UNIVARITE MAXIMUM SELF-DECOMPOSABLE DISTRIBUTIONS

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ABSTRACT

A random variable X is said to be self-decomposable (benceforth, SD) if it satisfies tbe following equivalence relation in distribution

$$X \overset{D}{=} (\alpha * X') \; o \; (X_\alpha)$$

for all positive α in some open interval. The operation * is either multiplication or addition and tbe distribution of the co-random variable $X\alpha$ depends on tbe constant $_\alpha$. In this paper we study SD random variables where the operation o defined to be maximum. Some properties of such random variables are given and a representation theorem is stated for discrete and continuous random variables for the univariate case.

Keywords and phrases: self - decomposable, max-stable distributions, extreme values,

INTRODUCTION AND SUMMARY

The study of functions of sequences of independently distributed random variables has long been a preoccupation of probabilists. For sums of random variables these studies have led to, among other results, the central limit theorem, the characterization of the stable and infinitely–divisible laws, and later the characterization of the self-decomposable laws (see for example Gnedenko and Kolmogorov (1954), Loeve (1963), Lukacs (1970), and Laha and Rohatgi (1979)). A probability distribution is said to be sum-self-decomposable (henceforth, SSD), or of class L, if its characteristic function satisfies

$$\psi(t) = \psi(\alpha t)\psi_\alpha(t)$$

for all $\alpha \varepsilon (0,1)$, with $\psi_\alpha(.)$ being a characteristic function. For the corresponding random variables this means that

$$X \overset{D}{=} \alpha X' + X_\alpha \tag{1}$$

for all $\alpha \ \varepsilon \ (0,1)$, where X' and X_α are independent and X' is distributed as X.

The class of SSD distributions are the limit laws for the sum of independent but not identically distributed (INID) random variables. Clearly, apart from $X = 0$, no lattice random variable can satisfy equation (1); in fact, all nondegenerate SSD distributions are known to be absolutely continuous. A discrete analogue of SSD distributions is defined by Steutel and Van Harn (1979) as follows: A discrete distribution with probability generating function P is called discrete SSD if

$$P(z) = P \ (1 - \alpha + \alpha z) \ P_\alpha(z)$$

for $\alpha \ \varepsilon \ (0,1)$, with $P_\alpha(.)$ being a probability generating function. It turns out that discrete SSD distributions have properties that are quite similar to those of their continuous counterparts.

The problems involved in the theory of the limit distributions for the maximum of random variables are parallel to those encountered in the theory of limit laws for the sums of random variables. Gnedenko (1943) characterized the maximum stable (henceforth, MS) laws. He showed that the limit distributions for $F^n(a_n x + b_n)$, where $a_n > 0$ and b_n are suitably chosen real constants, are confined to the distributions of the three forms:

$$\Lambda(x) \ = \ \{\exp(-e^{-x}), \qquad -\infty < x < \infty \qquad (2)$$

$$\Phi_\alpha(x) \ = \ \begin{cases} \exp(-x^{-\alpha}), & \text{if } x > 0 \\ 0 \ , & \text{if } x < 0 \end{cases}$$

$$\psi_\alpha(x) \ = \ \begin{cases} \exp(-(-x)^\alpha), & \text{if } x < 0 \\ 0 \ , & \text{if } x > 0 \end{cases}$$

It has been shown by Mejzler (1965) that the analogue of the class of sum infinitely divisible distributions is the class of all probability distributions for the operation maximum. So that maximum infinitely divisible distributions is not particularly interesting. Thus, we are led here to a study of the class of maximum self-decomposable (MSD) distributions, the analogue of Class L.

Let

$$\gamma(F) \ = \ \inf \{x : F(x) > 0\}$$

be the lower end point of the distribution function F and let

$$\Omega(F) = \sup \ \{x : F(x) < 1\}$$

be the upper end point of the distribution function F. Throughout the paper we will only consider the cases

. $\gamma(F) = - \infty, \ \Omega(F) = 0;$

. $\gamma(F) = 0, \ \Omega \ (F) = + \infty;$

. $\gamma(F) = - \infty, \ \Omega(F) = + \infty.$

With obvious modifications some of the results can be extended to other values of $\gamma(.)$ and $\Omega(.)$.

