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Research Article

Faber polynomials coefficients estimates for a certain subclass of Bazilevic functions

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Abstract

For a certain subclass of Bazilevic functions, Faber polynomials expansions are used to obtain bi-univalent properties. Estimates on the nth Taylor-Maclaurin coefficients of functions in this class are found. Moreover, some special cases are also indicated.

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1. Introduction

Let A be the class of all analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ . \tag{1.1}$$

When the function $f \in A$ is univalent, we denote the subclass of these functions by S. The univalence property of the function $f \in S$ guarantees the existence of the inverse function f^{-1}, by using the Koebe one-quarter theorem [9] in $U^* = \left\{w \in \mathbb{C} : |w| < \frac{1}{4}\right\}$, which is defined by $f^{-1}(f(z)) = z \ \ (z \in U)$ and $f(f^{-1}(w)) = w \ (w \in U^*)$ with the power series

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \ .$$

For the function $f \in S$, if the inverse function f^{-1} is univalent in U, then f is called bi-univalent function in U. Let σ be the class of all bi-univalent functions in U which are given by (1.1). In 1967, Lewin [18] was the first author who studied the class of analytic and bi-univalent functions. Later, the first two coefficients $|a_2|$ and $|a_3|$ for different subclasses of analytic and bi-univalent functions were estimated by many authors, see for example [3,4,6,7,12,13,15–17,19,20,22–24,27,28]. In 1903, Faber [10] introduced Faber polynomials

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which have an effective role in some branches of mathematics. In addition, Airault and Bouali [1] determined the coefficients of the inverse function $g = f^{-1}$ as follow

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ..., a_n) w^n,$$

where $K_n^p(a_2, a_3, ..., a_n)$ are given by

$$\begin{aligned}
K_1^p &= pa_2, \qquad K_2^p = \frac{p(p-1)}{2}a_2^2 + pa_3, \\
K_3^p &= p(p-1)a_2a_3 + pa_4 + \frac{p(p-1)(p-2)}{3!}a_2^3, \\
K_4^p &= p(p-1)a_2a_4 + pa_5 + \frac{p(p-1)}{2}a_3^2 + \frac{p(p-1)(p-2)}{2}a_2^2a_3 + \frac{p!}{(p-4)!4!}a_2^4,
\end{aligned}$$

More generally,

$$\begin{aligned}
K_n^p &= \frac{p!}{(p-n)!n!}a_2^n + \frac{p!}{(p-n+1)!(n-2)!}a_2^{n-2}a_3 + \frac{p!}{(p-n+2)!(n-3)!}a_2^{n-3}a_4 \\
&\quad + \frac{p!}{(p-n+3)!(n-4)!}a_2^{n-4}\left(a_5 + \frac{p-n+3}{2}a_3^2\right) \\
&\quad + \frac{p!}{(p-n+4)!(n-5)!}a_2^{n-5}\left[a_6 + (p-n+4)a_3a_4\right] + \sum_{j\geq 6}^{\infty} a_2^{n-j}V_j,
\end{aligned}$$

where V_j is a homogeneous polynomial of degree j in the variables $a_2, a_3, ..., a_n$. In [1] and [2], we see that

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\mu} = 1 - \sum_{n=2}^{\infty} F_{n-1}^{\mu+n-1}(a_2, a_3, ..., a_n) z^{n-1} \tag{1.2}$$

and

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, ..., a_n) z^{n-1}, \tag{1.3}$$

where $F_{j-1}^k(a_2, a_3, ..., a_j)$, $j \geq 2$, are the generalized Faber polynomials given by $F_n^{n+j} = -\left(1 + \frac{n}{j}\right) K_n^j$ and $F_{n-1}(a_2, a_3, ..., a_n)$, $n \geq 2$, are the nth Faber polynomials such that $F_n = F_n^n$ (see [2, page 351] and [5, page 52]). We note that

$$\begin{aligned}
F_1^k &= -ka_2, \qquad F_2^k = \frac{k(3-k)}{2}a_2^2 - ka_3, \\
F_3^k &= \frac{k(4-k)(k-5)}{3!}a_2^3 + k(4-k)a_2a_3 - ka_4, \\
F_4^k &= \frac{k(5-k)(k-6)(k-7)}{4!}a_2^4 + \frac{k(5-k)(k-6)}{2!}a_2^2a_3 - k(5-k)a_2a_4 \\
&\quad + \frac{k(5-k)}{2}a_3^2 - ka_5, \\
F_5^k &= \frac{k(6-k)(k-7)(k-8)(k-9)}{5!}a_2^5 + \frac{k(6-k)(k-7)(k-8)}{3!}a_2^3a_3 \\
&\quad + \frac{k(6-k)(k-7)}{2}a_2^2a_4 + \frac{k(6-k)(k-7)}{2}a_2a_3^2 + k(6-k)a_3a_4 \\
&\quad + k(6-k)a_2a_5 - ka_6.
\end{aligned} \tag{1.4}$$

It is well known that $1 + zf''(z)/f'(z) = z(zf'(z))'/zf'(z)$, using (1.3) we have

$$\frac{zf''(z)}{f'(z)} = -\sum_{n=2}^{\infty} F_{n-1}(2a_2, 3a_3, ..., na_n) z^{n-1}. \tag{1.5}$$

For two analytic functions $f_1(z)$ and $f_2(z)$ in U, $f_1(z)$ is subordinate to $f_2(z)$, written $f_1 \prec f_2$ or $f_1(z) \prec f_2(z)$, if there exists a Schwarz function $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n$ which is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$ such that $f_1(z) = f_2(\omega(z))$.

