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SOME SPACES OF MATRIX OPERATORS

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ABSTRACT

The structural theory of infinite matrices in the classes $(\iota_\infty(p), \iota_\infty)$, $(\iota_\infty(p), c)$, $(\iota(p)\ \iota_\infty)$, $(c_0(p), \iota_\infty(q))$ or $(c, \iota_\infty(p))$ have been studied. Some of our results include as a special cases, the earlier results obtained by Rao.

I. INTRODUCTION

For a sequence $p = (p_k)$ of positive real numbers, the following classes of sequences have been introduced and studied in [1].

$$\iota(p) = \{x: \sum_{k=1}^{\infty} |x_k|^{p_k} < \infty\}.$$

$$\iota_\infty(p) = \{x: \sup_k |x_k|^{p_k} < \infty\}.$$

$$c(p) = \{x: |x_k - \iota|^{p_k} \to 0 \text{ for some } \iota\}.$$

$$c_0(p) = \{x: |x_k|^{p_k} \to 0\}.$$

When $p_k = p > 0$, for all k, then $\iota(p) = \iota_p$, $\iota_\infty(p) = \iota_\infty$, $c(p) = c$, $c_0(p) = c_0$, where ι_p, ι_∞, c and c_0 are respectively the spaces of p-summable, bounded, convergent and null sequences. In particular, if $(p_k) = \left(\frac{1}{k}\right)$ in $\iota_\infty(p)$ and $c_0(p)$ then these spaces are called spaces of analytic and entire sequences, respectively. The works on these spaces has been carried out by Rao in [6], [7], and by other authors. The spaces $\iota(p)$, $\iota_\infty(p)$, $c(p)$ and $c_0(p)$ are linear spaces under coordinatewise addition and scalar multiplication if and only if $p \in \iota_\infty$ see [4].

Let λ and μ be two nonempty subsets of the space ω of all complex sequences. Then we denote the class of all infinite matrices $A:\lambda \to \mu$ by (λ, μ) such that

$$(A_n(x))_{n=1}^{\infty} = \left(\sum_{k=1}^{\infty} a_{nk}x_k\right)_{n=1}^{\infty} \in \mu,$$

whenever $x \in \lambda$, the convergence of $\sum_{k=1}^{\infty} a_{nk}x_k$ $(n = 1, 2, \ldots)$ being assumed.

Recently, the structure theory of infinite matrices transforming spaces of the analytic, entire, bounded and convergent sequences has been studied by Rao [6]. The present paper is devoted to the structural theory of the infinite matrices in the classes $(\iota_\infty(p), \iota_\infty)$, $(\iota_\infty(p), c)$, $(\iota(p), \iota_\infty)$, $(c_0(p), \iota_\infty(q))$ and $(c, \iota_\infty(p))$. Our results include as a special case, the earlier results obtained by Rao [6]. To find the necessary and sufficient conditions for infinite matrices to be in above mentioned classes one may refer to Chaudhary and Nanda [1].

2. An infinite matrix $A \in (\iota_\infty(p), \iota_\infty)$ if and only if for all integer $N > 1$ we have

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| N^{1/p_k} < \infty.$$

Let us start with the following theorems:

Theorem 1. Let $p = (p_k) \in \iota_\infty$ and $N > 1$ be any integer then the class of matrix operators $(\iota_\infty(p), \iota_\infty)$ is a complete metric space with the metric

$$D_N(A, B) = \sup\{\sum_{k=1}^{\infty} |a_{nk} - b_{nk}| N^{1/p_k};\ n = 1, 2, \ldots\},$$

where $A = (a_{nk})$, $B = (b_{nk})$ are in $(\iota_\infty(p), \iota_\infty)$.

Proof. It can be proved by the standard arguments that D_N is a metric for every $N > 1$. Finally let $\alpha_k = N^{1/p_k}$ and $A^{(i)}$; $i = 1, 2, \ldots$ with $A^{(i)} = (a^i_{nk})$ be a Cauchy sequence in $(\iota_\infty(p), \iota_\infty)$. Then for a given $\varepsilon > 0$, there is a positive integer i_0 such that

(1) $$D(A^{(i)}, A^{(j)}) < \varepsilon, \ (i > i_0, j \geq i_0).$$

Since for each fixed k and n,

$$| a^{(i)}_{nk} - a^{(j)}_{nk} | < \sum_{k=1}^{\infty} \alpha_k | a^{(i)}_{nk} - a^{(j)}_{nk} | < \varepsilon \; (i \geq i_0, j \geq i_0),$$

therefore $(A^{(i)})$ is a Cauchy sequence of complex numbers and hence converge.

