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POWERS OF NONSYMMETRIC MATRICES

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ABSTRACT

The finite and the limiting behavior of the power of a real nonsymmetric matrix with distinct eigenvalues is analyzed through its spectral decomposition. Analytical results for all special cases of practical interest are obtained, and numerical examples are provided. The results are valid also for nonsymmetric matrices with repeated eigenvalues provided that a set of linearly independent eigenvectors exists.

INTRODUCTION

Powers of matrices occur in a variety of problems, especially those of a multivariate and recursive nature. In statistics, for example, transition matrices in Markov chain analysis, and covariance matrices in multivariate time series analysis provide examples involving powers of matrices. The limiting behavior of the power of a matrix is often of considerable interest as it relates to the long term behavior of the process defined by the matrix. We will consider here positive integer powers of real matrices and their limits as the power tends to infinity.

The power of a nonsymmetric matrix with distinct eigenvalues can be expressed in terms of the eigenvalues and eigenvectors of the matrix through its spectral decomposition, which proves to be very convenient for the analysis. Spectral decomposition can be used also for nonsymmetric matrices with repeated eigenvalues provided that a set of linearly independent eigenvectors exists. If this condition is not satisfied, which is usually the case, then one can use the Jordan canoical form [3], but that requires an analytical treatment quite different and more complicated than the spectral decomposition approach. We intend to treat that case in a future study. The analysis here will be restricted to nonsymmetric matrices with distinct eigenvalues, and repeated eigenvalues will be allowed only if a set of linearly independent eigenvectors exists. Powers of symmetric matrices, with or without distinct eigenvalues, can be analyzed as a special case with considerable simplifications due to the nonexistence of complex eigenvalues and the existence of real orthonormal eigenvectors.

CONVERGENCE OF A SEQUENCE OF MATRICES

Let $\{A^{(k)}\}$ denote a general sequence of $m \times n$ matrices, $A^{(1)}$, $A^{(2)}$,..., where $a_{ij}{}^{(k)}$ is the (ij)-th element of $A^{(k)}$. The sequence is said to converge to the $m \times n$ matrix $A = (a_{ij})$, or to have the limit A, if for each i and j, the sequence of scalars $\{a_{ij}{}^{(k)}\}$ converges to a_{ij}. This is denoted as $\lim_{k \to \infty} A^{(k)} = A$ or $\{A^{(k)}\} \to A$. Otherwise, the sequence is said to diverge. In some cases, divergence may arise from one or more of the component scalar sequences diverging to $+\infty$ or $-\infty$. This case of divergence to infinity is denoted as $\lim_{k \to \infty} A^{(k)} = \infty$ or $\{A^{(k)}\} \to \infty$. In other cases, divergence may arise from the matrix sequence oscillating between various limiting matrices. Due to its practical significance, this case of divergence by oscillation will receive considerable attention in our analysis. Another from of divergence is the continuous change of the matrix sequence without oscillation and without diverging to infinity. This mode of divergence will be referred to here as "divergence by drifting".

Since we will deal with complex matrices, some definitions for a sequence of complex scalars will be given here. Let $\{z_k\}$ denote a sequence of complex numbers z_1, z_2 ,..., where $z_k = a_k + i\, b_k$. The sequence is said to converge to $z = a + i\, b$ iff each sequence of real scalars $\{a_k\}$ and $\{b_k\}$ converges to the real numbers a and b, respectively. This is denoted as $\lim_{k \to \infty} z_k = z$ or $\{z_k\} \to z$. Otherwise, the sequence is said to diverge. The sequence is said to diverge to infinity, denoted as $\lim_{k \to \infty} z_k = \infty$ or $\{z_k\} \to \infty$, iff the sequence $\{|z_k|\}$ diverges to infinity, where $|z_k| = (a^2{}_k + b^2{}_k)^{1/2}$ is the absolute value of z_k.

An important theorem for the convergence of matrix sequences, which is relevant to our analysis, follows:

THEOREM 1 If the sequence $\{A^{(k)}\}$ converges to A, then the sequence $\{PA^{(k)}Q\}$ converges to PAQ [1].

In this study we consider the special case of the sequence of $n \times n$ matrices $\{A^k\}$, where A^k, $k = 1, 2, \ldots$, is the k-th power of A. Two important theorems for the convergence of such matrix sequences follow:

THEOREM 2 $\{A^k\} \to 0$ iff the eigenvalues of A are less than one in absolute value [1].

