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ABSOLUTE PRODUCT SUMMABILITY OF THE FOURIER SERIES AND ITS ALLIED SERIES BY

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ABSTRACT

In this paper, the authors have proved four theorems concerning $|(R, \exp\{\log w)^{\Delta}\}, \alpha)(C, 1)|(\Delta > 0, \alpha > 0)$ summability of the Fourier series, its factored conjugate series and their derived series. Earlier, these results were obtained by Chandra [1] for $|R, \exp\{\log w \ \log\log w\}, 1+\alpha|(\alpha > 0)$ summability. Also it has been shown that the sequence of factors $\{1/\log(n+1)\}$ used for the conjugate series and its derived series can not be dropped.

DEFINITIONS AND NOTATIONS

Let, throughout the paper, Σ stand for $\sum_{0}^{\infty}$ or $\sum_{1}^{\infty}$ in case the first term is either zero or not defined and let Σa_n be an infinite series with the partial sum $s_n = a_0 + a_1 + a_2 + \ldots + a_n$. Then t_n, the (C,1)–mean of (s_n), is given by

$$t_n = (n+1)^{-1} \sum_{m=o}^{n} s_m = \sum_{m=o}^{n} (n+1-m)a_m/(n+1).$$

Hence

$$t_n - t_{n-1} = \sum_{m=1}^{n} m a_m/(n+(n+1)) \qquad (n \geqslant 1).$$

Let

$$d_n = \begin{cases} t_n - t_{n-1} & (n \geqslant 1), \\ t_o & (n=0), \end{cases}$$

and let $\lambda(w)$ be a differentiable, monotonic increasing, function of w tending to infinity with w. Then $(R, \lambda(w), \alpha)$ mean of $\Sigma\, d_n$, which is the same thing as $(R, \lambda(w), \alpha)$ (C, 1) mean of Σa_n, where $\alpha > 0$, is given by (see [5] and [6])

$$A_\alpha(w) = (\lambda(w))^{-\alpha} \sum_{n \leq w} \{\lambda(w) - \lambda(n)\}^\alpha \, d_n$$

$$= (\lambda(w))^{-\alpha} \sum_{n \leq w} \{\lambda(w)-\lambda(n)\}^{\alpha} \frac{1}{n(n+1)} \sum_{m=1}^{n} m\, a_m$$

The series $\Sigma\, a_n$ is said to be summable $|(R,\lambda(w),\alpha)\ (C,1)|$ where $\alpha > 0$, if

$$\int_k^{\infty} \frac{\lambda^{(1)}(w)}{\{\lambda(w)\}^{\alpha+1}} \left| \sum_{n \leq w} \{\lambda(w)-\lambda(n)\}^{\alpha-1} \frac{\lambda(n)}{n(n+1)} \sum_{m=1}^{n} m\, a_m \right| dw < \infty,$$

where h is a positive number (see [7] and [8]) and $\lambda^{(1)}(w)$ stands for

$\frac{d}{dw}\lambda(w)$. For $\alpha>0$, we further define that

$\Sigma\, c_n(t) = O(1)\ |(R,\lambda(w),\alpha)\ (C,1)|$, uniformly in $0 < t < \pi$, if

$$\int_k^{\infty} \frac{\lambda^{(1)}(w)}{\{\lambda(w)\}^{\alpha+1}} \left| \sum_{n \leq w} \{\lambda(w)-\lambda(n)\}^{\alpha-1} \frac{\lambda(n)}{n(n+1)} \sum_{m=1}^{n} m\, c_m(t) \right| dw = O(1),$$

uniformly in $0 < t < \pi$. Silmilarly

$\Sigma\, c_n(t) = O(1)\ |C, \alpha|\ (\alpha > 0)$, uniformly in $0 < t < \pi$, if

$$\Sigma\, (nA_n^{\alpha})^{-1} \left| \sum_{m=1}^{n} A_{n-m}^{\alpha-1}\, m\, c_m(t) \right| = O(1),$$

uniformly in $0 < t < \pi$.

Let f be 2π–periodic function and L–integrable over $(-\pi, \pi)$. Then we may suppose, without loss of generality, the Fourier series of f, at a point x, is given by

(1.1) $\Sigma\, (a_n \cos nx + b_n \sin nx) = \Sigma\, A_n(x)$.