Again $\frac{\varepsilon}{\alpha_k 2^k} > 0$, gives the existence of a positive integer i_0, and $A=(a_{nk})$ such that for each fixed k

$$\alpha_k | a^{(i)}_{nk} - a_{nk} | < \frac{\varepsilon}{2^k}, \; (i \geq i_0).$$

Thus

$$\sum_{k=1}^{\infty} \alpha_k | a^{(i)}_{nk} - a_{nk} | < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} < \varepsilon, \; (i \geq i_0).$$

It remains to show that $A = (a_{nk}) \in (\iota_\infty(p), \iota_\infty)$.

Letting $j \to \infty$ in (1), we have

$$\sum_{k=1}^{\infty} \alpha_k | a_{nk} - a^{(i)}_{nk} | < \varepsilon,$$

this implies that

$$\varepsilon > \sum_{k=1}^{\infty} \alpha_k | a_{nk} - a^{(i)}_{nk} | \geq \sum_{k=1}^{\infty} \alpha_k | a_{nk} | - \sum_{k=1}^{\infty} \alpha_k | a^{(i)}_{nk} |.$$

Now, $A^{(i)} \in (\iota_\infty(p), \iota_\infty)$ gives us the required result.

Corollary 2. Let $p = (p_k) \in \iota_\infty$ and $Y(p) = (\iota_\infty(p), c)$, Then the class $Y(p)$ is a closed subset of $(\iota_\infty(p), \iota_\infty)$ and hence a complete metric space with the metric D_N for each $N > 1$.;

Proof. The set c is a subspace of the BK-space ι_∞, therefore $Y(p)$ is a subset of $(\iota_\infty(p), \iota_\infty)$. Let $Y(p)$ denotes the closure of $Y(p)$ in the metric topology D_N. Let $A \in Y(p)$, then there exists a sequence $(A^{(i)})$ in $Y(p)$ such that

(1) $$D_N(A^i, A) \to 0 \text{ as } i \to \infty.$$

Thus for each $\varepsilon > 0$ there exists $i_0 > 0$ such that

$$\sum_{k=1}^{\infty} \alpha_k | a^{(i)}_{nk} - a_{nk} | < \varepsilon. \; (i \geq i_0).$$

This implies that

$$\sum_{k=1}^{\infty} \alpha_k \mid a_{nk} \mid < \sum_{k=1}^{\infty} \alpha_k \mid a_{nk}^{(i)} \mid + \varepsilon, \ (i \geq i_0).$$

Hence, $A = (a_{nk}) \in (\iota_\infty(p), \iota_\infty)$. Finally to prove $(a_{nk}) \in Y(p)$: $(A^{(i_0)}) \in Y(p)$ gives column limits of $A^{(i_0)}$ exists, hence for each $\varepsilon > 0$ there exists a positive integer n_0 such that for each fixed k

$$|a_{nk}^{i_0} - a_{mk}^{i_0}| < \frac{\varepsilon}{3\alpha_k 2^k} \ (m \geq n_0, n \geq n_0),$$

then

$$\sum_{k=1}^{\infty} \alpha_k \mid a_{nk}^{i_0} - a_{mk}^{i_0} \mid < \frac{\varepsilon}{3} .$$

Now from (1) there is a positive integer i_0 such that

$$\sum_{k=1}^{\infty} \alpha_k \mid a_{nk} - a_{nk}^{i_0} \mid < \frac{\varepsilon}{3} .$$