THEOREM 3 If $B = PAP^{-1}$, then $\{B^k\}$ converges iff $\{A^k\}$ converges.

Theorem 3 easily follows from Theorem 1 by first noting that $B^k = PA^kP^{-1}$ and $A^k = P^{-1} B^kP$.

POWERS OF NONSYMMETRIC MATRICES

SPECTRAL DECOMPOSITION

Let A be an $n \times n$ real nonsymmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and a set of corresponding $n \times 1$ eigenvectors $p_1, \ldots, p_n$. Some of the eigenvalues may be complex, and complex eigenvalues occur in conjugate pairs. For a real eigenvalue, a real eigenvector always exists. Eigenvectors corresponding to conjugate pairs of eigenvalues can also be expressed as conjugate pairs, elementwise. Assume that the eigenvalues are all distinct. It can be shown in that case that the eigenvectors are linearly independent.

Let

$$\Lambda = \text{Diag.} (\lambda_1, \ldots, \lambda_n) \tag{1}$$

$$P = (p_1, \ldots, p_n) \tag{2}$$

It follows from the linear indepence of $p_1, \ldots, p_n$ that P is nonsingular.

Let

$$P^{-1} = \begin{pmatrix} q'_1 \\ \vdots \\ q'_n \end{pmatrix}$$

where q'_i is the i-th row of P^{-1} and "′" denotes the transpose. It can be shown that the matrix A can be expressed as

$$A = P \Lambda P^{-1}$$

$$= \sum_{i=1}^{n} \lambda_i p_i q'_i \tag{3}$$

which is known as the spectral decomposition of A [3]. This simply follows from $AP = P\Lambda$, which is the matrix representation of $Ap_i = \lambda_i p_i$, $i = 1, ..., n$. The k-th power of A can be obtained by multiplying A (3) by itself k times, giving

$$A^k = P\Lambda^k P^{-1}, \qquad k = 1, 2, ...$$
$$= \sum_{i=1}^{n} \lambda_i^k p_i q'_i \tag{4}$$

where

$$\Lambda^k = \text{Diag.}\ (\lambda_1^k, ..., \lambda_n^k), \qquad k = 1, 2, ... \tag{5}$$

is the k-th power of Λ. It should be noted that the matrices Λ, Λ^k, P and P^{-1} may be complex but A and A^k are real. For a nonsymmetric matrix with repeated eigenvalues, a set of linearly independent eigenvectors, and hence P^{-1}, may or may not exist. If it exists, then the spectral decomposition of A as given by (3) is applicable and all the results in the paper are valid, unless otherwise stated.

When P is a complex matrix, the matrix P^{-1} may be difficult to obtain through the usual inverse operations. There is an easier way of obtaining P^{-1}. The matrix A' has the same eigenvalues as A, but not necessarily the same eigenvectors. Let $s_1, ..., s_n$ be a set of $n \times 1$ eigenvectors of A' corresponding to the eigenvalues $\lambda_1, ..., \lambda_n$. These are also called the left eigenvectors of A, as $A' s_i = \lambda_i s_i$, by transpose, is equivalent to $s'_i A = \lambda_i s'_i$, which is different from $Ap_i = \lambda_i p_i$ satisfied by the (right) eigenvectors $p_1, ..., p_n$ of A. If the eigenvalues of A are all distinct, it can be shown that $s_1, ..., s_n$ are linearly independent, $p'_i s_j = 0$ for $i \neq j$ and $p'_i s_i = d_i \neq 0$, where d_i may be a complex number. These conditions may or may not be satistfied by a nonsymmetric matrix with repeated eigenvalues. Let $S = (s_1, ..., s_n)$ and $D = \text{Diag.}\ (d_1, ..., d_n)$. It follows from the above results that S and D are nonsingular and that $P' S = S' P = D$, from which we obtain $P^{-1} = D^{-1} S'$. This provides an easy way of obtaining P^{-1} from the eigenvectors of the A and A' matrices.