Then the series conjugate to (1.1) is given by

(1.2) $\Sigma\, (b_n \cos nx - a_n \sin nx) = \Sigma\, B_n(x)$.

The differentiated series of the Fourier series at a point x will be

$$\Sigma\, n(b_n \cos nx - a_n \sin nx) = \Sigma\, nB_n(x)$$

and

$$\Sigma -n(a_n \cos nx + b_n \sin nx) = -\Sigma\, n\, A_n(x),$$

respectively.

We use the following notations throughout this paper:

(1.3) $\varnothing(t) = \frac{1}{2}\{f(x+t)+f(x-t)-2s\}$ (s is suitable constant)

(1.4) $\psi(t) = \frac{1}{2}\{f(x+t)-f(x-t)\}$

(1.5) $\beta(w) = \exp\{(\log w)^{\Delta}\}$ $(\Delta > 0)$

(1.6) $\beta^{(1)}(w) = \frac{d}{dw}\beta(w)$

(1.7) $F(w) = \sum_{m \leq w} \{\beta(w)-\beta(n)\}^{\alpha-1}\beta(n)/\{n(n+n1)\}$ $(\alpha > 0)$

(1.8) $E(w,t) = \sum_{n \leq w} \{\beta(w)-\beta(n)\}^{\alpha-1} n^{-1}\beta(n)\exp(int)$ $(\alpha > 0)$

(1.9) $K(w,t) = \sum_{n \leq w} \{\beta(w)-\beta(n)\}^{\alpha-1}\{n\log(n+1)\}^{-1}\beta(n)\exp(int)$ $(\alpha > 0)$

(1.10) $\Delta g_n = g_n - g_{n+1}$ $(n \geqslant 0)$.

Throughout the paper, we take $K \geqslant \pi e^5$ for the convenience in the analysis.

INTRODUCTION

In this paper, we prove the following theorems concerning the absolute summability of the Fourier series and allied series at a point x:

THEOREM 1. Let $t\,\varnothing_1(t) = \int_0^t \varnothing(u)\,du$. Then

(2.1) $\varnothing_1(t)\log\log(k/t) \in BV(0,\pi)$

implies that

(2.2) $\sum A_n(x) \in |(R, \beta(w), \alpha)(C,1)|$ $(\alpha > 0)$.

THEOREM 2. Let $t\,\psi_1(t) = \int_0^t \psi(u)\,du$. Then

(2.3) (i) $\psi_1(t)\log\log(k/t) \in BV(0,\pi)$; (ii) $\frac{\psi_1(t)}{t\log(k/t)} \in L(0,\pi)$

imply that

(2.4) $\sum B_n(x)/\log(n+1) \in |(R,\beta(w),\alpha)(C,1)|$ $(\alpha > 0)$.

The factor $(\log(n+1))^{-1}$ in (2.4) cannot be dropt.

THEOREM 3. Let $U(t) = \psi(t)/t$. Then

(2.5) $U(t)\log\log(k/t) \in BV(0, \pi)$

implies that

(2.6) $\Sigma\, n\, B_n(x) \in |(R, \beta(w), \alpha)(C, 1)| \quad (\alpha > 0)$.

THEOREM 4. Let $V(t) = \varnothing(t)/t$. Then

(2.7) (i) $V(t)\log\log \frac{k}{t} \in BV(0,\pi)$; (ii) $\frac{V(t)}{t\log\frac{k}{t}} \in L(0, \pi)$

imply that

(2.8) $\Sigma\, -nA_n(x)/\log(n+1) \in |(R, \beta(w), \alpha)(C,1)| \quad (\alpha > 0)$.

The factor $\{\log(n+1)\}^{-1}$ in (2.8) can not be dropt.

Earlier, Chandra [1] established these results for $|R, \exp\{\log w \log\log w\}, 1+\alpha|(\alpha>0)$ summability. Since there has not been any known relation between the summability methods $|R, \exp\{\log w \log\log w\}, 1+\alpha|(\alpha>0)$ and $|(R, \beta(w), \alpha)(C, 1)|$ $(\alpha>0)$ therefore it remains open to settle the problem about the relationship of these two methods.

