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Some Characterizations of Pseudo-Complex Space Forms

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Some Characterizations of Pseudo-Complex Space Forms

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ABSTRACT

B.Y. Chen and K. Ogiue [2] have proved that a Kahler manifold is a complex space form if and only if it has constant antiholomorphic sectional curvature. In this note we extend this result to a Pseudo-Kahler manifold and give another characterization of a Pseudo-Complex space form, which may be considered as a complex version of the criterian for constancy of sectional curvature of a Riemannian Manifold, obtained by Cartan [1].

1. INTRODUCTION

Definition: A Kahler Manifold (M^{2n},J) with structure Tensor J, endowed with a Pseudo-Riemannian metric g shall be called a Pseudo -Kahler manifold.

A vector field X on M is space-like, time-like, null if $g(X,X) > 0$, < 0, $= 0$ respectively.

A submanifold N of M shall be called non-degenerate (degenerate) if the restriction of g to N is non-degenerate (degenerate).

First, we state the following lemma which will be useful in our discussion.

Lemma: (1.1) [3]. The plane $p = sp\ \{X,Y\}$ is non–degenerate if and only if

$$g(X,X)\ g(Y,Y) - g(X,Y)^2 \neq 0.$$

Corollary (1.2): The plane $p = sp\ \{X,JX\}$ is non–degenerate if and only if $g(X,X) \neq 0$.

For a non–degenerate plane p, the sectional curvature K(X,Y) is defined by

$$\frac{R(X,Y,\ X,Y)}{g(X,X)\ g(Y,Y)\ -g(X,Y)^2}\ .$$

A plane section p is called holomorphic (anti–holomorphic) if Jp = p (Jp is perpendicular to p). The sectional curvature holomorphic (anti–holomorphic) plane is called holomorphic (anti–holomorphic) sectional curvature. A Pseudo–Kahler manifold of constant holomorphic sectional curvature called a Pseudo–complex space form.

Let R be the curvature tensor field of M. Then it is well known that R satisfies the following properties:

$$R(JX,\ JY) = R(X,Y), \tag{1}$$

$$R(X,Y)\ JZ = JR(X,Y)\ Z. \tag{2}$$

If K(X,Y) is the sectional curvature of M determined by orthonormal vectors X and Y, then it is easy to prove that

$$K(JX,\ JY) = K(X,Y), \tag{3}$$

$$K(X,\ JY) = K(JX,\ Y). \tag{4}$$

We remark that orthonormal vectors X and Y span and anti–holomorphic plane section if and only if X,Y and JX form an orthonormal set. Moreover, for such a plane we have

$$R(X,\ JX,\ JY,\ Y) = K(X,Y) + K(X,JY). \tag{5}$$

2. **Constancy of anti-Holomorhic Sectional Curvature.**

It is well known that a complex space form has constant anti–holomorphic sectional curvature. Conversely, Chen and Ogiue [2] proved that 'A Kahler manifold of real dimension $\geq$ 6 is a complex space form if and only if it has constant antiholomorphic sectional curvature. In the following theorem we extend this result to Pseudo–Kahler manifolds.

Theorem (2.1). Let (M,J) be a Pseudo–Kahler manifold of real dim $\geq$ 6. Then M is a Pseudo–complex space form if and only if the anti–holomorphic sectional curvatures of M are constant.

Proof. It is obvious that a complex space form has constant anti–holomorphic sectional curvature.

Conversely, assume that the anti–holomorphic sectional curvature of M is a constant c. Let X and Y be orthonormal vectors which span an anti–holomorphic plane section. Then we shall consider the following two cases:

Case I. [2]. Take $g(X,X) = g(Y,Y)$. In this case let us define $X' = \frac{X+Y}{2}$, and $Y' = \frac{JX - JY}{2}$. Then sp $\{X', Y'\}$ is again an anti-holomorphic plane section. Thus we have $c = K(X', Y')$. Using the relations (1) – (5) we can get

$$H(X) + H(Y) = 8c. \tag{6}$$

Case II. Let $g(X,X) = - g(Y,Y)$. In this case we define $X' = aX + bY$ and $Y' = bJX + aJY$, where a and b are two nos, such that $a^2 - b^2 = 1$. Then, again X', Y' and JX' form on orthonormal triplet. Therefore we have $c = K(X', Y')$. Again, as above we get

$$H(X) + H(Y) = 8c \tag{7}$$

Now, let m be any arbitrary point of M and let U and V be arbitrary unit vectors. Then we can always choose a unit vector W in $sp\{U,JU\}^{\perp} \cap sp\{V,JV\}^{\perp}$ such that sp $\{U,W\}$ and sp $\{W,V\}$ are anti –holomorphic. Therefore from (6) and (7) we have

$$H(U) + H(W) = 8c$$

and $H(W) + H(V) = 8c$, which gives $H(U) = H(V)$. Thus all non–degenerate planes have same holomorphic sectional curvature. Since the holomorphic sectional curvature is independent of the choice of the plane section, the complex version of well known theorem of F. Schur implies that M is a Pseudo–complex space form.