BEHAVIOR OF COMPLEX EIGENVALUES

It follow from (4) that the dependence of A^k on k is only through λ_i^k, the powers of the eigenvalues of A. For a complex eigenvalue $\lambda = a + bi$, the conjugate of λ is $\bar{\lambda} = a - bi$ and the absolute value of λ

and $\bar{\lambda}$ is $|\lambda| = |\bar{\lambda}| = (a^2 + b^2)^{1/2}$. By the polar representation of complex numbers,

$$\lambda = |\lambda| (\cos\theta + i \sin\theta)$$
$$\bar{\lambda} = |\lambda| (\cos\theta - i \sin\theta)$$

where $\cos\theta = a / |\lambda|$ and $\sin\theta = b / |\lambda|$. The powers of λ and $\bar{\lambda}$ can then be expressed as

$$\lambda^k = |\lambda|^k (\cos k\theta + i \sin k\theta), \qquad k = 1, 2, \ldots$$
$$(6)$$
$$\bar{\lambda}^k = |\lambda|^k (\cos k\theta - i \sin k\theta), \qquad k = 1, 2, \ldots$$

where $|\lambda|^k = |\lambda^k| = |\bar{\lambda}^k|$.

It follows from (6) that

$$\lim_{k\to\infty} \lambda^k = \lim_{k\to\infty} \bar{\lambda}^k = 0, \qquad |\lambda| < 1 \qquad (7)$$

$$\lim_{k\to\infty} \lambda^k = \lim_{k\to\infty} \bar{\lambda}^k = \infty, \qquad |\lambda| > 1 \qquad (8)$$

Since $|\lambda^k| = |\bar{\lambda}^k| = |\lambda|^k$, it follows that $\lim_{k\to\infty} |\lambda^k| = \lim_{k\to\infty} |\bar{\lambda}^k| = \infty$ for $|\lambda| > 1$. As discussed previously for complex scalar sequences, this is actually what is being meant by (8) for complex λ. Obviously, (7) and (8) are also valid for real λ. For $\lambda = 0$, (7) is satisfied for all $k = 1, 2, \ldots$, without a need for a limit. For $|\lambda| = 1$,

$$\lambda = \cos\theta + i \sin\theta$$
$$\bar{\lambda} = \cos\theta - i \sin\theta \qquad (9)$$
$$\lambda^k = \cos k\theta + i \sin k\theta, \qquad k = 1, 2, \ldots$$
$$\bar{\lambda}^k = \cos k\theta - i \sin k\theta, \qquad k = 1, 2, \ldots$$

Due to periodicity of sine and cosine functions, λ^k and $\bar{\lambda}^k$ in this case may exhibit periodic behavior under certain conditions. If these conditions are not satisfied, then λ^k and $\bar{\lambda}^k$ change continuously with k. We will now investigate in more detail the behavior of λ^k and $\bar{\lambda}^k$ when $|\lambda| = 1$.

Since sine and cosine functions have a period of 2π, $\cos(\theta + 2\pi m) = \cos\theta$ and $\sin(\theta + 2\pi m) = \sin\theta$ for $m = 1, 2, \ldots$, where θ is in radians, which can be restricted to the range $0 < \theta < 2\pi$. It then follows from (9) that $\lambda^k = \lambda$ and $\bar{\lambda}^k = \bar{\lambda}$ for $k \neq 1$ if there exist some integers $k \geq 2$ such that $k\theta = \theta + 2\pi m$ or $k = 1 + (2\pi / \theta)m$ for some positive integers m. In that case, λ^k and $\bar{\lambda}^k$ will have a period of $\iota = (2\pi / \theta)\ m_0$, where m_0 is the smallest positive integer to make ι an integer. That is, $\lambda^k = \lambda$ and $\bar{\lambda}^k = \bar{\lambda}$ for $k = 1, 1 + \iota, 1 + 2\iota, \ldots$. Note that $\iota > m_0$, since $2\pi / \theta \geq 1$. Since $(2\pi / \theta) = (\iota / m_0)$, a period ι will exist iff $2\pi / \theta$ is a rational number. Therefore a period ι for λ^k and $\bar{\lambda}^k$ may or may not exist and even if it exists it may be very large.

Another property of sine and cosine functions is that they change sign with a period of π, that is, in addition to having a period of 2π as discussed above, $\cos\theta\ (\theta + \pi n) = -\cos\theta$ and $\sin\ (\theta + \pi n) = -\sin\theta$ for $n = 1, 3, \ldots$. It then follows from (9) that $\lambda^k = -\lambda$ and $\bar{\lambda}^k = -\bar{\lambda}$ for some k if there exist some positive integers k such that $k\theta = \theta + \pi n$ or $k = 1 + (\pi / \theta)\ n$ for some positive odd integers n. In that case. λ^k and $\bar{\lambda}^k$ will change sign with a period of $\iota^- = (\pi / \theta)\ n_0$, where n_0 is the smallest positive odd integer to make ι^- an integer. That is, $\lambda^k = -\lambda$ and $\bar{\lambda}^k = -\bar{\lambda}$ for $k = 1 + \iota^-, 1 + 3\iota^-, \ldots$. Note that ι^- may or may not exist.

