# COMMUNICATIONS

## DE LA FACULTÉ DES SCIENCES DE L'UNIVERSITÉ D'ANKARA

Série $A_1$ : Mathématiques

TOME : 33 ANNÉE : 1984


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Ankara, Turquie

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# Almost Kahler Manifolds Of Constant Holomorphic Sectional Curvature

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(Received March 19, 1984 accepted August 13. 1984)

The aim of the present paper is to study holomorphic curvature of an almost Kahler manifold. In particular, we further clarify the results of Farran [2] and obtain some results similar to those given by Leung and Nomizu [4] for Riemannian manifolds.

## INTRODUCTION

For many years, the differential geometers have studied different classes of almost Hermitian manifolds and have proved a lot of properties concerning Kahler, Almost Kahler, Quasi-Kahler, Nearly Kahler and Hermitian manifolds. More closely in [3], A. Gray has considered nearly Kahler manifolds and has generalized many results of geometry of Kahler manifolds to this class. Moreover, he has defined a very nice class of nearly Kahler manifolds namely those of constant type.

However, not very much is known about the geometry and topology of almost Kahler manifolds. Recently, Hani Farran [2] has proved some results conserning holomorphic sectional curvature and bisectional curvature of an almost Kahler manifold. The aim of this paper is to further clarify the results of Farran and to obtain more results similar to those obtained by Gray [3] and Leung and Nomizu [4].

Let (M, g, J) be an almost Kahler manifold and let K (X,Y), R (X, Y,Z,W) and H (X) be the sectional curvature, Riemannian curvature tensor, and holomorphic sectional curvature, respectively, for arbitrary veators X,Y,Z and W on M. Then the following identities have been proved in [2].

$$R(X,Y,JZ,W) + R(X,Y,Z,JW)$$
$$= \tfrac{1}{2} F((\nabla_X J)Y - (\nabla_Y J)X, (\nabla_Z J)W - (\nabla_W J)Z) \qquad (1)$$
$$R(X,Y,Z,W) - R(X,Y,JZ,JW) = \tfrac{1}{2} g((\nabla_X J)Y - (\nabla_Y J)X,$$
$$(\nabla_Z J)W - (\nabla_W J)Z)$$

and (2)

$$R(JX, JY, JZ, JW) = R(X,Y,Z,W) \qquad (3)$$

where $F(X,Y) = g(JX, Y)$.

Furthermore, we know the following:

**Proposition** (1.1) [2]. In an almost Kahler manifold we have

$$R(X,Y,X,Y) = \frac{1}{32} \{3Q(X+JY) + 3Q(X-JY) - Q(X+Y)$$

$$- Q(X-Y) - 4Q(X) - 4Q(Y)\} + \frac{3}{8} \| (\nabla_X J)Y - (\nabla_Y J)X \|^2 \qquad (4)$$

for any vectors $X,Y \in T_m(M)$, $Q(X) = R(X,JX,X,JX)$.

## 2. Constancy of holomorphic sectional curvature

First, using the proposition (1.1), we shall give an alternative proof of the following theorem due to Farran ([2], p. 714).

**Theorem** (2.1). Let (M, g, J) be an almost Kahler manifold of constant holomorphic sectional curvature C (m) at every point m of M, then the Riemannian curvature tensor of M is of the from:

$$R(X,Y,Z,W) =$$

$$\frac{C(m)}{4} [g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(X,JW)g(Y,JZ)$$

$$- g(X,JZ)g(Y,JW) - 2g(X,JY)g(Z,JW)] + \frac{1}{8} [2g((\nabla_X J)Y$$

$$- (\nabla_Y J)X, (\nabla_Z J)W - (\nabla_W J)Z) + g((\nabla_Z J)Y - (\nabla_Y J)Z,$$

$$(\nabla_X J)W - (\nabla_W J)X) + g((\nabla_Y J)W - (\nabla_W J)Y, (\nabla_X J)Z - (\nabla_Z J)X)]$$