Definition 1.1. Let $\Upsilon(\lambda, \mu, \phi)$ be the class of functions $f \in S$ satisfying the following subordination condition

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \lambda\left(\frac{zf''(z)}{f'(z)} + (1-\mu)\left(1 - \frac{zf'(z)}{f(z)}\right)\right) \prec \phi(z),$$

for some $\lambda, \mu \geq 0$ and ϕ is an analytic function with positive real part in U and $\phi(U)$ is symmetric with respect to the real axis such that

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \ (B_1 > 0).$$

By putting different values of λ, μ and ϕ, in the above definition, various previous results are deduced.

(1) Putting $\phi = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, the subclass of Bazilevic functions which was considered by Wang and Jing [29] is obtained.
(2) The classes $\Upsilon\left(0, 0, \frac{1+Az}{1+Bz}\right) = S[A, B]$ and $\Upsilon\left(1, 0, \frac{1+Az}{1+Bz}\right) = K[A, B](-1 \leq B < A \leq 1)$ are the well-known Janowski starlike and convex functions.
(3) The classes $\Upsilon\left(0, 0, \frac{1+(1-2\alpha)z}{1-z}\right) = S^*(\alpha)$ and $\Upsilon\left(1, 0, \frac{1+(1-2\alpha)z}{1-z}\right) = K(\alpha)$ are the classes of starlike and convex functions of order $\alpha (0 \leq \alpha < 1)$.
(4) The class $\Upsilon(0, 0, \sqrt{1+z}) = S_L^*$ was introduced and studied by Sokół and Stankiewicz [26].
(5) The class $\Upsilon\left(0, 0, z + \sqrt{1+z^2}\right) = S_{\nabla}^*$ was introduced and studied by Raina and Sokół [25].
(6) The class $\Upsilon\left(0, 0, \frac{1}{(1-z)^s}\right) = ST_{hpl}(s)$ $(0 < s \leq 1)$ was introduced and studied by Kanas et al. [14].
(7) The class $\Upsilon(0, 0, e^z) = S_e^*$ was introduced and studied by Mendiratta et al. [21].
(8) The class $\Upsilon\left(0, 0, \frac{2}{1+e^{-z}}\right) = S_G$ was introduced and studied by Goel and Kumar [11].

Definition 1.2. A function $f \in \sigma$ is said to be in the class $\Upsilon_\sigma(\lambda, \mu, \phi)$ if both f and its inverse map $g = f^{-1}$ are in $\Upsilon(\lambda, \mu, \phi)$.

Remark 1.3. There are new classes if we take special cases for the function $\phi(z)$ $z \in U$ in Definition 1.2 such as

(1) If

$$\phi(z) = \frac{1+Az}{1+Bz} = 1 + (A-B)z - B(A-B)z^2 + B^2(A-B)z^3 + ..., -1 \leq B < A \leq 1,$$

then we get the new class $\Upsilon_\sigma(\lambda, \mu, A, B)$ which is defined by

$$\Upsilon_\sigma(\lambda, \mu, A, B) =$$

$$\left\{ f \in \sigma : f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \lambda\left(\frac{zf''(z)}{f'(z)} + (1-\mu)\left(1 - \frac{zf'(z)}{f(z)}\right)\right) \prec \frac{1+Az}{1+Bz}, \right.$$

$$\left. g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} + \lambda\left(\frac{wg''(w)}{g'(w)} + (1-\mu)\left(1 - \frac{wg'(w)}{g(w)}\right)\right) \prec \frac{1+Aw}{1+Bw} \right\};$$

(2) If

$$\phi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+2\alpha^2z^2+....,0<\alpha\leq 1,$$

then we obtain the new class $\Upsilon_{\sigma}(\lambda,\mu,\alpha)$ which is defined by

$$\Upsilon_{\sigma}(\lambda,\mu,\alpha)=$$
$$\left\{f\in\sigma:\left|\arg\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\lambda\left(\frac{zf''(z)}{f'(z)}+(1-\mu)\left(1-\frac{zf'(z)}{f(z)}\right)\right)\right)\right|<\frac{\pi}{2}\alpha,\right.$$
$$\left.\left|\arg\left(g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\lambda\left(\frac{wg''(w)}{g'(w)}+(1-\mu)\left(1-\frac{wg'(w)}{g(w)}\right)\right)\right)\right|<\frac{\pi}{2}\alpha\right\};$$

(3) If

$$\phi(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+....,0\leq\beta<1,$$

then we acquire the new class $\Upsilon_{\sigma}^{\beta}(\lambda,\mu)$ which is defined by

$$\Upsilon_{\sigma}^{\beta}(\lambda,\mu)=$$
$$\left\{f\in\sigma:\Re\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\lambda\left(\frac{zf''(z)}{f'(z)}+(1-\mu)\left(1-\frac{zf'(z)}{f(z)}\right)\right)\right)>\beta,\right.$$
$$\left.\Re\left(g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\lambda\left(\frac{wg''(w)}{g'(w)}+(1-\mu)\left(1-\frac{wg'(w)}{g(w)}\right)\right)\right)>\beta\right\};$$