If λ^k and $\bar{\lambda}^k$ change sign with a period of ι^-, then they should repeat themselves with a period of $\iota = 2\iota^-$. Therefore, if ι^- exists, then ι exists and $\iota = 2\iota^-$, which is even. Hence, if ι does not exist or is odd, then ι^- does not exist. It can be shown that if ι is even, then ι^- exists (and $\iota^- = \iota / 2$). To show this, note that $(m_0 / \iota) = (\theta / 2\pi)$, where m_0 is the smallest positive integer to make ι an integer. If m_0 and ι are both even for a given θ, then they can both be decreased by division by at least two and therefore the minimality requirement on m_0 is not satisfied. Therefore when ι is even, m_0 should be odd, and $(\iota / 2) = (\pi / \theta)m_0$ satisfies the definition of $\iota^- = (\pi / \theta)\ n_0$ (where n_0 is odd) with $n_0 = m_0$. It can therefore be concluded that ι^- exists iff ι exists and is even.

If ι^- exists, then the sequence $\{\lambda^k\}$ contains only ι^- distinct elements (disregarding sign) $\lambda, \lambda^2, \ldots, \lambda^{\iota^-}$ which change sign with a period ι^-. If ι^- does not exist, then $\{\lambda^k\}$ contains ι distinct elements, $\lambda, \lambda^2, \ldots, \lambda^{\iota}$ which repeat themselves with a period ι without changing sign.

As previously mentioned, complex eigenvalues occur in conjugate pairs. It should be noted from the above discussion that λ^k and $\bar{\lambda}^k$ have the same ι and ι^- (if they exist) and therefore they exhibit the same periodic behavior when periodicity exists. If ι does not exist, then it follows from (9) that λ^k and $\bar{\lambda}^k$ change continuously with k. In our presentation, when we refer to the existence of a complex eigenvalue λ_i, it should be understood that we are actually referring to the existence of a pair of complex eigenvalues λ_i and $\bar{\lambda}_i$.

The only real numbers which satisfy the condition $|\lambda| = 1$ are $\lambda = 1$ and $\lambda = -1$. For $\lambda = 1$, $\theta = 2\pi$, $m_0 = 1$, $\iota = 1$ and ι^- does not exist. For $\lambda = -1$, $\theta = \pi$, $m_0 = n_0 = 1$, $\iota = 2$ and $\iota^- = 1$. For $\lambda = 1$, $\lambda^k = 1$ for $k = 1, 2, \ldots$. Among all λ with $|\lambda| = 1$, $\lambda = 1$ is the only one with a fixed fpower. It can easily be verified that $\iota = 1$ only for $\lambda = 1$ and $\iota^- = 1$ only for $\lambda = -1$. Therefore, when $|\lambda| = 1$, the sequence $\{\lambda^k\}$ is fixed only for $\lambda = 1$, and the sequences $\{\lambda^k\}$ and $\{\bar{\lambda}^k\}$ diverge, either by oscillation or by drifting, when $\lambda \neq 1$.

As some examples of complex eigenvalues with unit absolute values, consider $\lambda = i$, $\lambda = -i$ and $\lambda = (-1 + i\sqrt{3})/2$. For $\lambda = i$, $\theta = \pi/2$, $m_0 = n_0 = 1$, $\iota = 4$ and $\iota^- = 2$. In this case, $\lambda^k = i$ for $k = 1, 5, 9, \ldots$, $\lambda^k = -i$ for $k = 3, 7, 11, \ldots$, and the sequence $\{\lambda^k\}$ contains 2 ($= \iota^-$) distinct elements $\lambda = i$ and $\lambda^2 = -1$ which change sign with period 2. That is, the sequence $\{\lambda^k\}$ expands as $i, -1, -i, 1, i, -1, \ldots$, which can easily be verified. For $\lambda = -i$, $\theta = 3\pi/2$, $m_0 = n_0 = 3$, $\iota = 4$ and $\iota^- = 2$. Note that $\lambda = -i$, being the conjugate of $\lambda = i$, has the same ι and ι^- values as $\lambda = i$. For $\lambda = (-1 + i\sqrt{3})/2$, $\theta = 2\pi/3$, $m_0 = 1$, $\iota = 3$ and ι^- does not exist. In this case, $\lambda^k = \lambda$ for $k = 1, 4, 7, \ldots$, and the sequence $\{\lambda^k\}$ contains 3 ($= \iota$) distinct elements λ, λ^2 and λ^3 which repeat themselves with period 3 without changing sign. That is, the sequence $\{\lambda^k\}$ expands as $\lambda, \lambda^2, \lambda^3, \lambda, \lambda^2, \lambda^3, \ldots$, which can easily be verified.