It may be observed (see Chandra [1]; Lemma 7) that the conditions (2.3) and (2.7) are equivalent to

(2.9) (i) $\psi_1(0+) = 0$; (ii) $\int_0^{\pi} \log\log(k/t)\,|u\,\psi_1(t)| < \infty$

and

(2.10) (i) $V(0+) = 0$; (ii) $\int_0^{\pi} \log\log(k/t)\,|d\,V(t)| < \infty$,

respectively.

INEQUALITIES

For the proof of the theorems, we shall require the following order-estimates, uniformly in $0<t\leq\pi$, whenever $\Delta > 1$ and $0 < \alpha \leq 1$:

(3.1) $\int_2^{w} \{\beta^{(1)}(y)(\log w)^{1-\Delta}/y\}\,dy = O\,\{\beta(w)\,w^{-1}(\log w)^{1-\Delta}\}$

(3.2) $F(w) = O\{\beta^{\alpha}(w)\ (\log\ w)^{1-\Delta}\ w^{-1}\}$

(3.3) $E(w,t) = O\{t^{-\alpha}\beta^{\alpha}(w)\ w^{-\alpha}\ (\log\ w)^{(\Delta-1)\ (\alpha-1)}\}\ (w > t^{-1})$

(3.4) $K(w,t) = O\{t^{-\alpha}\beta^{\alpha}(w)\ w^{-\alpha}\ (\log\ w)^{(\Delta-1)\ (\alpha-1)-1}\}\ (w > t^{-1})$

Inequality (3.3) is contained in Chandra [3]; (3.2) and inequality (3.4) may be obtained similarly. Thus, we furnish the proofs of (3.1) and (3.2) only.

Proof of (3.1). It is easlily verified that

$$\frac{d}{dw}\left\{\frac{\beta(w)}{w}(\log\ w)^{1-\Delta}\right\} \sim \frac{\beta^{(1)}(w)}{w}\quad(\log\ w)^{1-\Delta}$$

as $w \to \infty$. Hence if c is a constant with $0 < c < 1$, we have

$$\frac{d}{dw}\left\{\frac{\beta(w)}{w}(\log\ w)^{1-\Delta}\right\} \geq c\ \frac{\beta^{(1)}(w)}{w}\ (\log\ w)^{1-\Delta}$$

for sufficiently large w. On integrating this inequality, we obtain (3.1).

Proof of (3.2). We have

$$F(w) = O(1) + \int_2^{w_1} \{\beta(w)-\beta(y)\}^{\alpha-1}\ \frac{\beta(y)}{y(y+1)}\ dy$$

$$= O(1) + \left(\int_2^{w_1} + \int_{w_1}^{w}\right)\left(\{\beta(w)-\beta(y)\}^{\alpha-1}\ \frac{\beta(y)}{y(y+1)}\ dy\right)$$

$$= O(1) + J_1 + J_2, \text{ say,}$$

where w_1 is determined by the equation

$$(\log\ w)^{\Delta} - (\log\ w_1)^{\Delta} = 1.$$

Now

$$J_1 = \frac{1}{\Delta}\int_2^{w_1} \{\beta(w)-\beta(y)\}^{\alpha-1}\ \frac{\beta^{(1)}(y)}{(y+1)\ (\log\ y)^{\Delta-1}}\ dy$$

$$\leq \frac{1}{\Delta}\ \{\beta(w)-\beta(w_1)\}^{\alpha-1}\int_2^{w_1} \beta^{(1)}(y)\ (\log\ y)^{1-\Delta}\ (y+t1)^{-1}\ dy$$

$$= O\{\beta^{\alpha}(w)\ w^{-1}\ (\log\ w)^{1-\Delta}\},$$

by (3.1). And

$$\Delta J_2 = \int_{w_1}^{w} \{\beta(w)-\beta(y)\}^{\alpha-1} \frac{\beta^{(1)}(w)}{(y+1)\ (\log\ y)^{\Delta-1}}\ dy$$

$$= O\{(w_1)^{-1}\ (\log\ w_1)^{1-\Delta} \int_{w_1}^{w} \{\beta(w)-\beta(y)\}^{\alpha-1}\ \beta^{(1)}(y)\ dy\}$$

$$= O\{\beta^{\alpha}(w)\ w^{-1}\ (\log\ w)^{1-\Delta}\}.$$

Combining J_1 and J_2, we obtain the required result.