Now, as an application of the above result, we prove the following important theorem, which may be considered a complex version result of a well known theorem of Cartan [1].

Theorem (2.2). Let (M,J) be a Pseudo–Kahler manifold with real dim ≥ 6. Then M is a Pseudo–complex space form if and only if $R(X,Y,Z,X) = 0$ for all orthonormal vectors X,Y and Z at any point m of M which span an anti–holomorphic subspace of $T_m(M)$.

Proof. Let X,Y and Z be orthonormal vectors which span an anti–holomorphic subspace p of the tangent space $T_m(M)$ at an arbitrary point m. If M is a space of constant holomorphic sectional curvature C, then R is given by

$$R(A,B,C,D) = \frac{c}{4} [g(A,D)\ g(B,C) - g(A,C)\ g(B,D)$$
$$+ g(JA,D)\ g(JB,C) - g(JA,C)\ g(JB,D)$$
$$+ 2g(A,JB)\ g(JC,D)] \tag{8}$$

for all vectors A,B,C and D tangent to M. From (8), it is clear that

$$R(X,Y,Z,X) = 0, \tag{9}$$

for all vectors X,Y and Z of above type.

Conversely, assume that M satisties $R(X,Y,Z,X) = 0$, for all vectors X,Y and Z which span an anti–holomorphic subspace and consider the case when $g(Y,Y) = g(Z,Z)$. We choose non–zero numbers a and b with $a^2 + b^2 = 1$ and define $Y' = aY + bZ$, $Z' = -bY + aZ$. Now, clearly X, Y′ and Z′ form an anti–holomorphic subspace. So we have

$$\begin{aligned} 0 &= R(X,Y',Z',X) \\ &= R(X,\ aY + bZ,\ -bY + aZ,\ X) \\ &= -\ ab\ R(X,Y,\ Y,X) + a^2(X,Y,\ Z,X) \\ &\quad - b^2\ R(X,Z,\ Y,X) + ab\ R(X,Z,\ Z,X) \end{aligned}$$

which implies that

$$R(X,Y,\ Y,X) = R(X,Z,\ Z,X). \tag{10}$$

When $g(Y,Y) = -g(Z,Z)$. we define Y′ and Z′ by the following:

$Y' = aY + bZ$ and $Z' = bY + aZ$ with $a^2 - b^2 = 1$. Then again X, Y′ and Z′ span an anti–holomorphic subspace. So the hypothesis implies

$$\begin{aligned} 0 &= R(X,Y',\ Z',X) \\ &= ab\ R(X,Y,\ Y,X) + ab\ R(X,Z,\ Z,X). \end{aligned}$$

from which we get

$$R(X,Y,\ Y,X) = -R(X,Z,\ Z,X). \tag{11}$$

Therefore, from (10) and (11) we conclude that

$$K(X,Y) = K(X,Z). \tag{12}$$

Now, let W be any unit vector which tugether with X defines an anti-holomorphic plane. We can write $W = c_1W_1 + c_2W_2$, where $W_1 \in sp\{Z,JZ\}$ and $W_2 \in sp\{Z,JZ\}^{\perp}$. Now using the hypothesis of the theorem and relations (10) and (11) we have

$$\begin{aligned} R(X,W,W,X) &= R(X,c_1W_1 + c_2W_2,\ c_1W_1 + c_2W_2,\ X) \\ &= R(X,Y,\ Y,X). \end{aligned} \tag{13}$$

Thus, from (12) and (13) we conclude that sectional curvatures of M are equal for all anti-holomorphic non-degenerate plane sections containing the vector X. Let U be any other unit vector belonging to $T_m(M)$. Then we can find a unit vector $V \in T_m(M)$ which is orthogonal to X and Y and the planes $sp\{X,V\}$ and $sp\{U,V\}$ are anti-holomorphic Therefore, from (13) we have

$$R(X,V,\ V,X) = R(U,V,\ V,U). \tag{14}$$

Hence, we conclude that all anti-holomorphic non-degenerate planes have same sectional curvatures at m. So, our theorem follows from Theorem (2.1).

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