots,U_n\}$ be linearly independent, where

$$U_i = D_{U_1} U_{i-1} , 1 < i \leq n .$$

Then the orthonormal system $\{V_1, V_1,\ldots, V_n\}$ which is obtained by Gram-Schmidt process from $\{U_1, U_1,\ldots,U_n\}$ is called the Frenet frame field of the curve α in E^n, here note that $U_1 = V_1$.

DEFINITION 1: Let $\{V_1, V_2,\ldots,V_n\}$ be the Frenet frame field of a curve $\alpha : I \longrightarrow E^n$, $I \subset$ IR. Then, for each i, $1 \leq i < n$, the function

$$k_i : I \longrightarrow \mathbb{R}$$

defined for $s \in I$ by

$$k_i(s) = \langle V'_i(s) , V_{i+1}(s) \rangle$$

is called the i^{th} curvature function of the curve α and $k_i(s)$ is called the i^{th} curvature of the curve α at $\alpha(s)$ [Hacısalihoğlu (1983)].

Henee we have the following theorem [Hacısalihoğlu (1983)]:

THEOREM 1: (Frenet formulas): If $\alpha : I \longrightarrow E^n$ is a unitspeed curve then

$$D_{V_1} V_i = V'_i = -k_{i-1} V_{i-1} + k_i V_{i+1},$$

where $1 \leq i \leq n$, $k_o = k_n = 0$.

It is possible to write the Frenet formulas in the form

$$\begin{bmatrix} V'_1 \\ V'_2 \\ \cdot \\ \cdot \\ \cdot \\ V'_{n-1} \\ V'_n \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & \dots & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -k_{n-2} & 0 & k_{n-1} \\ 0 & 0 & 0 & \dots & 0 & -k_{n-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_{n-1} \\ V_n \end{bmatrix}$$

or simply

$$V' = K(V)\, V .$$

The matrix $K(V)$ is known as the (higher) curvature matrix of the curve α in E^n [Hacısalihoğlu (1983)].

Now we consider a hypersurface M in E^n and a curve α which lies on M. Let $\{X_1,\dots,X_{n-1}\}$ be the Frenet frame field of α in M and X_n be the unit normal vector field to M. Then the orthonormal system $\{X_1,\dots,X_{n-1}, X_n\}$ is called natural frame field for the curve-hypersurface pair (α, M) or the strip (α, M) [Guggenheimer (1963)].

DEFINITION 2: Let M be a hypersurface in E^n and α be a curve on M. Then, for each i, $1 \leq i < n-1$, the function

$$k_{ig} : I \longrightarrow IR$$

defined for $s \in I$ by

$$k_{ig}(s) = \langle X'_i(s) , X_{i+1}(s) \rangle$$

is called the i^{th} geodesic curvature function of the curve α and $k_{ig}(s)$ is called the i^{th} geodesic curvature of the curve α at $\alpha(s)$ [Guggenheimer (1963)] .

THEOREM 2: Let M be a hypersurface in E^n and α be a curve on M. Then the derivative formulas of the natural frame field $\{X_1, \ldots, X_{n-1}, X_n\}$ are

$$D_{X_1} X_i = X_i' = -k_{(i-1)g} X_{i-1} + k_{ig} X_{i+1} + II(X_1, X_i) X_n ,$$

$$D_{X_1} X_n = -II(X_1,X_1) X_1 - II(X_1,X_2)X_2 - \ldots - II(X_1,X_{n-1})X_n$$

where $1 \leq i \leq n-1$ and $k_{og} = k_{(n-1)g} = 0$ [Guggenheimer 1963].

We can write these derivative formulas in the matrix form

$$\begin{bmatrix} X'_1 \\ X'_2 \\ \cdot \\ \cdot \\ \cdot \\ X'_{n-1} \\ X'_n \end{bmatrix} = \begin{bmatrix} 0 & k_{1g} & 0 & \ldots & 0 & 0 & II(X_1,X_1) \\ -k_{1g} & 0 & k_{2g} & \ldots & 0 & 0 & II(X_1,X_2) \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & -k_{(n-2)g} & 0 & II(X_1,X_{n-1}) \\ -II(X_1,X_1) & \cdot & \cdot & \cdot & \cdot & -II(X_1,X_{n-1}) & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_{n-1} \\ X_n \end{bmatrix}$$

or simply

$$X' = K(X)\, X .$$

The matrix K(X) is known as the (higher) curvature matrix (or the Cartan matrix) for the pair (α, M) [Guggenheimer (1963)].