(4) If

$$\phi(z)=\sqrt{1+z}=1+\frac{1}{2}z-\frac{1}{8}z^2+....,$$

then we get the new class $\Upsilon_{L\sigma}(\lambda,\mu)$ which is defined by

$$\Upsilon_{L\sigma}(\lambda,\mu)=$$
$$\left\{f\in\sigma:\left|\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\lambda\left(\frac{zf''(z)}{f'(z)}+(1-\mu)\left(1-\frac{zf'(z)}{f(z)}\right)\right)\right)^2-1\right|<1,\right.$$
$$\left.\left|\left(g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\lambda\left(\frac{wg''(w)}{g'(w)}+(1-\mu)\left(1-\frac{wg'(w)}{g(w)}\right)\right)\right)^2-1\right|<1\right\};$$

(5) If

$$\phi(z)=z+\sqrt{1+z^2}=1+z+\frac{1}{2}z^2-\frac{1}{8}z^4+....,$$

then we obtain the new class $\Upsilon_{\sigma}^{\Delta}(\lambda,\mu)$ which is defined by

$$\Upsilon_{\sigma}^{\Delta}(\lambda,\mu)=$$
$$\left\{\begin{array}{l} f\in\sigma:\left|\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\lambda\left[\frac{zf''(z)}{f'(z)}+(1-\mu)\left(1-\frac{zf'(z)}{f(z)}\right)\right]\right)^2-1\right| \\ \quad<2\left|f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\lambda\left[\frac{zf''(z)}{f'(z)}+(1-\mu)\left(1-\frac{zf'(z)}{f(z)}\right)\right]\right|,\end{array}\right.$$
$$\left.\begin{array}{l}\left|\left(g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\lambda\left[\frac{wg''(w)}{g'(w)}+(1-\mu)\left(1-\frac{wg'(w)}{g(w)}\right)\right]\right)^2-1\right| \\ \quad<2\left|g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\lambda\left[\frac{wg''(w)}{g'(w)}+(1-\mu)\left(1-\frac{wg'(w)}{g(w)}\right)\right]\right|\end{array}\right\};$$

(6) If

$$\phi(z)=\frac{1}{(1-z)^{s}}=1+sz+\frac{s(s+1)}{2!}z^{2}+\frac{s(s+1)(s+2)}{3!}z^{3}+\ldots,\ 0<s\leq 1,$$

then we acquire the new class $\Upsilon_{\sigma}(\lambda,\mu,s)$ which is defined by

$$\Upsilon_{\sigma}(\lambda,\mu,s)=$$
$$\left\{f\in\sigma: f^{'}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\lambda\left[\frac{zf^{''}(z)}{f^{'}(z)}+(1-\mu)\left(1-\frac{zf^{'}(z)}{f(z)}\right)\right]\prec\frac{1}{(1-z)^{s}},\right.$$
$$\left.g^{'}(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\lambda\left[\frac{wg^{''}(w)}{g^{'}(w)}+(1-\mu)\left(1-\frac{wg^{'}(w)}{g(w)}\right)\right]\prec\frac{1}{(1-w)^{s}}\right\};$$

(7) If

$$\phi(z)=e^{z}=1+z+\frac{1}{2!}z^{2}+\frac{1}{3!}z^{3}+\ldots,$$

then we get the new class $\Upsilon_{\sigma e}(\lambda,\mu)$ which defined by

$$\Upsilon_{\sigma e}(\lambda,\mu)=$$
$$\left\{f\in\sigma:\left|\log\left(f^{'}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\lambda\left(\frac{zf^{''}(z)}{f^{'}(z)}+(1-\mu)\left(1-\frac{zf^{'}(z)}{f(z)}\right)\right)\right)\right|<1,\right.$$
$$\left.\left|\log\left(g^{'}(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\lambda\left(\frac{wg^{''}(w)}{g^{'}(w)}+(1-\mu)\left(1-\frac{wg^{'}(w)}{g(w)}\right)\right)\right)\right|<1\right\}.$$

In this paper, Faber polynomials expansions are used to find estimate of the nth $(n\geq 3)$ Taylor-Maclaurin coefficients $|a_n|$ of functions belong to the class $\Upsilon_{\sigma}(\lambda,\mu,\phi)$. Moreover, estimates of the first coefficients $|a_2|$ and $|a_3|$ are also obtained.

2. The estimates of the coefficients for the class $\Upsilon_{\sigma}(\lambda,\mu,\phi)$

In the next theorem, estimate of the nth $(n\geq 3)$ Taylor-Maclaurin coefficients $|a_n|$ of functions belong to the class $\Upsilon_{\sigma}(\lambda,\mu,\phi)$ is found by using Faber polynomials expansions.