In the next section the definition of MSD distributions and some examples will be given. The MSD distributions can be characterized in several different ways, one of the such characterization will be introduced in Section 3. By the analogy to Class L, we can discover the limits in distribution of the maximum of INID random variables which corresponds to the class of MSD distributions. Such a characterization of MSD distributions will not be discussed in this study. For this characterization see Eddy and Sungur (1986). Some properties of MSD distributions will be discussed in Section 4, and unimodality of MSD distributions will be considered in Section 5.

DEFINITION AND SOME EXAMPLES

DEFINITION 1. A distribution function (df) F on the real line is said to be positive scale MSD if it satisfies the following functional equation

$$F(x) = F \ (\beta x).G_\beta(x), \ \forall \beta > 1$$

if $\gamma (F) = 0$ with $G_\beta(.)$, called the co-distribution function (co–df) of F, being a df.

DEFINITION 2. A df F on the real line is said to be negative scale MSD if it satisfies the following functional equation

$$F(x) = F(\beta x).G_\beta(x), \ \forall \beta, \ 0 < \beta < 1$$

if $\Omega(F) = 0$ with $G_\beta(.)$ being a df.

DEFINITION 3. A df F on the real line is said to be location MSD if it satisfies the following functional equation

$$F(x) = F(x + \gamma). H_\gamma(x), \ \forall \gamma, \ \gamma > 0$$

with $H_\gamma(.)$ being a df.

For the corresponding random variables location MSD means that

$$X \overset{=}{_D} \max \{X' - \gamma, X_\gamma\}$$

where X' and X_γ are independent, and X and X' are identically distributed.

If the functional equation specified for location MSD holds only for some fixed γ, then the random variable and its df is called location γ–MSD. Similarly, we define positive (negative) scale β–MSD distributions. On the other hand, if a df satisfies the specified relation for a subset of values of γ or β, then it is called "incomplete location MSD", or "incomplete positive (negative) scale MSD".

EXAMPLES: (a) Every degenerate random variable is location MSD.

(b) The uniform distribution on $[0,1]$ is location MSD.

(c) Not every random variable is MSD. For example, the random variable defined by $P(X = 0) = P(X = 1) = 1/2$ is not.

(d) Extreme-value distributions given in Equation (2) are MSD.

In order to be able to assert that a random variable and its df is location MSD, we need to show that the function defined by

$$H_\gamma(x) = \frac{F(x)}{F(x + \gamma)}$$

is a df for all $\gamma > 0$. Hence,

(a) Obvious.

(b) For $\gamma > 1$

$$H_\gamma(x) = F(x) = I_{[0,1]}(x). \tag{3}$$

For $0 < \gamma < 1$,

$$H_\gamma(x) = 0 \qquad \text{if } x < 0$$

$$= \frac{x}{x + \gamma} \qquad \text{if } 0 < x < 1 - \gamma \tag{4}$$

$$= x \qquad \text{if } 1 - \gamma < x < 1$$

$$= 1 \qquad \text{if } x > 1.$$

It is clear that both of the functions given in Equations (3) and (4) are df's. Therefore, uniform distribution on the interval [0,1] is location MSD.

(c) Note that for $\gamma = 1/2$

$$H_{1/2}(1/4) = \frac{F(1/4)}{F(1/4 + 1/2)} = 1$$

but

$$H_{1/2}(3/4) = \frac{F(3/4)}{F(3/4 + 1/2)} = 1/2$$

which shows that $H_\gamma(x)$ is not a nondecreasing function, therefore it is not a df.

To show that extreme-value distributions are MSD, we will follow a different approach. We will define a function $H\gamma(x)$ which is a df for all $\gamma > 0$, and by solving the resulting functional equation we will show that $F(x)$ should have the form of extreme-value distributions. Here we will show that extreme-value distribution $\Lambda(x)$ is location MSD. Similar approach can be used to show that $\Phi_\alpha(x)$ is positive scale MSD, and $\psi_\alpha(x)$ is negative scale MSD.