**Proof.** Since, $Q(X) = -H(X) \|X\|^4$, we have

$$Q(X+JY) = -H(X+JY) \| X+JY \|^4$$

$$= -H(X+JY)\{(g(X,X)+g(Y,Y))^2+4(g(X,X)+g(Y,Y))g(X,JY)+4g(X,JY)^2\}$$

Substituting the values of Q (X+JY), Q (X—JY), Q (X+Y) and Q (X—Y) in (4), we get

$$\begin{aligned} R(X,Y,X,Y) = -\frac{1}{32}\, & [3H(X+JY)\{(g(X,X)+g(Y,Y))^2 \\ & +4(g(X,X)+g(Y,Y))g(X,JY)+4g(X,JY)^2\} \\ & +3H(X-JY)\{(g(X,X)+g(Y,Y))^2 \\ & -4(g(X,X)+g(Y,Y))g(X,JY)+4g(X,JY)^2\} \\ & -H(X+Y). \\ & \{(g(X,X)+g(Y,Y))^2+4(g(X,X)+g(Y,Y)) \\ & +4g(X,Y)^2\} \\ & -H(X-Y)\{(g(X,X)+g(Y,Y))^2-4(g(X,X) \\ & +g(Y,Y))g(X,Y)+4g(X,Y)^2\}-4H(X) \\ & g(X,X)^2-4H(Y)g(Y,Y)^2] \\ & +\frac{3}{8}\|(\nabla_X J)Y-(\nabla_Y J)X\|^2. \end{aligned} \tag{5}$$

Thus, for constant holomorphic sectional curvature C (m), we have

$$R(X,Y,X,Y) = \frac{C(m)}{4}[g(X,Y)^2-g(X,X)g(Y,Y)-3g(X,JY)^2] + \frac{3}{8}\|(\nabla_X J)Y-(\nabla_Y J)X\|^2 \tag{6}$$

Replacing Y by Y+W in (6), we get

$$\begin{aligned} R(X,Y,X,W) = \frac{C(m)}{4}\, & [g(X,Y)g(X,W)-g(X,X)g(Y,W) \\ & -3g(X,JY)g(X,JW)] \\ & +\frac{3}{8}[(g((\nabla_X J)Y-(\nabla_Y J)X,(\nabla_X J)W-(\nabla_W J)X)] \end{aligned} \tag{7}$$

Again, replacing X by X+Z in (7), we have

$$R(X,Y,Z,W) + R(Z,Y,X,W)$$
$$= \frac{C(m)}{4} [g(X,W)\, g(Y,Z) + g(X,Y)\, g(Z,W)$$
$$- 2g(X,Z)\, g(Y,W)$$
$$- 3g(X,JW)\, g(Z,JY) - 3g(X,JY)\, g(Z,JW)]$$
$$+ \frac{3}{8} [g((\nabla_X J) Y, (\nabla_Z J) W) + g((\nabla_Z J) Y, (\nabla_X J) W)$$
$$+ g((\nabla_Y J) X, (\nabla_W J) Z) + g((\nabla_Y J) Z, (\nabla_W J) X)$$
$$- g((\nabla_X J) W, (\nabla_Y J) Z) - g((\nabla_Z J) W, (\nabla_Y J) X)$$
$$- g((\nabla_X J) Y, (\nabla_W J) Z) - g((\nabla_Z J) Y, (\nabla_W J) X)] \quad (8)$$

Interchanging the role of X and Y in last equation and subtracting the equation thus obtained from (8), we have

$$R(X,Y,Z,W) + R(Z,Y,X,W) - R(Z,X,Y,W) - R(Y,X,Z,W)$$
$$= \frac{3C(m)}{4} [g(X,W)\, g(Y,Z) - g(X,Z)\, g(Y,W)$$
$$+ g(X,JW)\, g(Y,JZ) - g(X,JZ)\, g(Y,JW)$$
$$- 2g(X,JY)\, g(Z,JW)]$$
$$+ \frac{3}{8} [2g((\nabla_X J) Y - (\nabla_Y J) X, (\nabla_Z J) W - (\nabla_W J) Z)$$
$$+ g((\nabla_Z J) Y - (\nabla_Y J) Z, (\nabla_X J) W - (\nabla_W J) X)$$
$$+ g((\nabla_Y J) W - (\nabla_W J) Y, (\nabla_X J) Z - (\nabla_Z J) X)] \quad (9)$$

Now, our theorem follows from (9) and the first Bianchi's identity.