Theorem 2.1. *Let the function $f\in\Upsilon_{\sigma}(\lambda,\mu,\phi)$ and $a_k=0$ for $2\leq k\leq n-1$. Then*

$$|a_n|\leq\frac{B_1}{(\mu+n-1)\left[1+\lambda(n-1)\right]},\quad n\geq 3,\ \lambda,\mu\geq 0. \tag{2.1}$$

Proof. If u and v are Schwarz functions in U such that

$$u(z)=b_1z+\sum_{n=2}^{\infty}b_nz^n \quad\textit{and}\quad v(z)=c_1z+\sum_{n=2}^{\infty}c_nz^n,\qquad (z\in U), \tag{2.2}$$

then

$$|b_n|\leq 1 \quad\textit{and}\quad |c_n|\leq 1 \quad \text{for all } n=1,2,3,\ldots \tag{2.3}$$

which are proved by Duren [9]. Since $f\in\Upsilon_{\sigma}(\lambda,\mu,\phi)$, then there are two analytic functions $u,v:U\to U$ given by (2.2) such that

$$f^{'}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\lambda\left(\frac{zf^{''}(z)}{f^{'}(z)}+(1-\mu)\left(1-\frac{zf^{'}(z)}{f(z)}\right)\right)=\phi(u(z)) \tag{2.4}$$

and

$$g^{'}(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\lambda\left(\frac{wg^{''}(w)}{g^{'}(w)}+(1-\mu)\left(1-\frac{wg^{'}(w)}{g(w)}\right)\right)=\phi(v(w)), \tag{2.5}$$

where $g(w) = f^{-1}(w)$. By using (1.2),(1.3) and (1.5), we get

$$\begin{aligned} & f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \lambda\left(\frac{zf''(z)}{f'(z)} + (1-\mu)\left(1 - \frac{zf'(z)}{f(z)}\right)\right) \\ & = 1 - \sum_{n=2}^{\infty}\Big(F_{n-1}^{\mu+n-1}(a_2, a_3, ..., a_n) + \lambda F_{n-1}(2a_2, 3a_3, ..., na_n) \\ & -\lambda(1-\mu)F_{n-1}(a_2, a_3, ..., a_n)\Big) z^{n-1}, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} & g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} + \lambda\left(\frac{wg''(w)}{g'(w)} + (1-\mu)\left(1 - \frac{wg'(w)}{g(w)}\right)\right) \\ & = 1 - \sum_{n=2}^{\infty}\Big(F_{n-1}^{\mu+n-1}(d_2, d_3, ..., d_n) + \lambda F_{n-1}(2d_2, 3d_3, ..., nd_n) \\ & -\lambda(1-\mu)F_{n-1}(d_2, d_3, ..., d_n)\Big) w^{n-1}, \end{aligned} \tag{2.7}$$

where $d_n = \frac{1}{n}K_{n-1}^{-n}(a_2, a_3, ..., a_n)$. Simple calculation yields

$$\begin{aligned} \phi(u(z)) &= 1 - B_1\sum_{n=1}^{\infty}K_n^{-1}(b_1, b_2, ..., b_n, B_1, B_2, ..., B_n) z^n \\ &= 1 + B_1 b_1 z + \left(B_1 b_2 + B_2 b_1^2\right) z^2 + ..., \qquad (z \in U), \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \phi(v(w)) &= 1 - B_1\sum_{n=1}^{\infty}K_n^{-1}(c_1, c_2, ..., c_n, B_1, B_2, ..., B_n) w^n \qquad (2.9) \\ &= 1 + B_1 c_1 w + \left(B_1 c_2 + B_2 c_1^2\right) w^2 + ..., \qquad (w \in U), \end{aligned}$$

where the coefficients $K_n^p(k_1, k_2, ..., k_n, B_1, B_2, ..., B_n)$ are given by(see [8])

$$\begin{aligned} & K_n^p(k_1, k_2, ..., k_n, B_1, B_2, ..., B_n) = \frac{p!}{(p-n)!n!}k_1^n\frac{(-1)^{n+1}B_n}{B_1} \\ & +\frac{p!}{(p-n+1)!(n-2)!}k_1^{n-2}k_2\frac{(-1)^n B_{n-1}}{B_1} \\ & +\frac{p!}{(p-n+2)!(n-3)!}k_1^{n-3}k_3\frac{(-1)^{n-1}B_{n-2}}{B_1} \\ & +\frac{p!}{(p-n+3)!(n-4)!}k_1^{n-4} \\ & \left(k_4\frac{(-1)^{n-2}B_{n-3}}{B_1} + \frac{p-n+3}{2}k_2^2 k_3\frac{(-1)^{n-1}B_{n-2}}{B_1}\right) \\ & +\sum_{j\geq 5}^{\infty}k_1^{n-j}X_j, \end{aligned}$$

where X_j is a homogeneous polynomial of degree j in the variables $k_1, k_2, ..., k_n$. By comparing the corresponding coefficients of (2.6) and (2.8), we obtain

$$\begin{aligned} & F_{n-1}^{\mu+n-1}(a_2, a_3, ..., a_n) + \lambda F_{n-1}(2a_2, 3a_3, ..., na_n) - \lambda(1-\mu)F_{n-1}(a_2, a_3, ..., a_n) \\ = \; & B_1 K_{n-1}^{-1}(b_1, b_2, ..., b_{n-1}, B_1, B_2, ..., B_{n-1}). \end{aligned} \tag{2.10}$$

Now comparing the corresponding coefficients of (2.7) and (2.9), we get

$$\begin{aligned} & F_{n-1}^{\mu+n-1}(d_2, d_3, ..., d_n) + \lambda F_{n-1}(2d_2, 3d_3, ..., nd_n) - \lambda(1-\mu) F_{n-1}(d_2, d_3, ..., d_n) \\ = \; & B_1 K_{n-1}^{-1}(c_1, c_2, ..., c_{n-1}, B_1, B_2, ..., B_{n-1}). \end{aligned} \tag{2.11}$$