If there are two or more eigenvalues λ_i (which occur together with their conjugates $\bar{\lambda}_i$) with $|\lambda_i| = 1$ and with individual periods ι_i and ι_i^- for their powers λ_i^k, their simultaneous periods ι_g and ι_g^- (if exist) are of interest, where ι_g is the period at which all λ_i^k will repeat themselves and ι_g^- is the period at which all λ_i^k will change sign. That is, $\lambda_i^k = \lambda_i$ for all i for $k = 1, 1 + \iota_g, 1 + 2\iota_g$, and $\lambda_i^k = -\lambda_i$ for all i for $k = 1 + \iota_g^-, 1 + 3\iota_g^-, \ldots$. If ι_i exists for each λ_i^k, then ι_g also exists and it can be defined as the smallest integer for which ι_g/ι_i is an integer for all i. Note that $\iota_g = \Pi\,\iota_i$ can always staisfy this condition if a smaller ι_g does not exist. For example, if $\iota_1 = 2$, $\iota_2 = 3$ and $\iota_3 = 6$, then $\iota_g = 6$. If $\iota_1 = 3$, $\iota_2 = 4$ and $\iota_3 = 5$, then $\iota_g = 60$. If ι_i^- exists for each λ_i^k, then ι_g^- may or may not exist. The period ι_g^- exists iff there exist odd integers n_i such that $n_i \iota_i = \iota_g^-$ for all i. This follows from the fact that for a λ_i^k with period ι_i^-, $\lambda_i^k = -\lambda_i$ for $k = 1 + n_j \iota_i^-$ where $n_j = 1, 3, \ldots$. If ι_i^-'s consist of both even and odd integers, than ι_g^- cannot exist because $n_i \iota_i^-$ is odd for an odd ι_i^- and $n_j \iota_j^-$ is even for an even ι_j^- and therefore $n_i \iota_i^- \neq n_j \iota_j^-$. For example, if $\iota_1^- = 2$ and $\iota_2^- = 3$, then ι_g^- does not exist because the equality $2n_1 = 3n_2$ cannot be satisfied by any odd integers n_1 and n_2. If ι_i^-'s are all odd integers, then ι_g^- always exists and it can be defined as the smallest integer for which ι_g^- / ι_i^- is an odd integer for all i. This follows from the requirement $n_i \iota_i^- = \iota_g$ for odd n_i and for all i. Note that $\iota_g^- = \Pi \iota_i^-$ can always satisfy this condition if a smaller ι_g^- does not exist. For example, if $\iota_1^- = 3$, $\iota_2^- = 5$ and $\iota_3^- = 9$, then $\iota_g^- = 45$. If $\iota_1^- = 3$, $\iota_2^- = 5$ and $\iota_3^- = 7$, then $\iota_g^- = 105$. If ι_i^-'s are all even integers, then ι_g^- may or may not exist. For example, if $\iota_1^- = 4$ and $\iota_2^- = 6$, then ι_g^- does not exist because the equality $4n_1 = 6n_2$, or $2n_1 = 3n_2$, cannot be satisfied by any odd integers n_1 and n_2. If $\iota_1^- = 2$ and $\iota_2^- = 6$, then $\iota_g^- = 6$ because the equality $2n_1 = 6n_2$, or $n_1 = 3n_2$, can be satisfied by $n_1 = 3$ and $n_2 = 1$. It is again true that if ι_g^- exists, then ι_g exists and $\iota_g = 2\iota_g^-$, which is even. Hence, if ι_g does not exist or is odd, then ι_g^- does not exist. Even when ι_g is even, ι_g^- may not exist. For example, for $\iota_1 = 8$ and $\iota_2 = 12$, $\iota_1^- = 4$, $\iota_2^- = 6$, $\iota_g = 24$ but ι_g^- deos not exist. As a concrete example, consider $\lambda_1 = i$ with $\iota_1 = 4$ and $\iota_1^- = 2$, and $\lambda_2 = (-1 + i\sqrt{3})/2$ with $\iota_2 = 3$ and ι_2^- does not exist. Here $\iota_g = 12$ and ι_g^- does not exist. In this case, $\lambda_1^k = \lambda_1$ and $\lambda_2^k = \lambda_2$ simultaneously for $k = 1, 13, 25, \ldots$.

