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by

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Athwart Immersions Into Hyperbolic Space

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ABSTRACT

Athwart immersions into hyperbolic space have been defined. A fundamental theorem relating athwart immersions in Euclidean space to athwart immersions in hyperbolic space has been establised. Some concluding results concerning athwart immersions into hyperbolic space have been proved.

INTRODUCTION

In this work we are concerned with a problem, namely, the athwart immersions into the (n+1)-dimensional hyperbolic space H. In [3], the same problem has been discussed in Euclidean space R^{n+1}. Athwart immersions into Euclidean space may be defined as follows [3]:

Let M and N be C^∞ closed, connected n-manifolds and let f and g be smooth immersions of M and N, respectively into Euclidean space R^{n+1}. We say that f is athwart to g - written f $\pitchfork$ g - if and only if f(M) and g(N) have no tangent hyperplane in common.

In what circumstances is f $\pitchfork$ g? was the main question which has been answered in [3]. Actually, in [3] the following theorems have been proved:

Theorem (i): Let f: M $\to R^{n+1}$ and g: N $\to R^{n+1}$ be immersions. If f(M) has two tangent n-planes such that one meets g(N) and the other does not, then f is not athwart to g.

Theorem (ii): Let f: M $\to R^{n+1}$ and g: N $\to R^{n+1}$ be immersions such that f(M) $\cap$ g(N) $\neq \varnothing$. Then f is not athwart to g.

Theorem (iii): Let f and g be two immersions of the unit circle S^1 in R^2. If $f \pitchfork g$, then the image of one of the immersions is inside all the loops of the other.

Theorem (iv): Let f: $M \rightarrow R^{n+1}$ and g: $N \rightarrow R^{n+1}$ be immersions such that $f \pitchfork g$. Then one of the manifolds, say M, is diffeomorphic to the n-dimensional unit sphere S^n, f is an imbedding with starshaped inside and g(N) is contained in the interior of the kernel of the inside of f.

The main purpose of this work is to define the athwart immersion and to prove similar theorems to the above ones in the (n+1)-hyperbolic space H.

The fundamental theorem we are going to prove may be stated as follows:

Theorem (1): Let f: $M \rightarrow H$ and g: $N \rightarrow H$ be immersions into the (n+1)-hyperbolic space H. Then $f \pitchfork g$ if and only if $\beta of \pitchfork \beta og$. (The map β: $H \rightarrow R^{n+1}$ - as defined below - is the central projection).

1. DEFINITIONS AND BACKGROUNDS

Aiming to our study we give some important definitions and some notes. When dealing with (n+1) - hyperbolic space H one might use totally geodesic k-submanifolds to do the same business of k-planes in Euclidean space R^{n+1}.

A loop is a C^∞ map f: $[a, b] \rightarrow H$

such that $f \mid [a, b)$ is injective, $f'(t) \neq 0$ for $t \in [a, b]$, and $f(a) = f(b)$. Consequently, a loop in a 2-dimensional hyperbolic space is a Jordan curve and therefore the complement of its image in H consists of two disjoint open connected subsets of H according to the Jordan curve theorem [2]. One of these two subsets is bounded and will be called the inside of the loop f, or the inside of f, while the other is unbounded and is called the outside.

Definition (1.1):

(i) A subset $A \subseteq H$ is said to be starshaped set with respect to the point x in A if for every point y in A the geodesic segment joining x to y is contained inside A.

(ii) A subset $A \subseteq H$ is starshaped set if it is starshaped with respect to all of its points.

According to this definition, a subset $A \subseteq H$ is starshaped if and only if it is convex.

Definition (1.2): The kernel set $B \subseteq A$ of a subset $A \subseteq H$ is the set of all points x in A such that the geodesic segment from x to any point y in A lies in A.

One might see that the above two definitions (i) and (ii) are in consistence with the corresponding ones in Euclidean space.