Under the assumption $a_k = 0$ for $2 \leq k \leq n-1$, $d_n = -a_n$ and $F_{n-1}^{\mu+n-1} = -(\mu+n-1)$, (2.10) and (2.11) become

$$[(\mu+n-1) + \lambda(n-1)n - \lambda(1-\mu)(n-1)]\, a_n = B_1 b_{n-1} \tag{2.12}$$

and

$$[-(\mu+n-1) - \lambda(n-1)n + \lambda(1-\mu)(n-1)]\, a_n = B_1 c_{n-1}\ . \tag{2.13}$$

From (2.12), (2.13) and (2.3), we get

$$|a_n| \leq \frac{B_1}{(\mu+n-1)[1+\lambda(n-1)]},$$

which completes the proof. □

Lemma 2.2. [8] *Let the function* $\Phi(z) = \sum_{n=1}^{\infty} \Phi_n z^n$ *be a Schwarz function with* $|\Phi(z)| < 1$, $z \in U$. *Then for* $-\infty < \rho < \infty$

$$\left|\Phi_2 + \rho\Phi_1^2\right| \leq \begin{cases} 1-(1-\rho)\left|\Phi_1^2\right| & \rho > 0 \\ 1-(1+\rho)\left|\Phi_1^2\right| & \rho \leq 0 \end{cases}$$

In the following theorem, Faber polynomials expansions are also used to find estimates of the first coefficients $|a_2|$ and $|a_3|$ of functions belong to the class $\Upsilon_\sigma(\lambda, \mu, \phi)$.

Theorem 2.3. *Let the function* $f \in \Upsilon_\sigma(\lambda, \mu, \phi)$. *Then*

$$|a_2| \leq \begin{cases} \frac{B_1\sqrt{2B_1}}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)B_1^2+2(\mu+1)(\lambda+1)^2(B_1+B_2)\right)}} & B_2 \leq 0, B_1+B_2 \geq 0 \\ \frac{B_1\sqrt{2B_1}}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)B_1^2+2(\mu+1)(\lambda+1)^2(B_1-B_2)\right)}} & B_2 > 0, B_1-B_2 \geq 0 \end{cases}$$

and

$$\left|a_3 - a_2^2\right| \leq \begin{cases} \frac{B_1}{(\mu+2)(2\lambda+1)} & B_1 \geq |B_2| \\ \frac{|B_2|}{(\mu+2)(2\lambda+1)} & B_1 < |B_2| \end{cases} \tag{2.14}$$

Proof. Put $n = 2$ and $n = 3$ in (2.10) and (2.11), respectively, we obtain that

$$(\mu+1)(\lambda+1)\, a_2 = B_1 b_1 \tag{2.15}$$

$$(\mu+2)(2\lambda+1)\, a_3 + \left(\frac{(\mu-1)(\mu+2)}{2} - \lambda(\mu+3)\right) a_2^2 = B_1 b_2 + B_2 b_1^2 \tag{2.16}$$

$$-(\mu+1)(\lambda+1)\, a_2 = B_1 c_1 \tag{2.17}$$

$$-(\mu+2)(2\lambda+1)\, a_3 + \left(\frac{(\mu-1)(\mu+2)}{2} - \lambda(\mu+3) + 2(\mu+2)(2\lambda+1)\right) a_2^2 = B_1 c_2 + B_2 c_1^2. \tag{2.18}$$

From (2.15) and(2.17), we get

$$b_1 = -c_1. \tag{2.19}$$

Adding (2.16) and (2.18), we find that

$$[(\mu+1)(\mu+2\lambda+2)]\, a_2^2 = B_1(b_2+c_2) + B_2\left(b_1^2+c_1^2\right). \tag{2.20}$$

Thus

$$\left|a_2^2\right| \leq \frac{B_1}{(\mu+1)(\mu+2\lambda+2)}\left(\left|b_2+\frac{B_2}{B_1}b_1^2\right|+\left|c_2+\frac{B_2}{B_1}c_1^2\right|\right).$$

Case 1. If $B_2 \leq 0$ and $B_1+B_2 \geq 0$, using Lemma 2.2 with $\rho = \frac{B_2}{B_1} \leq 0$ and (2.19), we have

$$\left|a_2^2\right| \leq \frac{2B_1}{(\mu+1)(\mu+2\lambda+2)}\left(1-\left(\frac{B_1+B_2}{B_1}\right)\left|b_1^2\right|\right).$$

Using (2.15), we find that

$$|a_2| \leq \frac{B_1\sqrt{2B_1}}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)B_1^2+2(\mu+1)(\lambda+1)^2(B_1+B_2)\right)}}. \tag{2.21}$$

Case 2. If $B_2 > 0$ and $B_1 - B_2 \geq 0$, using Lemma 2.2 with $\rho = \frac{B_2}{B_1} > 0$ and (2.19), we have

$$\left|a_2^2\right| \leq \frac{2B_1}{(\mu+1)(\mu+2\lambda+2)}\left(1-\left(\frac{B_1-B_2}{B_1}\right)\left|b_1^2\right|\right).$$