Let y and x be the position vectors, represented by column matrices, of a point P in the fixed space Σ^n and the moving space E^n, respectively. A continuous series of displacements, given by

$$y = Ax + b,$$

where the orthogonal matrix A and the translation vector b are functions of a parameter s which may be identified with the time, is called a motion. Now we consider the rotational motion, given by

$$y = Ax.$$

The matrix

$$W = A'A^t$$

is called the angular velocity matrix or the Darboux matrix of the motion [Bottema and Roth (1979)].

DARBOUX MATRICES OF SOME MOTIONS IN THE EUCLIDEAN n-SPACE

We consider now the moving space E^n and the fixed space Σ^n as reference space and so we describe the Frenet-Serret motion in E^n which is defined in terms of a curve α fixed in Σ^n.

Let $E = \{O; e_1,\ldots,e_n\}$ be the standard orthonormal frame and $H = \{o; x_1,\ldots,x_n\}$ be the moving orthonormal frame. We denote the Frenet frame of the curve α with the arc-length parameter by $V = \{Q; V_1,\ldots,V_n\}$ at a point Q. The Frenet-Serret motion is such that the moving frame H moves with o along α while rotating so that the $x_1,\ldots,x_{n-1}$ axes always coincide with, respectively, the $V_1,\ldots,V_{n-1}$ vectors of α. This means that as o coincides with the point Q of α, the frame H coincides with the Frenet-Serret n-handed at Q: V. Obviously, the geometry of this motion is completely defined by α.

Now we can give the Darboux matrix W(E,V), the angular velocity matrix of the Frenet-Serret motion along α in E^n, by the following theorem.

THEOREM 1: The entries of the Darboux matrix $W(E,V) = [w_{ij}]$ of the Frenet-Serret motion along a curve α in E^n can be given by

$$w_{ij} = \sum_{r=1}^{n-1} \det\,[P_{ij}(V_{r+1}),\ P_{ij}(V_r)]\, k_r\,, \tag{1}$$

where, for each i and j, $1 \leq i, j \leq n$, P_{ij} denotes the orthogonal projection which is defined by

$$P_{ij} : T_{E^n}(p) \longrightarrow Sp\ \{e_i, e_j\}$$

$$P_{ij}\left(\sum_{s=1}^{n} u_s e_s\right) = u_i e_i + u_j e_j$$

and k_r is the r^{th} curvature of α.

PROOF: Let $(v_1^q,\ldots,v_n^q)$ represent any vector V_q, $1 \leq q \leq n$, of the frame V. The position of V relative to E is represented by

$$E = AV,$$

where $A \in SO(n)$, $A = [v^j_i]$.

From

$$W\ (E,V) = A'A^t,$$

it follows that

$$[w_{ij}] = [v_i^{r'}] \; [v^r{}_j] = \left[\sum_{r=1}^{n} v_i^{r'} \; v^r{}_j \right]$$

$$w_{ij} = \sum_{r=1}^{n} v_i^{r'} \, v_j^r \, .$$

This together with the Frenet formulas implies

$$w_{ij} = \sum_{r=1}^{n} (-k_{r-1} v_i^{r-1} + k_r v_i^{r+1}) \, v_j^r$$

or with $k_o = k_n = 0$

$$w_{ij} = \sum_{r=1}^{n-1} (v_i^{r+1} v^r{}_j - v_i^r \, v_j^{r+1}) \, k_r.$$

It is easily seen that $w_{ij} = -w_{ji}$. On the other hand, for the tangent space $T_{E^n}(p)$ at a point p we can write

$$T_{E^n}(p) = Sp\{e_i, e_j\} \oplus Sp\{e_1, \ldots, e_{i-1}, \; e_{i+1}, \ldots, e_{j-1}, \; e_{j+1}, \ldots, e_n\}.$$

Thus using the orthogonal projections P_{ij} we obtain

$$w_{ij} = \sum_{r=1}^{n-1} \det \, [P_{ij}(V_{r+1}), P_{ij}(V_r)] \, k_r$$

since

$$v_i^{r+1} \, v_j^r - v_i^r \, v_j^{r+1} = \det \, [P_{ij}(V_{r+1}) \, , P_{ij}(V_r)] \, . \qquad \text{QED}$$

We now give a relation between the curvature matrix of a curve in E^n and the Darboux matrix of the Frenet-Serret motion in E^n by the following theorem.