(d) Let F be a location MSD with the co–df

$$H_\gamma = F^{K(\gamma)}$$

where $K : \gamma \to [0,1]$ is a continuous function. By using the definition of location MSD distributions

$$F^{1-K(\gamma)}(x) = F(x + \gamma) \tag{5}$$

Equation (5) can be written as

$$[1 - K(\gamma)].h(x) = h(x + \gamma) \tag{6}$$

where

$$h(x) = \log F(x).$$

The solution of the functional equation (6) will give us the form of this special class of MSD df's.

LEMMA 1. If $h(.)$ is a nonpositive real-valued function and $K: \gamma \to [0,1]$ is continuous, then the only solution of the functional equation [6] is

$$h(x) = -e^{cx+a}$$

and

$$K(\gamma) = 1 - e^{c\gamma}$$

where a and c are arbitrary constants.

PROOF: Introduce the functions

$$h^*(x) = \log(-h(x))$$

and

$$K^*(\gamma) = \log(1 - K(\gamma)].$$

Then we have

$$h^*(x + \gamma) = K^*(\gamma) + h^*(x). \tag{7}$$

Equation (7) has the form of Pexider's equation which has drawn much attention in mathematical literature and arises in different contexts (see Aczel (1966)). Its general solution is stated in Aczel (1966) only for positive or nonnegative x and γ. Since Equation (7) must hold for all $-\infty < x < +\infty$, and $\gamma > 0$, its solution requires further justification.

First, by repeated use of Equation (7),

$$h^*(x + \alpha + \gamma) = K^*(\alpha) + K^*(\gamma) + h^*(x).$$

On the other hand, by substitution of $\alpha + \gamma$ in Equation (7), we find

$$h^*(x + \alpha + \gamma) = K^*(\alpha + \gamma) + h^*(x).$$

From the last two equations we obtain

$$K^*(\alpha + \gamma) = K^*(\alpha) + K^*(\gamma), \ \forall \, \alpha, \gamma > 0$$

which is known as Cauchy's functional equation, and its general solution is

$$K^*(\gamma) = c \cdot \gamma.$$

So, the functional equation that we must solve becomes

$$h^*(x + \gamma) = c \cdot \gamma + h^*(x), \ \forall \, -\infty < x < +\infty, \ \gamma > 0.$$

We will solve the above functional equation by treating for nonnegative and negative x seperately. Firts, let x to be nonnegative If we set x = 0, the equation becomes

$$h^*(\gamma) = c \cdot \gamma + h^*(0)$$

which imlies that

$$h^*(x) = c \cdot x + a \qquad \text{for } x > 0.$$

Now, let x < 0. If we set $\gamma = -x$, then

$$h^*(0) = c \cdot (-x) + h^*(x)$$

i.e.,

$$h^*(x) = c \cdot x + a \qquad \text{for } x < 0. \qquad (8)$$

Note that Equation (8) also holds for x = 0, since $h^*(0) = a$.

Therefore, the general solution of Equation (7) is

$$h^*(x) = c \cdot x + a$$

and

$$K^*(\gamma) = c \cdot \gamma$$

where, a and c are arbitrary constants.

So, we have

$$h(x) = -e^{cx+a}$$

and

$$K(\gamma) = 1 - e^{c\gamma}$$

as the general solution of the functional Equation (6).   QED

Now, if we impose the condition that $h(+\infty) = 0$, and $h(-\infty) = -\infty$ (which makes the $F(x) = e^{h(x)}$ be a df), we find that c < 0. By using the Lemma 2.1 we find that

$$F(x) = \exp(-e^{cx+a})$$

where, c < 0, and a are arbitrary constants. Note that, the extreme-value distribution $\Lambda(x)$ has the above specified form for c = -1, and a = 0.