Definition (1.3): Let M and N be C^∞ closed, connected n-manifolds and let $f : M \to H$ and $g : N \to H$ be smooth immersions of M and N, respectively, in the (n+1)-hyperbolic space H. The immersion f is called athwart to g-written $f \pitchfork g$-if and only if f(M) and g(N) have no tangent totally geodesic hypersurface in common.

From this definition, we can see easily that two concentric geodesic spheres in H are athwart while two intersecting geodesic spheres in H are not.

Now, we define and discuss briefly the main properties of the central projection (Beltrami map) $\beta : H \to R^{n+1}$ as it represents an important tool for proving the fundamental theorem (1). The most convenient model for H is the spherical one which might be defined as follows:

$$H = \{(x^{(o)}, x^{(1)}, \ldots\ldots, x^{(n+1)}) \in V^{n+2} : -(x^{(o)})^2 + (x^{(1)})^2 + \ldots\ldots + (x^{(n+1)})^2 = -1\}$$

where V^{n+2} denotes the Minkowski space $(R^{n+2}, <, >)$ with the metric

$$<, > = -dx^o \otimes dx^o + \sum_{i=1}^{n+1} dx^i \otimes dx^i.$$

The Beltrami map $\beta : H \to R^{n+1}$ is defined to be the map which takes $x \in H$ to the intersection of the Euclidean space R^{n+1} defined by $x^{(o)} = 1$ with the straight line through x and the origin 0 of V^{n+1} as indicated in the following figure.

In this case $\beta(H)$ is the open (n+1)-ball D(p, 1) of radius 1 in the above R^{n+1} centered at the point $p = (1,0,0, \ldots\ldots, 0)$.

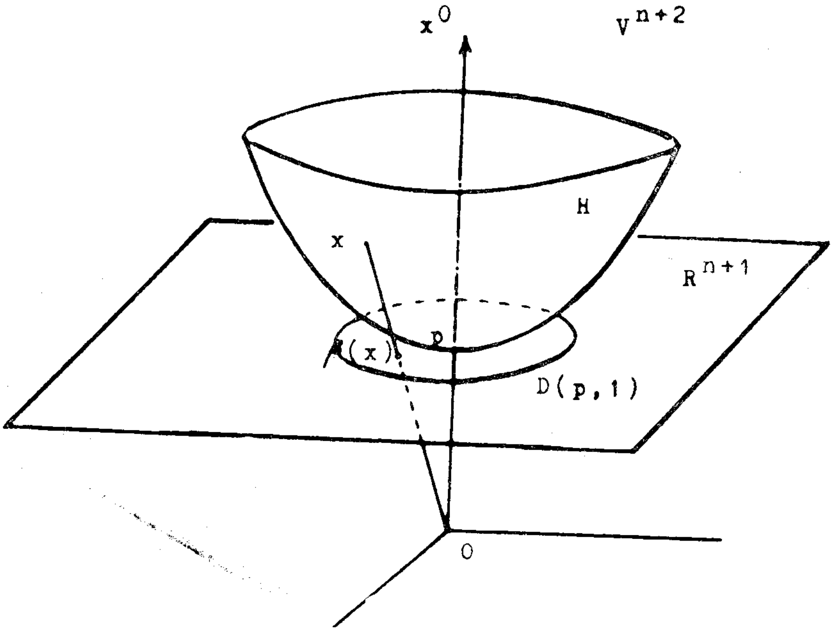

Fig. 1

Take $(x^{(o)}, x^{(1)}, \ldots, x^{(n+1)}) \in H$, the map β can be expressed mathematically as follows [1]:

$$\beta(x) = - x / < x, e_o > = \frac{x}{x^{(o)}}$$

$$= \left\{ 1, \frac{x^{(1)}}{x^{(o)}}, \ldots\ldots, \frac{x^{(n+1)}}{x^{(o)}} \right\}$$

where $e_o = (1, 0, 0, \ldots\ldots, 0)$.

