Commun. Fac. Sci. Univ. Ank. Series A_1
V. 37, pp. 9-15 (1988)

ON THE GAUSSIAN AND MEAN CURVATURES OF PARALLEL HYPERSURFACES

M. KEMAL SAĞEL and H. H. HACISALİHOĞLU

Department of Mathematics, Ankara University, Ankara, Turkey

ABSTRACT

In this study, we give a generalization of the higher order Gaussian curvature function and the mean curvature function about the parallel surfaces in E^3, for E^{n+1}.

BASIC CONCEPTS

DEFINITION 1: Let M_1 and M_2 are two hypersurfaces in E^{n+1}, with unit normal vector N_1 of M_1

$$N_1 = \sum_{i=1}^{n+1} a_i \frac{\partial}{\partial x_i} .$$

Where each a_i is a C^∞ function on M_1. If there is a function f, from M_1 to M_2 such that

$$\begin{aligned} f : M_1 &\longrightarrow M_2 \\ P &\longrightarrow f(P) = (p_1 + ra_1(P), \ldots, p_{n+1} + ra_{n+1}(P)). \end{aligned}$$

Then M_2 is called a parallel hypersurfaces of M_1, where $r \in R$ [1].

DEFINITION 2: Let M be a hypersurface in E^{n+1} and S denotes the shape operator on M, at $P \in M$. The function H defined by

$$\begin{aligned} H : M &\longrightarrow R \\ P &\longrightarrow H(P) = \text{İz } S(P) \end{aligned}$$

is called the mean curvature function of M and the real number H(P) is called mean curvature of M at the point P [3].

DEFINITION 3: Let M be a hypersurface in E^{n+1} and S denotes the shape operator on M, at $P \in M$. The function K defined by

$$K : M \longrightarrow R$$
$$P \longrightarrow K(P) = \det S(P)$$

is called the Gaussian curvature function of M and the K(P) is called Gaussian curvature of M at the point P [3].

DEFINITION 4: Let M be a hypersurface in E^{n+1} and $T_M(P)$ be a tangent space on M, at $P \in M$. If S_P denotes the shape operator on M, at $P \in M$, then

$$S_P : T_M(P) \longrightarrow T_M(P)$$

is a linear mapping. If we denote the characteristic values by $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the corresponding characteristic vectors by $x_1, x_2, \ldots, x_n$ of S_P then $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the principal curvatures and $x_1, x_2, \ldots, x_n$ are the principal directions of M, at $P \in M$. On the other hand, if we use the notations

$$K_1(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^{n} \lambda_i$$

$$K_2(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i<j}^{n} \lambda_i \lambda_j$$

$$K_3(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i<j<t}^{n} \lambda_i \lambda_j \lambda_t$$

$$\vdots$$

$$K_n(\lambda_1, \lambda_2, \ldots, \lambda_n) = \prod_{i=1}^{n} \lambda_i$$

then the characteristic polinomial of S(P) becomes

$P_{S(P)}(\lambda) = \lambda^n + (-1) K_1 \lambda^{n-1} + \ldots + (-1)^n K_n$ and K_i, $1 \leq i \leq n$ are uniquely determined, where the functions K_i are called the higher ordered Gaussian curvatures of the hypersurface M [4], [5].

THEOREM 1: Let f be a mapping from M to M_r and M_r be a parallel hypersurface of M. Then f preserves the principal curvature directions [2]

THEOREM 2: Let M_r be a parallel surface of the surface $M \subset E^3$. Let the Gaussian curvature and mean curvature of M be denoted by K and H at $P \in M$, respectively and the Gaussian curvature and mean curvature of M_r be denoted by K_r and H_r at $f(P) \in M_r$, respectively. Then we know that

$$K_r = \frac{K}{1 + r\,H + r^2\,K}$$

and

$$H_r = \frac{H + 2\,r\,K}{1 + r\,H + r^2\,K}$$

[2].

GENERALIZED THEOREMS

THEOREM 1: Let M be a hypersurface of E^{n+1} and $K_1, K_2, \ldots, K_n$ are the higher order Gaussian curvatures and $k_1, k_2, \ldots, k_n$ the principal curvatures at the point $P \in M$.

Let us define a function

$$\Phi : M \longrightarrow R$$

such that $P \longrightarrow \Phi(P) = \Phi(r, k_1, k_2, \ldots, k_n) = \prod_{i=1}^{n} (1 + rk_i)$

$$\Phi(r, k_1, k_2, \ldots, k_n) = 1 + r \sum_{i=1}^{n} k_i + r^2 \sum_{i<j}^{n} k_i k_j + \ldots + r^n \prod_{i=1}^{n} k_i$$

or

$$\Phi(r, k_1, k_2, \ldots, k_n) = 1 + r\,K_1 + r^2\,K_2 + \ldots + r^n\,K_n,$$

where $r \in R$ is given in Definition 1.

PROOF: We prove the theorem by induction method.

For $n = 1$, the theorem holds.