For each fixed n and k we have the following,

$$\sum_{k=1}^{\infty} \alpha_k \mid a_{nk} - a_{mk} \mid \leq \sum_{k=1}^{\infty} \alpha_k \mid a_{nk} - a_{nk}^{i_0} \mid$$

$$+ \sum_{k=1}^{\infty} \alpha_k \mid a_{nk}^{i_0} - a_{mk}^{i_0} \mid + \sum_{k=1}^{\infty} \alpha_k |a_{mk}^{i_0} - a_{mk}|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence

$$|a_{nk} - a_{mk}| < \frac{\varepsilon}{\alpha_k}, \text{ for all k.}$$

This shows that the column limit of the matrix A exists. Thus the matrix A belongs to Y(p). Arbitrariness of A in $\overline{Y(p)}$ shows that Y(p) is closed in the complete metric space $(\iota_\infty(p), \iota_\infty)$, which completes the proof.

Theorem 3. The space $(\iota_\infty(p), \iota_\infty)$ is separable.

Proof. Let M denotes the set of all matrices $B = (b_{nk})$ with rational (complex) entries for which integers n_1, q_1 exists such that $b_{nk} = 0$ whenever $n \geq n_1$, or $k > q_1$ or both, Then M is a countable subset of $(\iota_\infty(p), \iota_\infty)$. It is sufficient to prove that M is dense in $(\iota_\infty(p), \iota_\infty)$. Let $A = (a_{nk})$ be any element of $(\iota_\infty(p), \iota_\infty)$, then for each $\varepsilon > 0$ there exists $n_1 > 0$ such that

$$\sum_{j=n_1+1}^{\infty} |a_{nj}| N^{1/p_j} < \frac{\varepsilon}{2} .$$

Since rationals (complex) are dense in C, therefore for each entry a_{nj} in A there is a rational entry b_{nj} close to it. So we can find a matrix $B = (b_{nk}) \in M$ satisfying

$$\sum_{j=1}^{n_1} | a_{nj} - b_{nj} | N^{1/p_j} < \frac{\varepsilon}{2} .$$

It follows that

$$D_N(A,B) = \sum_{j=1}^{n_1} |a_{nj} - b_{nj}| N^{1/p_j} + \sum_{j=n1+1}^{\infty} |a_{n1} - b_{nj}| N^{1/p_j}$$

$$= \sum_{j=1}^{n_1} |a_{nj} - b_{nj}| N^{1/p_j} + \sum_{j=n1+1}^{\infty} |a_{n1}| N^{1/p_j}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so $(\ell_\infty(p), \ell_\infty)$ is separable.

3. For the remainder of this paper $q = (q_k)$ will denote a sequence of strictly positive real numbers such that

$$\frac{1}{p_k} + \frac{1}{q_k} = 1 \text{ for all } k.$$

Let Q denotes the set of all $p = (p_k)$ for which there exists $N = N(p) > 1$ such that

$$\sum_{k=1}^{\infty} N^{-1/p_k} < \infty.$$

Also it is easy to prove that $p \in Q$ implies $p_k \to 0$ [2].

A more general proof of the following lemma may be found in [3].

Lemma 4. Let $p \in Q$, then $A \in (\ell(p), \ell_\infty)$ if and only if

$$D = \sup_{n,k} |a_{nk}|^{\frac{q_k}{q_k+1}} < \infty.$$

Lemma 5 [2]. Let $p \in Q$, then $A \in (c_0(p), \ell_\infty(p'))$ if and only if

$$\sup_{n,k} |a_{nk}|^{\left(\frac{1}{p_k} + \frac{1}{p'_n}\right)^{-1}} < \infty.$$

Now we prove the following theorems.

Theorem 6. Let $p \in Q$, then the class of matrix operators $(\iota(p), \iota_\infty)$ is complete metric space with the metric.