Using (2.15), we find that

$$|a_2| \leq \frac{B_1\sqrt{2B_1}}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)B_1^2+2(\mu+1)(\lambda+1)^2(B_1-B_2)\right)}}. \tag{2.22}$$

Therefore, (2.21) and (2.22) are the required estimate of $|a_2|$ .To estimate the next part of this theorem, we subtract (2.18) from (2.16) to obtain

$$2(\mu+2)(2\lambda+1)\left(a_3-a_2^2\right) = B_1(b_2-c_2)+B_2\left(b_1^2-c_1^2\right). \tag{2.23}$$

Then

$$\left|a_3-a_2^2\right| \leq \frac{B_1}{2(\mu+2)(2\lambda+1)}\left(\left|b_2+\frac{B_2}{B_1}b_1^2\right|+\left|c_2+\frac{B_2}{B_1}c_1^2\right|\right).$$

Case 1. If $B_2 \leq 0$, using Lemma 2.2 with $\rho = \frac{B_2}{B_1} \leq 0$ we have

$$\left|a_3-a_2^2\right| \leq \frac{B_1}{2(\mu+2)(2\lambda+1)}\left(\left(1-\frac{B_1+B_2}{B_1}\left|b_1^2\right|\right)+\left(1-\frac{B_1+B_2}{B_1}\left|c_1^2\right|\right)\right).$$

Using the assumption $B_1+B_2 \geq 0$, we get

$$\left|a_3-a_2^2\right| \leq \frac{B_1}{(\mu+2)(2\lambda+1)}. \tag{2.24}$$

But if $B_1+B_2 < 0$ and by using (2.3), we get

$$\left|a_3-a_2^2\right| \leq \frac{-B_2}{(\mu+2)(2\lambda+1)}. \tag{2.25}$$

Case 2. If $B_2 > 0$, using Lemma 2.2 with $\rho = \frac{B_2}{B_1} > 0$ we have

$$\left|a_3-a_2^2\right| \leq \frac{B_1}{2(\mu+2)(2\lambda+1)}\left(\left(1-\frac{B_1-B_2}{B_1}\left|b_1^2\right|\right)+\left(1-\frac{B_1-B_2}{B_1}\left|c_1^2\right|\right)\right).$$

Using the assumption $B_1-B_2 \geq 0$, we get

$$\left|a_3-a_2^2\right| \leq \frac{B_1}{(\mu+2)(2\lambda+1)}. \tag{2.26}$$

But if $B_1+B_2 < 0$ and by using (2.3), we get

$$\left|a_3-a_2^2\right| \leq \frac{B_2}{(\mu+2)(2\lambda+1)}. \tag{2.27}$$

Therefore, (2.24),(2.25),(2.26) and (2.27) are the desired estimations of $\left|a_3-a_2^2\right|$ and this completes the proof. □

In Theorem 2.1 and Theorem 2.3, taking the special cases for the function $\phi(z)$ as in Remark 1 leads to the following corollaries.

Corollary 2.4. *If the function* $f \in \Upsilon_\sigma(\lambda, \mu, A, B)$ *and* $a_k = 0$ *for* $2 \le k \le n-1$, *then*

$$|a_n| \le \frac{A-B}{(\mu+n-1)(1+\lambda(n-1))}, \quad n \ge 3,\ \lambda, \mu \ge 0.$$

Corollary 2.5. *If the function* $f \in \Upsilon_\sigma(\lambda, \mu, A, B)$, *then*

$$|a_2| \le \begin{cases} \frac{(A-B)\sqrt{2}}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)(A-B)+2(\mu+1)(\lambda+1)^2(1-B)\right)}} & 0 \le B < 1 \\ \frac{(A-B)\sqrt{2}}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)(A-B)+2(\mu+1)(\lambda+1)^2(1+B)\right)}} & -1 \le B < 0 \end{cases}$$

and

$$\left|a_3 - a_2^2\right| \le \frac{A-B}{(\mu+2)(2\lambda+1)}.$$

Corollary 2.6. *If the function* $f \in \Upsilon_\sigma(\lambda, \mu, \alpha)$ *and* $a_k = 0$ *for* $2 \le k \le n-1$, *then*

$$|a_n| \le \frac{2\alpha}{(\mu+n-1)(1+\lambda(n-1))}, \quad n \ge 3,\ \lambda, \mu \ge 0.$$

Corollary 2.7. *If the function* $f \in \Upsilon_\sigma(\lambda, \mu, \alpha)$, *then*

$$|a_2| \le \frac{2\alpha}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)\alpha+(\mu+1)(\lambda+1)^2(1-\alpha)\right)}}$$

and

$$\left|a_3 - a_2^2\right| \le \frac{2\alpha}{(\mu+2)(2\lambda+1)}.$$

Corollary 2.8. *If the function* $f \in \Upsilon_\sigma^\beta(\lambda, \mu)$ *and* $a_k = 0$ *for* $2 \le k \le n-1$, *then*

$$|a_n| \le \frac{2(1-\beta)}{(\mu+n-1)(1+\lambda(n-1))}, \quad n \ge 3,\ \lambda, \mu \ge 0.$$