A CHARACTERIZATION OF MSD DISTRIBUTIONS

The Laws of class L have been determined by solving a functional equation for the characteristic function of the distribution. Similarly, MSD distributions can be characterized by solving the functional equation for the distribution function of the distribution.

THEOREM 1. (i) A df $F(x)$ is positive scale MSD if and only if

$$F(x) = \exp\{g(\log x)\}$$

when $\gamma(F) = 0$ and $\Omega(F) = +\infty$.

(ii) A df $F(x)$ is negative scale MSD if and only if

$$F(x) = \exp\{g(-\log(-x))\}$$

when $\gamma(F) = -\infty$ and $\Omega(F) = 0$.

(iii) A df $F(x)$ is location MSD if and only if

$$F(x) = \exp\{g(x)\}$$

when $\gamma(F) = -\infty$ and $\Omega(F) = +\infty$.

In each case, $g(.)$ is a nonpositive, nondecreasing, right continuous and concave function satisfying $g(\Omega) = 0$ and $g'(\gamma) = +\infty$.

The proof of these results is given in Eddy and Sungur (1986), and Sungur (1985).

Suppose that $F(.)$ is location MSD , i.e.,

$$F(x) = F(x + \gamma).H_\gamma(x), \qquad \forall \gamma \geq 0 \qquad (9)$$

which imlies that

$$F(x + \gamma) = F(x + 2\gamma).H_\gamma(x + \gamma)$$

by using the above relation iteratively and substitutirg back in Equation (9), we end up with

$$F(x) = F(x + n\gamma). \prod_{i=1}^{n} H_\gamma(x + (i-1)\gamma) \quad \forall\ n \text{ and } \gamma \geq 0.$$

Taking the limits as n increases, we get

$$F(x) = \prod_{i=1}^{\infty} H_\gamma(x + (i-1)\gamma).$$

Similarly, one can show that

$$F(x) = \prod_{i=1}^{\infty} G_\beta(\beta^{i-1}x)$$

for positive and negative scale MSD distributions, for appropriately defined β. These results plays very important role on the characterization of MSD distributions as the limit laws of maximum of INID random variables.

SOME PROPERTIES OF MSD DISTRIBUTIONS

If X is location MSD, then a location and scale transform of X, i.e., $aX + b$, for $a > 0$, is also location MSD. For the positive and negative scale MSD random variables, it can be shown that $Y = cX$ for $c > 0$, is also scale MSD. On the other hand, if X is scale MSD, then the transformed random variable $Y = X - b$, $b > 0$, will not be a scale MSD, but its df will satisfy the following relation:

$$F(y) = F(\beta y + b(\beta - 1)).G_{\beta,b}(y).$$

In terms of the corresponding random variable this means that

$$Y \overset{=}{_D} \max(\beta^{-1}(Y - b(\beta - 1)), \; Y_{\beta,b}).$$

If we consider other transformations of MSD random variables, the following property of log-concave functions, which is proven by Klinger and Mangasarian (1968) will be helpful.

LEMMA 2. Let $\varnothing(x)$ be a nondecreasing log–concave function and let $f(x)$ be a concave function. Then the function which is defined as

$$\eta(x) = \varnothing(f(x))$$

is log-concave.

By using Lemma 2, and the representation of location MSD df's, we can prove the following result on the transformations of location MSD random variables.

THEOREM 2. Let X be a location MSD random variable, and let $Y = \psi(X)$, where $\psi(.)$ is increasing and $\psi^{-1}(.)$ is concave on the range of X. Then, Y is location MSD.

PROOF: Since $\psi(.)$ is increasing on the range of X

$$F_Y(y) = F_X(\psi^{-1}(y)).$$

Therefore, the result follows from the Lemma 4.1.

As an example, consider the transformed random variable $Y = e^X$, by Theorem 2, if X is location MSD, then Y is also location MSD.

UNIMODALITY OF MSD DISTRIBUTIONS

In mathematical statistics a df, $F(.)$, is called unimodal if its derivative, $F'(X)$, exists everywhere and has a unique (finite) maximum. We use the following more restrictive definition of unimodality of df's which is given by Gnedenko and Kolmogorov (1954).