THEOREM 2: Let K(V) be the curvature matrix of a curve in E^n and W (E,V) be the Darboux matrix of the Frenet-Serret motion in E^n. Then

$$K(V) = -A^t \, W(E,V) \, A. \tag{2}$$

PROOF: Derivating E = AV, with respect to s, we have

$$O = A' \, V + AV'$$
$$V' = -A^t \, A' \, V.$$

If we write

$$V = A^t \, E, \; W(E, V) = A' \, A^t \text{ and } V' = K(V) \, V$$

then we obtain

$$\begin{aligned} V' &= -A^t A' A^t E \\ &= -A^t W(E, V) E \\ K(V) V &= -A^t W(E, V) E \\ K(V) A^t E &= -A^t W(E, V) E \\ K(V) A^t &= -A^t W(E, V) \\ K(V) &= -A^t W(E, V) A . \end{aligned}$$

QED.

COROLLARY 1: Let W_o (E, V) and $K_o(V)$ be, respectively, the Darboux matrix of the Frenet-Serret motion in E^n and the curvature matrix of the curve in E^n for the initial moment s=0. Then

$$K_o(V) = -W_o (E, V) . \tag{3}$$

PROOF: Without any loss of generality we may suppose that, for s=0, the origins in E^n and Σ^n coincide so that $A_o = I_n$. First writing (2) according to s=0 we have

$$K_o(V) = -A^t_o W_o(E, V) A_o.$$

This together with $A_o = I_n$ implies

$$K_o(V) = -W_o(E, V) .$$

QED.

Next we consider a curve α which lies on a hypersurface M in E^n. Then we can speak of a natural frame field $X = \{X_1, \ldots, X_{n-1}, X_n\}$ for the pair (α, M). The motion of the frame X along α also is similar to the motion of the frame V.

THEOREM 3: The entries of the Darboux matrix $W(E,X) = [w_{ij}]$ of the motion of natural frame field for the pair (α, M) along α are given by

$$w_{ij} = \sum_{r=1}^{n-2} \det[P_{ij}(X_{r+1}), P_{ij}(X_r)] k_{rg} - \sum_{s=1}^{n-1} a_s x_i^s x_j^n, \tag{4}$$

where P_{ij} $(1 \leq i,j \leq n, i \neq j)$, k_{rg}, x_i^k $(1 \leq k \leq n)$ and II denote the orthogonal projection, the r^{th} geodesic curvature of α, the i^{th} component of X_k and the second fundamental form of M, respectively.

PROOF: Let $(x_1^q, \ldots, x_n^q)$ represent any vector X_q, $1 \leq q \leq n$, with respect to the standard frame E. Thus the relation between X and E can be stated in the form

$$E = BX,$$

where $B \in SO(n)$, $B = [x_i^j]$.

From

$$W(E,X) = B' B^t$$

it follows

$$[w_{ij}] = [x_i^{r'}] [x_j^r] = \left[\sum_{r=1}^{n} x_i^{r'} x_j^r \right]$$

$$w_{ij} = \sum_{r=1}^{n} x_i^{r'} x_j^r$$

$$w_{ij} = \sum_{r=1}^{n-1} x_i^{r'} x_j^r + x_i^{n'} x_j^n .$$

Here if we use the derivative formulas for X then we get

$$w_{ij} = \sum_{r=1}^{n-1} (-k_{(r-1)g} x_i^{r-1} x^r_j + k_{rg} x_i^{r+1} x^r_j) - \sum_{s=1}^{n-1} a_s x_i^s x^n_j$$

or with $k_{og} = k_{(n-1)g} = 0$

$$w_{ij} = \sum_{r=1}^{n-2} (x_i^{r+1} x^r_j - x^r_i x_j^{r+1}) k_{rg} - \sum_{s=1}^{n-1} a_s x_i^s x^n_j .$$

In addition, we can write that

$$x_i^{r+1} x^r_j - x^r_i x_j^{r+1} = \det [P_{ij} (X_{r+1}), P_{ij} (X_r)]$$

by using the orthogonal projection P_{ij}. So finally we obtain

$$w_{ij} = \sum_{r=1}^{n-2} \det [P_{ij} (X_{r+1}), P_{ij} (X_r)] k_{rg} - \sum_{s=1}^{n-1} a_s x_i^s x_j^n$$

which completes the proof of the theorem.