Corollary 2.9. *If the function* $f \in \Upsilon_\sigma^\beta(\lambda, \mu)$, *then*

$$|a_2| \le 2\sqrt{\frac{1-\beta}{(\mu+1)(\mu+2\lambda+2)}}$$

and

$$\left|a_3 - a_2^2\right| \le \frac{2(1-\beta)}{(\mu+2)(2\lambda+1)}.$$

Corollary 2.10. *If the function* $f \in \Upsilon_{L\sigma}(\lambda, \mu)$ *and* $a_k = 0$ *for* $2 \le k \le n-1$, *then*

$$|a_n| \le \frac{1}{2(\mu+n-1)(1+\lambda(n-1))}, \quad n \ge 3,\ \lambda, \mu \ge 0.$$

Corollary 2.11. *If the function* $f \in \Upsilon_{L\sigma}(\lambda, \mu)$, *then*

$$|a_2| \leq \frac{1}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)+3(\mu+1)(\lambda+1)^2\right)}}$$

and

$$\left|a_3 - a_2^2\right| \leq \frac{1}{2(\mu+2)(2\lambda+1)}.$$

Corollary 2.12. *If the function* $f \in \Upsilon_{\sigma}^{\Delta}(\lambda,\mu)$ *and* $a_k = 0$ *for* $2 \leq k \leq n-1$, *then*

$$|a_n| \leq \frac{1}{(\mu+n-1)(1+\lambda(n-1))}, \quad n \geq 3,\ \lambda,\mu \geq 0.$$

Corollary 2.13. *If the function* $f \in \Upsilon_{\sigma}^{\Delta}(\lambda,\mu)$, *then*

$$|a_2| \leq \frac{\sqrt{2}}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)+(\mu+1)(\lambda+1)^2\right)}}$$

and

$$\left|a_3 - a_2^2\right| \leq \frac{1}{(\mu+2)(2\lambda+1)}.$$

Corollary 2.14. *If the function* $f \in \Upsilon_{\sigma}(\lambda,\mu,s)$ *and* $a_k = 0$ *for* $2 \leq k \leq n-1$, *then*

$$|a_n| \leq \frac{s}{(\mu+n-1)(1+\lambda(n-1))}, \quad n \geq 3,\ \lambda,\mu \geq 0.$$

Corollary 2.15. *If the function* $f \in \Upsilon_{\sigma}(\lambda,\mu,s)$, *then*

$$|a_2| \leq \frac{s\sqrt{2}}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)s+(\mu+1)(\lambda+1)^2(1-s)\right)}}$$

and

$$\left|a_3 - a_2^2\right| \leq \frac{s}{(\mu+2)(2\lambda+1)}.$$

Corollary 2.16. *If the function* $f \in \Upsilon_{\sigma e}(\lambda,\mu)$ *and* $a_k = 0$ *for* $2 \leq k \leq n-1$, *then*

$$|a_n| \leq \frac{1}{(\mu+n-1)(1+\lambda(n-1))}, \quad n \geq 3,\ \lambda,\mu \geq 0.$$

Corollary 2.17. *If the function* $f \in \Upsilon_{\sigma e}(\lambda,\mu)$, *then*

$$|a_2| \leq \frac{\sqrt{2}}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)+(\mu+1)(\lambda+1)^2\right)}}$$

and

$$\left|a_3 - a_2^2\right| \leq \frac{1}{(\mu+2)(2\lambda+1)}$$

3. Distortion theorem

An important consequence of Bieberbach's inequality $|a_2| \leq 2$ is that it provides sharp lower and upper bounds of $|f|$ and $\left|f'\right|$ usually referred to as growth and distortion theorems, respectively. In this section, we obtain the distortion theorem of functions in the class $\Upsilon_\sigma(\lambda, \mu, \phi)$

Theorem 3.1. *If $f \in \Upsilon_\sigma(\lambda, \mu, \phi)$ and $z = re^{i\theta}$, then*

$$\frac{(1-r)^{M-1}}{(1+r)^{M+1}} \leq \left|f'(z)\right| \leq \frac{(1+r)^{M-1}}{(1-r)^{M+1}}, \tag{3.1}$$

where

$$M := \begin{cases} \frac{B_1\sqrt{2B_1}}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)B_1^2+2(\mu+1)(\lambda+1)^2(B_1+B_2)\right)}} & B_2 \leq 0, B_1+B_2 \geq 0 \\ \frac{B_1\sqrt{2B_1}}{\sqrt{(\mu+1)\left((\mu+2\lambda+2)B_1^2+2(\mu+1)(\lambda+1)^2(B_1-B_2)\right)}} & B_2 > 0, B_1-B_2 \geq 0. \end{cases}$$

Proof. Using the same method and technique given by Duren [9, Theorem 2.5, Page 32], we have

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{2Mr}{1-r^2}.$$

In particular,

$$\frac{2r^2-2Mr}{1-r^2} \leq \Re\left(\frac{zf''(z)}{f'(z)}\right) \leq \frac{2r^2+2Mr}{1-r^2}.$$

This leads to

$$\frac{2r-2M}{1-r^2} \leq \frac{\partial}{\partial r}\log\left|f'(re^{i\theta})\right| \leq \frac{2r+2M}{1-r^2}.$$

Integrating and exponentiating, we find that

$$\frac{(1-r)^{M-1}}{(1+r)^{M+1}} \leq \left|f'(z)\right| \leq \frac{(1+r)^{M-1}}{(1-r)^{M+1}}.$$

□

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References

[1] H. Airault and A. Bouali, *Differential calculus on the Faber polynomials*, Bull. Sci. Math. **130**, 179–222, 2006.