DEFINITION 4. The df $F(.)$ is called unimodal if there exists at least an x_0 such that $F(x)$ is convex for $x < x_0$ and concave for $x > x_0$.

It is easy to verify that the Normal, the Cauchy, and the uniform distribution on a finite interval are all unimodal in the sense of Definition 5.1. Also, $\Lambda(x)$, which is defined in Equation (2), is unimodal and has mode at $x = 0$. Since, $\Lambda(x)$ is a special case of location MSD df's, it suggests that location MSD df's might be unimodal. In this section, we will show that in fact this proposition is true for location MSD df's. Before we prove the result, we need the following theorem which is proven by Gnedenko and Kolmogorov (1954, pg. 157).

THEOREM 3. (i) If the df $F(.)$ is unimodal with vertex at $x = 0$, then there exists a df $V(x)$ such that

$$F(x) - xF'_+(x) = V(x) \text{ and,}$$

$$\lim_{z \uparrow x}[F(z) - x F'_-(z)] - \lim_{z \uparrow x}V(z)$$

for every x (a product $0.\infty$ is taken to be 0), where F'_+ and F'_- are the right and left derivatives of F respectively.

(ii) Let the df $F(x)$ be continuous except possibly at $x = 0$. Suppose that there is a denumerable set D of points x and a df $V(x)$ such that if x is not in D, the right or left derivative of $F(x)$ (possibly different ones at different points) exists and satisfies the equation

$$F(x) - xF'(x) = V(x);$$

Then $F(x)$ is unimodal.

On the other hand, if $F(x)$ is continuous and differentiable, then the Theorem takes the following form.

THEOREM 4. In order that a df $F(x)$ be unimodal (at vertex $x = 0$). it is necessary and sufficient that the function

$$V(x) = F(x) — xF'(x)$$

be a df.

LEMMA 3. If $F(x) = e^{h(x)}$ is location MSD, then $h'(+ \infty) = 0$.

PROOF: If $F(.)$ is location MSD then

$$F(x) = F(x + \gamma)H_\gamma(x), \ \forall \gamma > 0,$$

where, $H_\gamma(.)$ is a df. It follows that $H_\gamma(+ \infty) = 1$ if and only if

$$\lim_{x \to +\infty}(h(x) — h(x + \gamma)) = 0, \qquad \forall \gamma > 0.$$

Consequently,

$$\lim_{x \to +\infty}\gamma^{-1}(h(x) — h(x + \gamma)) = 0, \qquad \forall \gamma > 0,$$

which implies that

$$\lim_{x \to +\infty}\lim_{\gamma \to 0}\gamma^{-1}(h(x) — h(x + \gamma)) = 0$$

which proves the Lemma.

Now, we are ready to prove the following result on the unimodality of MSD df's.

THEOREM 5. If $F(x) = e^{h(x)}$ is a nondegenerate location MSD df such that $xh(x) \to 0$, as $x \to + \infty$, and $x \to -\infty$, then it is unimodal with vertex at $x = 0$.

PROOF: To prove the theorem, we have to verify that

$$V(x) = F(x) — xF'(x)$$

$$= e^{h(x)} — xh'(x)e^{h(x)}$$

is a df, where $h(.)$ is a nonpositive, nondecreasing, right continuous concave function satisfying $h'(-\infty) = + \infty$, and $h(+ \infty) = 0$.

First, note that, since $H(x)$ is concave, $h'(x)$ is decreasing (so, $-h'(x)$ will be increasing).)Therefore

$$V(x) = [1 — xh'(x)]e^{h(x)}$$

$$= (1 + x(-h'(x))]e^{h(x)}$$

is a nondecreasing function of x.

Therefore, by the assumption that $xh(x) \to 0$, as $x \to +\infty$ and $x \to -\infty$, the result follows from the Theorem 4.

By using the properties of quasiconcave functions strong unimodality MSD of df's can be studied.

ACKNOWLEDGEMENTS

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