LEMMAS

We shall use the following lemmas in the proof of the theorems:

LEMMA 1. For $p=1,2$ and $q=2,3$

$$\int_0^t \frac{\sin\ nu}{u(\log(k/u)^p(\log\log(k/u))^q}\ du = O\{(\log\ n)^{-p}(\log\log\ n)^{-q}\},$$

uniformly in $0 < t \leq \pi$.

This may be deduced from (3.2) of Chandra [2].

LEMMA 2. Uniformly in $0 < t < \pi$,

$$\Sigma = \Sigma\ (nA_n^{\alpha})^{-1}\ \left|\ \sum_{m=o}^{n} A_{n-m}^{\alpha-1} \sin\ mt\ \right| = O(1),$$

$$\text{where } A_n^{\alpha} = \binom{n+\alpha}{\alpha} \sim \frac{n^{\alpha}}{\Gamma(\alpha+1)}$$

Proof. Writing T for the integral part of k/t, we obtain by Lemma 5.1 of McFadden [6]

$$\Sigma \leq t \sum_{n=1}^{T} (nA_n^{\alpha})^{-1} \sum_{m=1}^{n} mA_{n-m}^{\alpha-1} + \sum_{n=T}^{\infty} (nA_n^{\alpha})^{-1} \left| \sum_{m=o}^{n} A_{n-m}^{\alpha-1}\right.$$

$$\left.\sin\ mt\ \right| = O(1) + O\ (t^{-\alpha} \sum_{n=T}^{\infty} n^{-1-\alpha}$$

$$= O(1), \text{ uniformly in } 0 < t < \pi.$$

This completes the proof of the lemma.

LEMMA 3. Uniformly in $0 < t \leq \pi$,

$$\Sigma\ (nA_n^{\alpha})^{-1}\ \left|\ \sum_{m=1}^{n} A_{n=m}^{\alpha-1} \frac{\cos\ mt}{\log\ (m+1)}\ \right| = O\ \{\log\log(k/t\}.$$

This may be proved on proceeding as in Lemma 2.

LEMMA 4 ([10]). Let F be measurable over $(0, \infty)\times(0, \infty)$. Then in order that for every $h \in L'(0, \infty)$, the function

$$H(y) = \int_0^\infty F(y,t)h(t)\, dt$$

should be defined almost everywhere and

$$\int_0^\infty |G(y)|\, dy < \infty,$$

it is necessary and sufficient that

$$\operatorname*{ess\ sup}_{0<t\leq\pi} \int_0^\infty |F(y,t)|\, dy < \infty.$$

LEMMA 5. For all t in $0 < t < \pi$,

$$(4.1)\qquad \sum_{n=1}^{\infty} n^{-1} \cos nt = -\log \{2 \sin\tfrac{1}{2}t\}.$$

Proof. We know that

$$\sum_{n=1}^{\infty} n^{-1} \sin nt = \tfrac{1}{2}(\pi-t)$$

for all t in $0 < t < \pi$ and hence

$$\sum_{n=1}^{\infty} n^{-1} \cos nt = \sum_{n=1}^{\infty} n^{-1} \exp(int) - i \sum_{n=1}^{\infty} n^{-1} \sin nt$$

$$(4.2)\qquad = \sum_{n=1}^{\infty} n^{-1} \exp(int) - \tfrac{1}{2} i (\pi-t).$$

Also

$$\sum_{n=1}^{\infty} i \int_t^\pi \exp(int)\, dt = -\log 2 - \sum_{n=1}^{\infty} n^{-1} \exp(int),$$

so that

$$\sum_{n=1}^{\infty} n^{-1}\exp(int) = -\log 2 - i \sum_{n=1}^{\infty} \exp(inu)\, du$$

$$= - \log 2 - i \int_t^{\pi} \{\exp(iu)/(1-\exp(iu))\} \, du$$
$$= - \log 2 + \log (1-\exp(i\pi)) - \log (1-\exp(it))$$
$$= - \log(1-\exp(it))$$
$$= - \log(2\sin\tfrac{1}{2}t) + \tfrac{1}{2}i(\pi-t).$$

Using this in (4.2), we get (4.1).