If (M, g, J) is a Kahler manifold i.e. $(\nabla_X J) Y = 0$, then as a consequence of above theorem we get the following well known result of Yano and Mogi [5].

**Corollary** (2.2). If a Kahler manifold (M, g, J) is of constant holomorphic sectional curvature C (m) at $m \in M$, then R has the following form:

$$R(X,Y,Z,W) = \frac{C(m)}{4} [g(X,W)\, g(Y,Z) - g(X,Z)\, g(Y,W) + g(X,JW)\, g(Y,JZ) - g(X,JZ)\, g(Y,JW) - 2g(X,JY)\, g(Z,JW)].$$

The notion of constant type for nearly Kahler manifold has been defined by Gray [3]. For an arbitrary almost Hermitian manifold it may be defined as follows: We say that an almost Hermitian manifold (M, g, J) is of constant type at $m \in M$ provided that for all $X \in T_m(M)$ we have

$$\lambda(X,Y) = \lambda(X,Z)$$

with $\lambda(X,Y) = R(X,Y,X,Y) - R(X,Y,JX,JY)$,

whenever the planes defined by X,Y and X,Z are anti-holomorphic and $g(Y,Y) = g(Z,Z)$. If this holds for all $m \in M$, we say that M has (pointwise) constant type. Finally, if $X,Y \in \chi(M)$ with $g(X,Y) = g(X,JY) = 0$, $\lambda(X,Y)$ is constant whenever $g(X,X) = g(Y,Y) = 1$, then M is said to have global constant type.

Now, in what follows, we consider only those almost Kahler manifolds which satisfy (3). From (2) it is clear that

$$\lambda(X,Y) = \tfrac{1}{2} \| (\nabla_X J) Y - (\nabla_Y J) X \|^2 \tag{11}$$

Then we have the following:

**Proposition** (2.3). Let (M, g, J) be an almost Kahler manifold. Then it has (pointwise) constant type if and only if there exists an $\alpha \in f(M)$ such that

$$\| (\nabla_X J) Y - (\nabla_Y J) X \|^2 = \alpha \{ g(X,X)\, g(Y,Y) - g(X,Y)^2 - g(X,JY)^2 \} \tag{12}$$

for all $X,Y \in \chi(M)$. Furthermore, M gas global constant type if and only if (12) holds with a constant function $\alpha$.

**Proof.** The sufficient part follows immediately from the definition of constant type. Conversely, let $X,Y \in \chi(M)$. Then we write

$$Y = aX + bJX + cZ,$$

where $g(Z,Z) = 1$ and $g(Z,X) = g(Z,JX) = 0$. Now we have $a = g(X,Y)/g(X,X)$, $b = g(Y,JX)/g(X,X)$ and $c = g(Y,Z)$. Then it follows that

$$\begin{aligned} \lambda(X,Y) &= R(X,Y,X,Y) - R(X,Y,JX,JY) \\ &= c^2 \lambda(X,Z) \\ &= \alpha \{g(X,X)\, g(Y,Y) - g(X,Y)^2 - g(X,JY)^2\} \end{aligned}$$

since $c^2 = g(Y,Y) - a^2 g(X,X) - b^2 g(X,X)$.

