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AN ALGEBRAIC CHARACTERIZATION OF CONFORMAL EQUIVALENCE OF RECTANGULAR DOMAINS

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SUMMARY

This paper presents a solution to a problem in the subject of rings of analytic functions. It was shown that [Bers (1948), Kakutani (1955)] two domains D_1 and D_2 in the complex plane were conformally equivalent (to within a certain equivalence relation) iff the rings $B(D_1)$ and $B(D_2)$ of all bounded analytic functions defined on them were algebraically isomorphic. In the case of rectangles, two are conformally equivalent iff the ration of the sides of one equals the same ratio for the other [Uluçay (1946)]. It follows that this ratio must be contained somewhere in the algebraic structure of the ring. The problem is to find it.

PROBLEM*. Let R* be a ring which is known to be isomorphic with the ring of bounded analytic functions of a rectangle

$$D = \{z : |\mathrm{Re} z| < r_2, \ |\mathrm{Im} z| < r_1, 0 < r_1 \leq r_2\},$$

where r_1 and r_2 are not known. From the ring R*, deduce the number r_2/r_1.

Actually the original problem is the one dealth with the algebra of all analytic functions. As we shall see later in the paper, this original problem has a solution also, and this solution is somewhat simpler.

To solve the problem, we will let $\varnothing$ be the isomorphism mapping B(D) onto R*, and will denote elements of B(D) by f, g, h, ..., and elements of R* by a, b, c, d, e, ... (e is the multiplicative identity). Let $1 \in B(D)$ be the function identically equal to 1 on D. Then clearly $e = \varnothing(1)$, and $ne = \varnothing(n1)$, so that $\pm (m/n)\, e = \varnothing(\pm (m/n)1)$.–e has two square roots in R*, one being the image of i 1, the other the image of –i 1. We choose one root of –e and make it correspond to i 1; denote it as ie. Then $\varnothing((r_1 + r_2 i)\, 1) = r_1 e + r_2\, ie$ for all rational r_1, r_2. Note

* In the case of annuli Problem was solved by Beck (1964).

now that $\alpha \in \bar{R}(f)$ (the closed range of f) iff f-α1 has no inverse in B(D), i.e., iff $\varnothing(f)-\alpha e$ has no inverse in R*, i.e., iff α is in the spectrum $\sigma(\varnothing(f))$ of $\varnothing(f)$. Thus, if $a \in R^*$, we know for each rational r_1, r_2 whether $r_1 + ir_2 \in \bar{R}(\varnothing^{-1}(a))$ by knowing whether $(r_1 + ir_2)$ e-a has an inverse in R*. Therefore, if $\varnothing^{-1}$ (a) is not a constant, we know $\bar{R}(\varnothing^{-1}(a))$. Specifically, if

$$\lambda(a) = \sup\{|\mathrm{Re}\ \alpha| : \alpha \in \sigma(a)\}$$

and

$$\theta(a) = \sup\{|\mathrm{Im}\alpha| : \alpha \in \sigma(a)\},$$

then for $\varnothing^{-1}(a) = f \in B(D)$

(i) $$|\mathrm{Re}f(z) = \max_{z_i \in \mathrm{bd}D} |\mathrm{Re}f(z_i)|$$

and

(ii) $$|\mathrm{Im}f(z)| = \max_{z_i \in \mathrm{bd}D} |\mathrm{Im}f(z_i)|$$

where bdD represents the boundary df D. Thus from (i) and (ii) we know that $\bar{R}(f)$ is a rectangle. That is, $\lambda(a)$ and $\theta(a)$ are the maximums of the real and imaginary part of $\varnothing^{-1}(a)$ respectively. At this point if we could find the function $f(z) = cz$, $0 \neq c \in \mathbf{R}$, it would be completed, for

$$\frac{\lambda(a)}{\theta(a)} = \frac{cr_2}{cr_1} = \frac{r_2}{r_1}.$$

RESULT. If $\varnothing^{-1}(a)(z) = cz$, $c \neq 0$, then $\dfrac{\lambda(a)}{\theta(a)} = \dfrac{r_2}{r_1}$.

Now we shall give an algebraic characterization of the function $f(z) = cz$. But before doing this we need the following lemma.

LEMMA. Let $D = \{z : |\mathrm{Re}z| < r_2, |\mathrm{Im}z| < r_1, 0 < r_1 \leq r_2\}$ be a rectangle and $f \in B(D)$. Suppose that

(1°) $f(0) = 0$,

(2°) f is univalent on $\bar{D}$,

(3°) $$|\mathrm{Re}f(z)| = \max_{z_i \in \mathrm{bd}D} |\mathrm{Re}f(z_i)| \text{ and}$$

$$|\mathrm{Im}f(z)| = \max_{z_i \in \mathrm{bd}D} |\mathrm{Im}f(z_i)|.$$

Then $f(z) = cz$ for some real number $c \neq 0$.