PROOF OF THE THEOREMS

In view of the first theorem of consistency (see [7] and [8]) and the second theorem of consistency (see [5]) for the absolute Riesz summability, we can assume, respectively, $0 < \alpha \leq 1$ and $\Delta > 1$, for the proof of all the theorems.

5.1. Proof of Theorem 1. We have

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \varnothing(t) \cos nt \, dt$$

$$= \frac{2}{\pi} \int_0^{\pi} \varnothing_1(t) \, nt \sin nt \, dt,$$

integrating by parts and using $\varnothing_1(\pi)=0$. Once again integrating by parts and using $\varnothing_1(\pi) = 0$, we obtain that

$$A_n(x) = - \frac{2}{\pi} \int_0^{\pi} h_n(t) \, d \, \{\varnothing_1(t) \operatorname{loglog}(k/t)\},$$

where

$$h_n(t) = \int_0^t \frac{nu \sin nu}{\operatorname{loglog}(k/u)} \, du .$$

Now, to prove that $\Sigma A_n(x) \in |(R, \beta(w), \alpha)(C, 1)|$, it is sufficient to show that

(5.1.1) $\Sigma h_n(t) = O(1) \; |(R, \beta(w), \alpha)(C, 1)|$, uniformly in $0 < t < \pi$, whenever (2.1) holds. However, integration by parts yields that

$$h_n(t) = - \frac{t \cos nt}{\operatorname{loglog}(k/t)} + \frac{\sin nt}{n} \frac{d}{dt} \left\{ \frac{t}{\operatorname{loglog}(k/t)} \right\}$$

$$- \int_0^t \frac{\sin nt}{n} \left(\frac{d}{dt}\right)^2 (t/\log\log(k/t)) \quad dt$$

$= h_{n,1}(t) + h_{n,2}(t) + h_{n,3}(t)$, say.

By Lemma 1, it follows that

$$\Sigma\ h_{n,3}(t) \in |C,0|$$

and hence by the absolute regularity of the method

$$\Sigma\ h_{n,3}(t) \in |(R,\beta(w),\alpha)\ (C,1)|.$$

Also, by Lemma 2,

$\Sigma\ h_{n,2}(t) = O(1)\ |R,\ \beta(w),\ \alpha)\ (C,\ 1)|$, uniformly in $0 < t < \pi$.
Thus, to complete the proof of (5.1.1), we only require to prove that

$$\int_{e^2}^{\infty} \frac{\beta^{(1)}(w)}{\beta^{1+\alpha}(w)} \left| \sum_{n\leq w} \{\beta(w)-\beta(n)\}^{\alpha-1} \frac{\beta\ (n)}{n(n+1)} \sum_{m=1}^{n} m \cos mt \right| dw$$

$$= O\ \{t^{-1} \log\log\ (k/t)\}, \text{ uniformly in } 0 < t < \pi.$$

Now, for $T=(k/t)\ (\log(k/t))^{\Delta}$, we split up the integral ${}_{e2}\int^{\infty}$ into sub-integrals ${}_{e2}\int^{k/t}$, ${}_{k/t}\int^{T}$ and ${}_{T}\int^{\infty}$. Let these sub-integrals be denoted by I_1, I_2 and I_3, respectively. Then, by using cos nt 1 and (3.2), we obtain that

$$I_1 = O\ \{{}_{e2}\int^{k/t} \{\beta^{(1)}(w)/\beta^{1+\alpha}(w)\}\ w^2\ F(w)\ dw\} = O\ (t^{-1}),$$

uniformly in $0 < t < \pi$. And using the inequality

$$\sum_{m=1}^{n} m \cos mt = O\ (n/t), \text{ uniformly in } 0 < t < \pi, \text{ and } (3.2)$$

once again, we obtain that

$$\begin{aligned} I_2 &= O\ \{t^{-1}\ {}_{k/t}\int^{T}\ \{\beta(^{(1)}(w)/\beta^{1+\alpha}(w)\}\ w\ F(w)\ dw\} \\ &= O\ \{t^{-1}\ {}_{k/t}\int^{T}\ w^{-1}\ dw\} \\ &= O\ \{t^{-1} \log\log(k/t)\}, \text{ uniformly in } 0 < t < \pi. \end{aligned}$$