Definition (1.4): A homeomorphism $\Psi : M \to \bar{M}$ from the manifold M into the manifold $\bar{M}$ is called a geodesic mapping if for every geodesic γ of M the composition $\Psi \circ \gamma$ is a reparametrization of a geodesic of $\bar{M}$.

Note that in accordance with the above definition a geodesic mapping $\Psi : M \to \bar{M}$ takes (totally) geodesic k-submanifolds of M to (totally) geodesic k-submanifolds of $\bar{M}$.

It is easy to show that:

Lemma (1.1): The central projection $\beta : H \rightarrow D(p, 1) \subset R^{n+1}$ is a diffeomorphism.

Lemma (1.2): The central projection β takes hypersurfaces of H with sectional curvatures $K \geq -1$ into hypersurfaces of R^{n+1} with sectional curvatures $K \geq 0$.
(For the proof of this lemma see [1]).

Lemma (1.3): The central projection map β takes starshaped subsets of H to starshaped subsets of R^{n+1}.

Proof: Take $A \subseteq H$ to be a starshaped subset and assume in contrary to the lemma that $\beta(A) = B \subset R^{n+1}$ is not a starshaped subset. Then there exists two points $y_1, y_2 \in B$ such that the straight line segment γ_1 joining y_1 and y_2 does not lie in B. It is easy to see that there exist two points $x_1, x_2 \in A$ such that $\beta(x_1) = y_1, \beta(x_2) = y_2$. It is clear that the geodesic segment γ joining x_1 and x_2 lies-by hypothesis-in A and consequently $\beta(\gamma) = \gamma_2$ is a straight line segment from y_1 to y_2 which is included in B. This argument shows that there are two straight line segments γ_1 and γ_2 in R^{n+1} joining y_1 and y_2 which is a contradiction. Thus, if A is a starshaped set, then $\beta(A)$ is a starshaped as well. In a similar way of discussions we can show that:

Lemma (1.4): The central projection map β takes the kernel set of a subset $A \subseteq H$ to the kernel set of the subset $\beta(A) \subseteq \beta(H)$.

Now, after this discussion we proceed to prove the fundamental theorem (1).

(i) **The necessity part:**

Let $f : M \rightarrow H$ and $g : N \rightarrow H$ be athwart immersions. Assume in contrary that $\beta o f : M \rightarrow R^{n+1}$ and $\beta o\ g : N \rightarrow R^{n+1}$ are not athwart. Then, there exists a hyperplane T in R^{n+1} which is tangent to both $\beta of(M)$ and $\beta og(N)$ at the points - say - p and q, respectively. Using β^{-1} and taking into account that β^{-1} has the same properties as β (Lemma (1.1) – (1.4)), we see that, there exists a totally geodesic hypersurface $\beta^{-1}(T)$ which is tangent to both f(M) and g(N) at the points $\beta^{-1}(P)$ and $\beta^{-1}(q)$, respectively. This would mean that f(M) and g(N) are not athwart which is a contradiction. Hence $\beta of(M) \not\pitchfork \beta o\ g(N)$.

(ii) **The sufficiency part:**

In a similar way of discussion we can show that athwartness of $\beta o f(M)$ and $\beta o g(N)$ in R^{n+1} implies athwartness of f(M) and g(N) in H.

2. SOME RESULTS

Theorem (2.1): Let $f : M \to H$ and $g : N \to H$ be immersions. If f(M) has two totally geodesic hypersurfaces such that one meets g(N) and the other does not then f is not athwart to g.

Proof: Under the above hypothesis and using the properties of the central projection map β, it is easy to see that $\beta o f(M)$ has two tangent n-planes such that one meets $\beta o\ g(N)$ and the other does not. According to theorem (i), $\beta o f$ is not athwart to $\beta o g$ and hence f is not athwart to g by the fundamental theorem (1).