Proof. Let $|\mathrm{Re}f(z)| = K$ and $|\mathrm{Im}f(z)| = k$. If every point in the boundary of R(f) has modulus K or k, then since f is univalent, R(f) is a rectangle centered at the origin which is conformal equivalent to D. Thus the ratio of the sides of the rectangle D is equal to the sides of the $\bar{R}(f)$ respectively and the mapping realizing this conformality is $f(z) = cz$, $0 \neq c \in \mathbf{R}$ [Uluçay (1946)].

We need only to show that the properties (2°) and (3°) on f follow from purely algebraic conditions on $\varnothing(f)$, and the problem is solved. We note that if f is not constant and univalent on D, then there is some complex number α such that for $z_1, z_2 \in D$ and $z_1 \neq z_2$, $f(z_1) = f(z_2) = \alpha$. Thus if for every complex constant $\alpha \in \bar{R}(f)$, $f-\alpha$ has only one zerro, then f is univalent on the rectangle D. It follows the following theorem.

THEOREM. Let R^* be a ring which is algebraically isomorphic with B(D), the ring of bounded analytic functions on the rectangle

$$D = \{z : |\mathrm{Re}z| < r_2, \ |\mathrm{Im}z| < r_1, \ 0 < r_1 \leq r_2\} .$$

Then there is an element $a \in R^*$ satisfying (1°), (2°) bellow, and for any such element $\dfrac{\lambda(a)}{\theta(a)} = \dfrac{r_2}{r_1}$.

(1°) For each complex constant $\alpha \in \bar{R}(f)$, $((a-\alpha))$ is a maximal principal ideal in R^*.

(2°) For every $a \in R^*$

$$\lambda(a) = \sup \{|\mathrm{Re}\ \alpha| : \alpha \in \sigma(a)\}$$

and

$$\theta(a) = \sup \{|\mathrm{Im}\ \alpha| : \alpha \in \sigma(a)\} .$$

Proof. For any $f \in B(D)$ let $\varnothing(f) = a$, $a \in R^*$ and suppose that $\varnothing(f)$ satisfies (1°) and (2°). We have to show that $f(z) = cz$, $0 \neq c \in \mathbf{R}$. From (1°), since for each $\alpha \in \bar{R}(f)$ $((a-\alpha))$ is a maximal principal ideal of R^*, then $((f-\alpha))$ is a maximal principal ideal in B(D), because $\varnothing$ is an isomorphism. Thus on D,$f-\alpha$ has only one zero. So f is univalent on D. In (2°), since for $a \in R^*$

$$\lambda(a) = \sup \{|\mathrm{Re}\alpha| : \alpha \in \sigma\ (a)\}$$

and

$$\theta(a) = \sup \{|\mathrm{Im}\alpha| : \alpha \in \sigma(a)\},$$

then for $\varnothing^{-1}(a) = f \in B(D)$

$$|\mathrm{Ref}(z)| = \max_{z_i \in \mathrm{bd}D} |\mathrm{Ref}(z_i)|$$

and

$$|\mathrm{Imf}(z)| = \max_{z_i \in \mathrm{bd}D} |\mathrm{Imf}(z_i)| .$$

As shown before, this implies that $\bar{R}(f)$ is a rectangle. If in addition, f is univalent, then we see that f is conformal. Thus $f(z) = cz$, $0 \neq c \in \mathbf{R}$ and

$$\frac{\lambda(a)}{\theta(a)} = \frac{cr_2}{cr_1} = \frac{r_2}{r_1} .$$

This completes the proof of the theorem.

Let R^* be a ring which is known to be isomorphic with the ring A(G) of all analytic functions on an unknown domain G. The main problem is finding a conformal or anticonformal image of G when R^* is given. We note that the spectrum of an element in A(G) is the actual range of the corresponding function, rather than its closure, and our methods will only yield the closure. If we actually knew all the irrational constant functions, then we could obtain the actual spectrum. Now let us obtain these constant functions. Denote the closure of the spectrum of an element a by $\bar{\sigma}(a)$. We see that if there is a nonconstant $f_0 \in A(G)$ which is bounded, then $z \in R(z1 + f_0 - w1)$, where $w \in R(f_0)$. Also

$$\bigcap_{n<0} \bar{R}\left((z1 + \frac{1}{n} f_0 - \frac{1}{n} w1\right)$$

is exactly the point z. Thus, if $a \in R^*$ is an element with no complex rational in its spectrum, then a must be the image of z1 for some irrational z. The value of z is, in fact the only point in $\cap\, \bar{\sigma}(a + b - \alpha e)$, where the intersection is taken over all $b \in R^*$ with at least two complex rationals in $\sigma(b)$, and all complex rationals $\alpha \in \sigma(b)$. Thus, for any $a \in R^*$, we can get $\sigma(a)$, which is the range of the corresponding function. Therefore, if a is univalent, $\sigma(a)$ is conformal or anticonformal with G.

In case domain G has no nonconstant bounded analytic functions, this method collapses completely, since $\bar{\sigma}\,(a + b - \alpha e)$ is always the whole plane.

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