Finally, we observe that

$$\sum_{m=1}^{n} m \cos mt = \frac{\cos(n+1)t-1}{(2\sin\frac{1}{2}t)^2} + (n+1)\ \frac{\sin(n+\frac{1}{2})t}{2\ \sin\frac{1}{2}t},$$

by Abel's transformation, and hence

$$I_3 = O(t^{-2})\, {}_T\!\int^{\infty} \{\beta^{(1)}(w)/\beta^{1+\alpha}(w)\}\, F(w)\, dw$$
$$+\ O(t^{-1})\, {}_T\!\int^{\infty} \{\beta^{(1)}(w)/\beta^{1+\alpha}(w)\}\ |E(w,t)|\ dw$$
$$= O(t^{-1}), \text{ uniformly in } 0 < t < \pi,$$

by (3.2) and (3.3). Thus, collecting the results, we obtain the required result.

This completes the proof of the theorem.

5.2. Proof of Theorem 2. We prove the theorem under (2.9). We have

$$B_n(x) = \frac{2}{\pi} \int_0^{\pi} \psi(t) \sin nt\, dt$$
$$= \frac{2}{\pi} \int_0^{\pi} \psi_1(t)\, nt \cos nt\, dt,$$

integrating by parts. Once again integrating by parts and using (2.9) (i), we obtain that

$$B_n(x) = -2\,\psi_1(\pi)\,\frac{\cos n\pi}{n\pi} + \frac{2}{\pi}\int_0^{\pi} \left\{ \frac{\cos nt}{n} + t \sin nt \right\} d\,\psi_1(t)$$
$$= (2/\pi)\, {}_0\!\int^{\pi} \{n^{-1}(\cos nt - \cos n\pi) + t \sin nt\}\, d\,\psi_1(t).$$

Now, $\Sigma B_n(x)/\log(n+1) \in |(R, \beta(w), \alpha)(C, 1)|$ if

(5.2.1) $\Sigma R_n(t) = O(1)\ |(R, \beta(w), \alpha)(C, 1)|$, uniformly in $0 < t \leq \pi$, whenever (2.9) (ii) holds, where

$$R_n(t) = (\log\log(k/t))^{-1}\, \frac{\cos nt}{n\log(n+1)} + t(\log\log(k/t))^{-1}\, \frac{\sin nt}{\log(n+1)}$$
$$= R_{n,1}(t) + R_{n,2}(t), \text{ say.}$$

However, it follows from Lemma 3 that

$$\Sigma R_{n,1}(t) = O(1)\, |C,1|, \text{ uniformly in } 0 < t \leq \pi,$$

and hence it is necessarily summable $|(R,\beta(w),\alpha)(C,1)|$. Thus, to complete the proof of (5.2.1), it remains to show that

$$\Sigma R_{n,2}(t) = O(1)\, |R,\beta(w), \alpha)\ (C, 1)|, \text{ uniformly in } 0 < t < \pi,$$

that is

$$J = \int_{e^2}^{\infty} \frac{\beta^{(1)}(w)}{\beta^{1+\alpha}(w)} \left| \sum_{n \leq w} \{\beta(w) - \beta(n)\}^{\alpha-1} \frac{\beta(n)}{n(n+1)} \sum_{m=1}^{n} \frac{m \sin mt}{\log(m+1)} \right| d\,w$$

$= O\ \{t^{-1} \log\log(k/t)\}$, uniformly in $0 < t < \pi$.