[2] H. Airault and J. Ren, *An algebra of differential operators and generating functions on the set of univalent functions*, Bull. Sci. Math. **126**, 343–367, 2002.

[3] S. Altınkaya, *Bounds for a new subclass of bi-univalent functions subordinate to the Fibonacci numbers*, Turkish J. Math. **44** (2), 553–560, 2020.

[4] S. Altınkaya and S. Yalcın, *Faber polynomial coefficient bounds for a subclass of bi-univalent functions*, C. R. Math. **353** (12), 1075–108, 2015.

[5] A. Bouali, *Faber polynomials, Cayley-Hamilton equation and Newton symmetric functions*, Bull. Sci. Math. **130**, 49–70, 2006.

[6] M. Caglar, H. Orhan and N. Yagmur, *Coefficient bounds for new subclasses of bi-univalent functions*, Filomat, **27** (7), 1165–1171, 2013.

[7] E. Deniz, *Certain subclasses of bi-univalent functions satisfying subordinate conditions*, J. Class. Anal. **2**, 49–60, 2013.

[8] E. Deniz, J.M. Jahangiri, S.G. Hamidi and S.K. Kina, *Faber polynomial coefficients for generalized bi-subordinate functions of complex order*, J. Math. Inequal. **12** (3), 645–653, 2018.

[9] P.L. Duren, *Univalent Functions*, Grundlehren Math. Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[10] G. Faber, *Über polynomische Entwickelungen*, Math. Ann. doi: 10.1007/BF01444293, 1903.

[11] P. Goel and S.S. Kumar, *Certain class of starlike functions associated with modified sigmoid function*, Bull. Malays. Math. Sci. Soc. **43**, 957–991, 2020.

[12] S.P. Goyal and R. Kumar, *Coefficient estimates and quasi-subordination properties associated with certain subclasses of analytic and bi-univalent functions*, Math. Slovaca, **65** (3), 533–544, 2015.

[13] T. Hayami and S. Owa, *Coefficient bounds for bi-univalent functions*, Pan Amer. Math. J. **22** (4), 15–26, 2012.

[14] S. Kanas, V.S. Masih and A. Ebadian, *Relations of a planar domain bounded by hyperbola with family of holomorphic functions*, J. Inequl. Appl. **246**, 1–14, 2019.

[15] A.Y. Lashin, *On certain subclasses of analytic and bi-univalent functions*, J. Egyptian Math. Soc. **24** (2), 220–225, 2016.

[16] A.Y. Lashin, *Coefficient estimates for two subclasses of analytic and bi-univalent functions*, Ukr. Math. J. **70** (9), 1484–1492, 2019.

[17] A.Y. Lashin and F.Z. EL-Emam, *Faber polynomial coefficients for certain subclasses of analytic and biunivalent functions*, Turkish J. Math. **44**, 1345–1361, 2020.

[18] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18**, 63–68, 1967.

[19] N. Magesh, T. Rosy and S. Varma, *Coefficient estimate problem for a new subclass of bi-univalent functions*, J. Complex Anal. **2013**, Art. ID 474231, 3pp., 2013.

[20] N. Magesh and J. Yamini, *Coefficient bounds for certain subclasses of bi-univalent functions*, Int. Math. Forum, **8**, 1337–1344, 2013.

[21] R. Mendiratta, S. Nagpal and V. Ravichandran, *On a subclass of strongly starlike functions associated with exponential function*, Bull. Malays. Math. Sci. Soc. **38** (1), 365–386, 2015.

[22] G. Murugusundaramoorthy, N. Magesh and V. Prameela, *Coefficient bounds for certain subclasses of bi-univalent function*, Abstr. Appl. Anal. Art. ID 573017, 3 pp., 2013.

[23] Z.-G. Peng and Q.-Q. Han, *On the coefficients of several classes of bi-univalent functions*, Acta Math. Sci. Ser. B Engl. Ed. **34**, 228–240, 2014.

[24] S. Porwal and M. Darus, *On a new subclass of bi-univalent functions*, J. Egypt. Math. Soc. **21** (3), 190–193, 2013.

[25] R.K. Raina and J. Sokół, *Some properties related to a certain class of starlike functions*, C. R. Acad. Sci. Paris Ser. I **353** (11), 973–978, 2015.

[26] J. Sokół and J. Stankiewicz, *Radius of convexity of some subclasses of strongly starlike functions*, Zeszyty Nauk. Politech. Rzeszowskiej Mat. **19**, 101–105, 1996.

[27] H.M. Srivastava, S. Bulut, M. Caglar and N. Yagmur, *Coefficient estimates for a general subclass of analytic and bi-univalent functions*, Filomat, **27** (5), 831–842, 2013.

[28] A. Zireh, E.A. Adegani and M. Bidkham, *Faber polynomial coefficient estimates for subclass of bi-univalent functions defined by quasi-subordinate*, Math. Slovaca, **68** (2), 369–378, 2018.

[29] Z-G. Wang and Y-P. Jiang, *Notes on certain subclass of p-valently Bazilevic functions*, J. Inequl. Pure Appl. Math. **9** (3), Art. 70, 7pp., 2008.