QED.

THEOREM 4: Let K(X) be the curvature matrix for the pair (α, M) and W(E, X) be the Darboux matrix of the motion of X along α. Then

$$K(X) = -B^t W(E, X) B. \tag{5}$$

PROOF: Derivating $E = BX$, with respect to s, we have

$$\begin{aligned} 0 &= B' X + B X' \\ X' &= -B^t B' X \\ &= -B^t B' B^t E \\ &= -B^t W(E, X) E . \end{aligned}$$

Furthermore, since $X' = K(X) X$ we obtain

$$\begin{aligned} K(X) X &= -B^t W(E, X) E \\ K(X) B^t E &= -B^t W(E, X) E \\ K(X) B^t &= -B^t W(E, X) \\ K(X) &= -B^t W(E, X) B . \end{aligned}$$

QED.

COROLLARY 2: Let $W_o(E, X)$ be the Darboux matrix of the motion of X along α for the initial moment $s=0$.

Then

$$K_o(X) = -W_o(E, X) , \tag{6}$$

where $K_o(X)$ is the curvature matrix for the pair (α, M) at the initial moment $s=0$.

PROOF: If we take $B_o = I_n$ in (5) for $s=0$ then we obtain (6).

QED.

Finally we can establish some relations among the Darboux matrices of the orthonormal frames along a curve. Thus we also establish some relations between the higher curvatures of a curve in E^n and the higher curvatures of the curve on a hypersurface in E^n. For these, we have to consider the position of X relative to V.

THEOREM 5: Let $W(V, X)$ be the Darboux matrix characterrized by

$$W(V, X) = C' C^t,$$

where $C \in SO(n)$ is given by $V = CX$. Then

$$-A^t W(E, V) A = W(V, X) - C B^t W(E, X) A. \tag{7}$$

PROOF: Derivating $V = CX$, with respect to s, we have

$$V' = C' X + C X' .$$

Moreover since

$$\begin{aligned} V' &= -A^t W(E, V) E, \\ X' &= -B^t W(E, X) E, \\ X &= C^t V \end{aligned}$$

the equation $V' = C' X + C X'$ reduces to

$$-A^t W(E\ V) A = W(V, X) - C B^t W(E, X) A$$

which completes the proof. QED.

COROLLARY 3: There exists the relation

$$-W_o(E, V) = W_o(V, X) - W_o(E, X) \tag{8}$$

among the Darboux matrices at the initial moment $s=0$.

PROOF: Since

$$A_o = B_o = C_o = I_n$$

for the initial moment $s=0$, from (7) the proof is clear.

QED.

COROLLARY 4: There exists the relation

$$K(V) = C' C^t + C K(X) C^t \tag{9}$$

between the curvature matrices.

PROOF: If we use the equations

$$\begin{aligned} -A^t W(E, V) A &= K(V), \\ C' C^t &= W(V, X), \\ A &= B C^t \end{aligned}$$

in (7), then we can write

$$K(V) = C' C^t - C B^t W(E, X) B C^t.$$

Moreover since

$$-B^t W(E, X) B = K(X)$$

we obtain

$$K(V) = C' C^t + C K(X) C^t.$$

QED.

COROLLARY 5: There exists the relation

$$K_o(V) = W_o(V, X) + K_o(X) \tag{10}$$

between the curvature matrices at the initial moment $s=0$.

PROOF: It is immediate from Corollary (4).

QED.

REFERENCES

BOTTEMA, O., and ROTH, B., 1979. *Theoretical Kinematics*, North Holland.

GUGGENHEIMER, H.W., 1963. *Differential Geometry*, Mc Graw-Hill, pp: 210, 338–340.

HACISALİHOĞLU, H.H., 1983. *Diferensiyel Geometri*, İnönü Üniversitesi Fen-Edebiyat Fak.