Actually,

$$\begin{aligned}\Phi(r, k_1, k_2, \ldots, k_n) &= \prod_{i=1}^{1} (1 + rk_i) \\ &= 1 + r\,k_1 \\ &= 1 + r \sum_{i=1}^{1} k_i \\ &= 1 + r\,K_1.\end{aligned}$$

now Suppose that the theorem holds for $n=p$ and show that it is true for $n=p+1$:

$$\Phi(r,k_1,k_2,\ldots,k_n) = \prod_{i=1}^{p} (1+rk_i)$$

$$= 1 + r \sum_{i=1}^{p} k_i + r^2 \sum_{i<j}^{p} k_i k_j + \ldots$$

$$+ r^p \prod_{i=1}^{p} k_i$$

$$= 1 + r K_1 + r^2 K_2 + \ldots + r^p K_p.$$

For $n = p + 1$, we have

$$\Phi(r,k_1,k_2,\ldots,k_n) = \prod_{i=1}^{p+1} (1+rk_i)$$

$$= \left[\prod_{i=1}^{p} (1+rk_i)\right](1+rk_{p+1})$$

$$= \left(1 + r \sum_{i=1}^{p} k_i + r^2 \sum_{i<j}^{p} k_i k_j + \ldots \right.$$

$$\left. + r^p \prod_{i=1}^{p} k_i\right)(1+rk_{p+1})$$

$$= 1 + r \left(\sum_{i=1}^{p} k_i + k_{p+1}\right) + r^2 \left(\sum_{i<j}^{p} k_i k_j \right.$$

$$\left. + k_{p+1} \sum_{i=1}^{p} k_i\right)$$

$$+ \ldots + r^p \left(\sum_{\substack{i<j<\ldots<s \\ p \text{ times}}}^{p} k_i k_j \ldots k_s \right.$$

$$\left. + k_{p+1} \sum_{\substack{i<j<\ldots<l \\ p-1 \text{ times}}}^{p} k_i k_j \ldots k_l\right)$$

$$+ r^{p+1} k_{p+1} \prod_{i=1}^{p} k_i$$

$$= 1 + r \sum_{i=1}^{p+1} k_i + r^2 \sum_{i<j}^{p+1} k_i k_j$$

$$+ \ldots + r^p \underbrace{\sum_{i<j<\ldots<s}^{p+1}}_{p \text{ times}} k_i k_j \ldots k_s + r^{p+1} \prod_{i=1}^{p+1} k_i$$

$$= 1 + r K_1 + r^2 K_2 + \ldots + r^p K_p + r^{p+1} K_{p+1},$$

which proves the theorem.

THEOREM 2: Let M_r be a hypersurface to the parallel hypersurface M of E^{n+1}. $K_1, K_2, \ldots, K_n$ denote the higher order Gaussian curvatures of M, at $P \in M$. K_r and H_r generalized Gaussian and mean curvature of M_r, at the point f (P).

Suppose the function $\Phi : M \longrightarrow R$

$$P \longrightarrow \Phi(P) = \Phi(r, k_1, k_2, \ldots, k_n) = \prod_{i=1}^{n} (1 + rk_i).$$

Then we have

$$K_r = \frac{\dfrac{\partial^n \Phi(r, k_1, k_2, \ldots, k_n)}{(\partial r)^n}}{(n!)\, \Phi(r, k_1, k_2, \ldots, k_n)}$$

and

$$H_r = \frac{\dfrac{\partial \Phi(r, k_1, k_2, \ldots, k_n)}{\partial r}}{\Phi(r, k_1, k_2, \ldots, k_n)} .$$

PROOF: Let $X_1, X_2, \ldots, X_n$ are the principal directions and $k_1, k_2, \ldots, k_n$ principal curvatures of M, at $P \in M$.

Let $f_*(X_1), f_*(X_2), \ldots, f_*(X_n)$ be principal directions at the point f(P) of M_r then we know that [2].

$$S_r(f_* X_1) = \frac{k_1}{1+rk_1} f_* X_1$$

$$S_r(f_* X_2) = \frac{k_2}{1+rk_2} f_* X_2$$

$$\vdots$$

$$S_r(f_* X_n) = \frac{k_n}{1+rk_n} f_* X_n.$$

Then, we know that, the shape operator of M_r is

$$S_r = \begin{bmatrix} \frac{k_1}{1+rk_1} & 0 & \cdots & 0 \\ 0 & \frac{k_2}{1+rk_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{k_n}{1+rk_n} \end{bmatrix}$$

and

$$\begin{aligned} K_r &= \det S_r \\ &= \frac{k_1}{1+rk_1} \cdot \frac{k_2}{1+rk_2} \cdots \frac{k_n}{1+rk_n} \\ &= \frac{k_1 . k_1 \ldots k_n}{\prod_{i=1}^{n} (1+rk_i)} \\ &= \frac{\prod_{i=1}^{n} k_i}{\prod_{i=1}^{n} (1+rk_i)} \end{aligned}$$

and

$$\Phi(r, k_1, k_2, \ldots, k_n) = \prod_{i=1}^{n} (1+rk_i)$$

then

$$K_r = \frac{\dfrac{\partial^n \Phi(r,k_1,k_2,\ldots,k_n)}{(\partial r)^n}}{(n!)\ \Phi(r,k_1,k_2,\ldots,k_n)} .$$

On the other hand

$$H_r = \mathrm{Iz}\ S_r$$

$$= \frac{k_1}{1+rk_1} + \frac{k_2}{1+rk_2} + \cdots + \frac{k_n}{1+rk_n}$$

$$= \frac{k_1 \prod\limits_{i=2}^{n} (1+rk_i) + (1+rk_1)k_2 \prod\limits_{i=3}^{n} (1+rk_i) + \ldots + k_n \prod\limits_{i=1}^{n-1} (1+rk_i)}{\prod\limits_{i=1}^{n} (1+rk_i)} .$$

Since

$$\Phi(r,k_1,k_2,\ldots,k_n) = \prod_{i=1}^{n} (1+rk_i) ,$$

we get that

$$H_r = \frac{\dfrac{\partial \Phi(r,k_1,k_2,\ldots,k_n)}{\partial r}}{\Phi(r,k_1,k_2,\ldots,k_n)} .$$

The case $n = 2$ of this theorem reduces to the result which is given in [2].

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