Theorem (2.2): Let $f : M \to H$ and $g : N \to H$ be immersions such that $f(M) \cap g(N) \neq \varnothing$. Then f is not athwart to g.

Proof: It is easy to show that by lemma (1.1) if $f(M) \cap g(N) \neq \varnothing$, then $\beta o f(M) \cap \beta o g(N) \neq \varnothing$. Hence, by theorem (ii), $\beta o f$ is not athwart to $\beta o g$ and consequently, by the fundamental theorem (1), f is not athwart to g.

3. SPECIAL CASES

Now we shall discuss the case when $n = 1$ as being represented in the following theorems.

Theorem (3.1): Let f and g be two immersions of S^1 into the 2-dimensional hyperbolic space H. If $f \nparallel g$ then the image of one of the immersions is inside all the loops of the other.

Proof: Since $f \nparallel g$, then by using the fundamental theorem (1) we see that $\beta o f \nparallel \beta o g$. According to theorem (iii) it is clear that the image of one of the immersions, say $\beta o f(M)$, is inside all the loops of $\beta o g(N)$. Using β^{-1} we obtain the result.

Remark: The converse of this theorem is not necessarily true even in Euclidean space R^2, i.e. if the image of one of the immersions of S^1

is inside all the loops of the other, then f is not necessarily athwart to g. The following example indicates this situation.

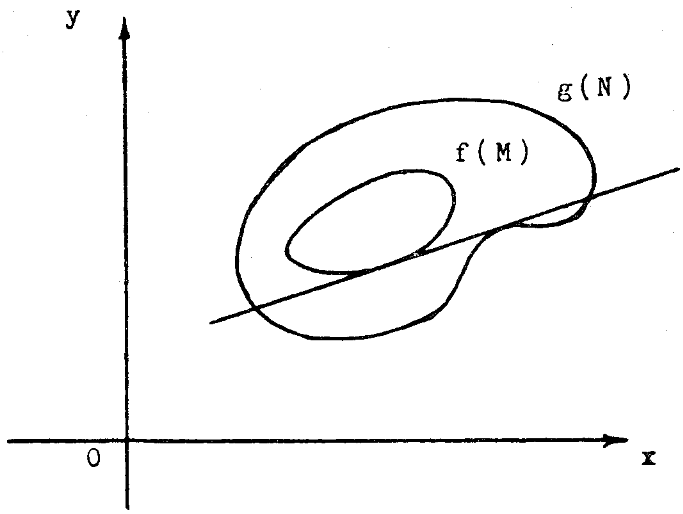

Fig. 2

Theorem (3.2): Let f : [a, b] → H be a loop in the 2-dimensional hyperbolic space H and p an outside point. Then there exists at least one geodesic tangent to the loop passing through p.

Proof: Using the properties of the central projection map β, there exists a loop βof : [a, b] → D (p, 1) ⊂ R^2 for which β(p) is an outside point. Then, there exists at least one tangent line to the loop βof passing through the point β(p) [3]. Hence, by lemma (1.2), we get at least one geodesic tangent to the loop f passing through p.

4. HYPERSURFACES

In case of immersions with codimension 1, we have

Theorem (4.1): Let f : M → H and g : N → H be immersions such that f ⋔̸ g. Then one of the manifolds, say M, is diffeomorphic to S^n, f is an imbedding with starshaped inside and g(N) is contained in the interior of the kernel of the inside of f.

Proof: As above, it is obvious that $\beta o f \pitchfork \beta o g$ (by fundamental theorem (1)) and consequently, by theorem (iv), one of the manifolds say $\beta o f(M)$ is diffeomorphic to S^n $\beta o f$ is an imbedding with starshaped inside and $\beta o g(N)$ is contained in the interior of the kernel of the inside of $\beta o g$. Taking into account that the map β is a diffeomorphism and using lemma (1.3) and (1.4) the proof is complete.

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