$$d(A, B) = \sup \{ |a_{nk} - b_{nk}|^{\frac{q_k}{q_k+1}}, n, k = 1, 2, ...\}.$$

Proof. It is obvious that $((\iota(p), \iota_\infty), d)$ is a metric space. Now let $(A^{(i)})$ be any Cauchy sequence in it, then for each $\varepsilon > 0$ there exisst a positive integer i_0 such that

$$d(A^{(i)}, A^{(j)}) < \varepsilon \quad i, j \geq i_0.$$

That is,

$$|a^i_{nk} - a^j_{nk}| < \varepsilon^{\frac{q_k+1}{q_k}} \quad i, j \geq i_0.$$

Hence for each fixed n,k we have

$$a^{(i)}_{nk} \to a_{nk} \ (i \to \infty).$$

Since $\varepsilon^{\frac{q_k+1}{q_k}} > 0$, therefore there exists a positive integer i_0, such that

$$|a^{(i)}_{nk} - a_{nk}| < \varepsilon^{\frac{q_k+1}{q_k}} \quad i \geq i_0.$$

Thus

$$d(A^i, A) < \varepsilon \quad (i, j \geq i_0).$$

Also $\frac{q_k}{q_k+1} < 1$ for all k, and

$$|a_{nk} - a^{(i)}_{nk}|^{\frac{q_k}{q_k+1}} < \varepsilon$$

gives

$$\varepsilon > |a_{nk} - a_{nk}^{(i)}|^{\frac{q_k}{q_k+1}} > |a_{nk}|^{\frac{q_k}{q_k+1}} - |a_{nk}^{(i)}|^{\frac{q_k}{q_k+1}}$$

It follows that

$$|a_{nk}|^{\frac{q_k}{q_k+1}} < |a_{nk}^{(i)}|^{\frac{q_k}{q_k+1}} + \varepsilon < \infty.$$

Hence, $A \in (\iota(p), \iota_\infty)$.

Theorem 7. Let $p \in Q$, then the class $(c_0(p), \iota_\infty(p'))$ is complete metric space with the metric

$$d'(A, B) = \sup \left\{ |a_{nk} - b_{nk}|^{\left(\frac{1}{p_k} + \frac{1}{p'_n}\right)^{-1}} ;\ n, k = 1, 2, \dots \right\}.$$

where $A = (a_{nk})$, $B = (b_{nk}) \in (c_0(p), \iota_\infty(p'))$.

Proof. It can be proved on the lines of Theorem 6. Now if we put $p_k = \frac{1}{k} \in Q$ for $N = 2$ and $p'_k = e$. Then the metric coincide with the metric given by Rao [6].

The following lemma may easily be obtained.

Lemma 8. An infinite matrix $A \in (c, \iota_\infty(p))$ if and only if A satisfies

$$\sup_n \left(\sum_{k=1}^{\infty} |a_{nk}| \right)^{p_n} < \infty.$$

Theorem 9. Let $\inf p_k > 0$, then $(c, \iota_\infty(p))$ is a complete linear metric space paranormed by g_p where

$$g_p(A) = \sup_n \left(\sum_{k=1}^{\infty} |a_{nk}| \right)^{p_n / M}$$

where $M = \max (1, \sup p_k)$, and $A = (a_{nk}) \in (c. \iota_\infty(p)$.

Proof. It can be proved by the standard arguments that g_p is a paranorm and also it is complete. Since, $g_p(A) = 0$ implies $A = 0$, therefore $(c, \iota_\infty(p))$ is a complete linear metric space.

Remark. The condition $\inf p_k > 0$ in Theorem 9 can not be dropped. It follows from the following example:

Example. Let $p_k = \frac{1}{k}$ for all k,

$$A = (A_{nk}) = (\delta_{nk})$$

where δ is Kronecker. Then $A \in (c, \ell_\infty(p))$. Now consider $0 < |\lambda| < 1$ then $|\lambda|^{1/k} < 1$ for all k and $|\lambda|^{1/k} \to 1$ as $k \to \infty$ so that

$$g_p(\lambda A) = \sup_n \left(\sum_k |\lambda \delta_{nk}| \right)^{1/n}$$

$$= \sup_n (|\lambda|)^{1/n} = 1.$$

Hence $\lambda A \not\to 0$ as $\lambda \to 0$ and thus g_p is not a paranorm.

Theorem 10. Let $E \subset (c, \ell_\infty(p))$ be compact then given $\varepsilon > 0$ there is some $i_0 = i_0(\varepsilon)$ such that for all n

$$\left(\sum_{k=i+1}^{\infty} |a_{nk}| \right)^{p_n / M} < \varepsilon$$

for all $A \in E$ and $i \geq i_0$.

Proof. Proof is easy one may see [5].

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