Now, for $T = (k/t)(\log(k/t))^{\Delta}$, we split up the integral ${}_{e2}\int^{\infty}$ into sub-integrals ${}_{e2}\int^{k/t}$, ${}_{k/t}\int^{T}$ and ${}_{T}\int^{\infty}$, and denote them, respectively, by J_1, J_2 and J_3. Proceeding as in I_1 and I_2 of Theorem 1, we may obtain that

$$J_i = O\ \{t^{-1} \log\log(k/t)\}\ (i=1, 2),$$

uniformly in $0 < t < \pi$. Also, by Abel's transformation,

$$\sum_{m=1}^{n} \frac{m \sin mt}{\log(m+1)} = O\ (t^{-2}) - \frac{n+1}{\log(n+2)} \cdot \frac{\cos(n+\frac{1}{2})t}{2 \sin\frac{1}{2}t}$$

and hence

$$J_3 = O(t^{-2})_T\int^{\infty} \{\beta^{(1)}(w) \mid \beta^{1+\alpha}(w)\}\ \{F(w) + t\,|K(w,t)|\}\ dt$$
$$= O\ (t^{-1}),\ \text{uniformly in } 0 < t < \pi,$$

by (3.2) and (3.4).

Collecting the results obtained for J_i $(i=1,2,3,)$, the proof of (2.4) may be completed.

Now we show that the factor $1/\log(n+1)$ in (2.4) can not be dropped. We have

$$B_n(x) = -2\,\psi_1(\pi)\ \frac{\cos n\pi}{n\pi} + \frac{2}{\pi}\ {}_0\int^{\pi} \log(n+1)\,(R_{n,1}(t)+R_{n,2}(t)) \times \log\log\frac{k}{t}\ d\,\psi_1(t)$$

$$= P_1(n) + P_2(n) + P_3(n), \text{ say.}$$

However, $\Sigma\, P_1(n) \in |C, 1|$. Also, proceeding as above in J_1, J_2, J_3, it may be proved that

$$\Sigma\ P_3(n) \in |(R, \beta(w), \alpha)\ (C, 1)|.$$

Thus in order that $\Sigma\, B_n(x) \in |(R, \beta(w), \alpha)\,(C,1)|$, it is necessary and sufficient that

$$\Sigma\ P_2(n) \in |(R, \beta(w), \alpha)\ (C,1)|$$

for which, by Lemma 4, it is necessary that

$$(5.2.2)\quad \underset{0<t<\pi}{\text{ess sup}} \int_{e^2}^{\infty} \frac{\beta^{(1)}(w)}{\beta^{1+\alpha}(w)} \Big| \sum_{n\leq w} \{\beta(w)-\beta(n)\}^{\alpha-1} \frac{\beta(n)}{n(n+1)} \sum_{m=1}^{n} mR_{m,1}(t)\log(m+1) \Big| dw$$

However, by Lemma 5,

$$\sum_{n=1}^{\infty} R_{n,1}(t)\log(n+1) = \left(\log\log \frac{k}{t}\right)^{-1} \log\left(\frac{1}{2\sin\frac{1}{2}t}\right)$$

which tends to infinity as $t \to 0+$. Therefore (5.2.2) does not hold since $|(R, \beta(w), \alpha)(C, 1)|$ is absolutely regular method.

5.3. Proof of Theorem 3. We have

$$n\, B_n(x) = \frac{2}{\pi} \int_0^{\pi} \psi(t)\, n \sin nt\, dt$$

$$= \frac{2}{\pi} \int_0^{\pi} U(t)\log\log(k/t) \frac{nt \sin nt}{\log\log(k/t)}\, dt.$$

Integrating by parts and using the fact that $\psi(\pi) = 0$, we obtain that

$$n\, B\, (x) = -\frac{2}{\pi} \int_0^{\pi} d\{v(t)\log\log(k/t)\} \int_0^{\pi} \frac{nu \sin nu}{\log\log(k/u)}\, du.$$

Now, whenever (2.5) holds, the proof of the theorem may be completed by using (5.1.1).

5.4. Proof of Theorem 4 .We shall prove the theorem under the equivalent condition (2.10). We have

$$-n\, A_n\, (x) = -\frac{2}{\pi} \int_0^{\pi} V(t)\; nt \cos nt\, dt$$

$$= \frac{2}{\pi} \int_0^{\pi} \{n^{-1}(\cos nt - \cos n) + t \sin nt\}\, dV(t),$$

integrating by parts and using $(V(0+) = 0$.

Now, the proof of (2.8) may be completed by using (5.2.1), whenever (2.10) (ii) holds.

The proof that the factor (1 /log(n+1) in (2.8) cannot be dropped may be followed from Theorem 2.

This completes the proof of the theorem.

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