**Remark.** The above proof is similar to that given by A. Gray [3].

## 3. Axion of holomorphic and anti-holomorphic planes (spheres)

In [1] E. Cartan defined the axiom of p-planes for Riemannian manifolds and proved that a manifold M which satisfies the axiom of p-planes for some p, is a space of constant curvature. Later, D.S. Leung and K. Nomizu [4] generalized this result by showing that a manifold M of dim $\geq 3$, satisfies the axiom of p-spheres for some p is also a space of constant curvature. In this section we study anti-holomorphic (resp. holomorphic) axiom of shperes for almost Kahler manifolds parallel to those studied by Leung and Nomizu for Riemannian manifolds.

**Definition:** An almost Hermitian manifold M is said to satisfy the anti-holomorphic (resp. holomorphic) axiom of p-spheres, for some p if for each $m \in M$ and any p-dimensional antiholomorphic (resp. holomorphic) subspace S of $T_m(M)$, there exists a p-dimensional totally unbilical submanifold N with mean curvature vector parallel in the normal bundle such that $m \in N$ and $T_m(N) = S$.

Now, let N be a submanifold of M and let $\nabla$ and $\overline{\nabla}$ be the covariant differentiation on N and M respectively. Then, the following results are well known in the theory of submanifolds.

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{12}$$

where X and Y are vector fields tangent to N, and h is the second fundamental from. Fro a normal vector field $\xi$ on N we write

$$\overline{\nabla}_X \xi = - A_\xi X + D_X \xi \tag{13}$$

where $- A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. normal) component of $\overline{\nabla}_X \xi$. Finally

$$\begin{aligned} R_{XY}\xi &= (\nabla_Y A_\xi) X - (\nabla_X A_\xi) Y + A_{D_X\xi} . Y \\ &\quad - A_{D_Y\xi} . X \text{ (modulo normal component)} \end{aligned} \tag{14}$$

We use these definitions and formulae to establish the following:

**Theorem** (3.1). Let (M, g, J) be an almost Kahler manifold with real dim $\geq 6$. If M satisfies the axiom of antiholomorphic p-spheres for some p, then M has pointwise constant holomorphic sectional curvature.

**Proof.** Let X and Y be arbitrary vectors which span on anti-holomorphic plane section at an arbitrary point m of M. Then the vectors X+Y and JX—JY also span an anti-holomorphic section Using the relation (14) and the arguments as given by Leung and Nomizu [4], we have

$$R(X,Y)\xi = 0 \text{ (modulo normal component)} \tag{15}$$

where $\xi$ is a vector field orthogonal to the plane section spanned by X and Y. Then we have, in particular,

$$R(X+Y, JX-JY)(JX+JY) = 0 \text{ (modulo normal component)}$$

from which, we have

$$R(X+Y, JX-JY, JX+JY, X+Y) = 0 \tag{16}$$

Now, expanding the left hand side of last relation and after some cancellation, we get

$$R(X,JX,JX,X) = R(Y,JY,JY,Y) \tag{17}$$

that is,

$$H(X) = H(Y) \tag{18}$$

Now, if for arbitrary vectors U and V, sp $\{U,V\}$ is holomorphic, then we can always choose a unit vector $W \in \text{sp}\{U,JU\}^{\perp} \cap \text{sp}\{V,JV\}^{\perp}$ Thus from (18) we have $H(U) = H(V) = H(W)$. That is M has pointwise constant holomorphic sectional curvature.

**Corollary** (3.2). Let (M, g, J) be as above. If M satisfies the axiom of anti-holomorphic p-planes for some p, then M has pointwise constant holomorphic sectional curvature.

**Theorem** (3.3). Let (M, g, J) be an almost Kahler manifold with real dim $\geq 6$. If M satisfies the axiom of holomorphic p-spheres for some p, then M has pointwise constant holomorphic sectional curvature.

**Proof.** The proof is similar to that of as in Theorem (3.1) except for the suitable choice of vectors.

**Corollary** (3.4). Let (M, g, J) be as above. If M satisfies the axiom of holomorphic p-planes for some p, then M has pointwise constant holomorphic sectional curvature.

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