SOME GENERALIZATIONS

Using (3), (4), (5) and the previous discussion of the behavior of λ_i^k, some general statements can be made regarding the convergence or divergence of the sequence $\{A^k\}$. It follows from (3) and Theorem 3 that $\{A^k\}$ converges iff $\{\Lambda^k\}$ converges. From (5) and (7), $\{\Lambda^k\}$ converges iff $|\lambda_i| < 1$, and $\lambda_i = 1$ being allowed since it has a fixed power. It also follows from Theorem 2 that $\{A^k\} \to 0$ iff $|\lambda_i| < 1$. Therefore, $\{A^k\} \to B$, where B is a nonzero matrix, iff $|\lambda_i| < 1$, and $\lambda_i = 1$ for at least one λ_i. Although we say "at least one" here, note that in dealing with nonsymmetric matrices with distinct eigenvalues there cannot be more than one unit eigenvalue. However, as mentioned before, our results are also valid for the case of repeated eigenvalues provided that a set of linearly independent eigenvectors exists. Therefore in our presentation we will reflect this possibility not to lose generality. From (4), $\{A^k\} \to \infty$ only if $\{\Lambda^k\} \to \infty$. Since $\Lambda^k = P^{-1} A^k P$, $\{\Lambda^k\} \to \infty$ only if $\{A^k\} \to \infty$. Therefore $\{A^k\} \to \infty$ iff $\{\Lambda^k\} \to \infty$. It follows in the same same way that $\{A^k\}$ diverges by oscillation or by drifting iff $\{\Lambda^k\}$ behaves so. From (8), $\{\Lambda^k\} \to \infty$, and hence $\{A^k\} \to \infty$, iff $|\lambda_i| > 1$ for at least one λ_i. The sequence $\{\Lambda^k\}$ diverges by oscillation, and so does $\{A^k\}$,iff $|\lambda_i| \leq 1$ with the equality holding for at least one λ_i other than $\lambda_i = 1$ and there exists a simultaneous period ι_g for the powers of eigenvalues with unit absolute values. If ι_g exists, a simultaneous period for change of sign, ι_g^-, may or may not exist. The sequence $\{\Lambda^k\}$ diverges by drifting, and so does $\{A^k\}$, iff $|\lambda_i| \leq 1$ and there exists no simultaneous period ι_g for the powers of eigenvalues with unit absolute values. As discussed previously, a simultaneous period ι_g for the powers of eigenvalues with unit absolute values exists iff the individual periods ι_i exist, and ι_i exist for $\lambda = 1$ and $\lambda = -1$ as 1 and 2, respectively. Therefore, the nonexistence of ι_g requires the existence of at least one complex eigenvalue with unit absolute value having no period.

SPECIAL CASES

For a more detailed analysis, lets restrict ourselves to the case $|\lambda_i| \leq 1$ where a simultaneous period ι_g exists for the powers of eigenvalues with unit absolute values. If ι_g does not exist, then $\{A^k\}$ diverges by drifting. If $|\lambda_i| > 1$ for at least one λ_i, then $\{A^k\} \to \infty$. Lets write Λ (1) in the partitioned diagonal form as

$$\Lambda = \text{Diag. } (I, U, D, O) \tag{10}$$

where I is the $n_1 \times n_1$ identity matrix containing n_1 unit eigenvalues on its diagonal, U is the $n_2 \times n_2$ diagonal matrix containing n_2 eigenvalues with unit absolute values other than $\lambda = 1$, D is the $n_3 \times n_3$ diagonal matrix containing n_3 nonzero eigenvalues such that $|\lambda_i| < 1$, and O is the $n_4 \times n_4$ null matrix containing n_4 zero eigenvalues. Therefore $n_1 + n_2 + n_3 + n_4 = n$, and it is possible that $n_i = 0$ for some i, meaning that the corresponding matrix does not appear in (10). Lets partition the P (2) and P^{-1} matrices accordingly as

$$P = (P_1, P_2, P_3, P_4)$$

$$P^{-1} = \begin{pmatrix} Q'_1 \\ Q'_2 \\ Q'_3 \\ Q'_4 \end{pmatrix} \tag{11}$$

where P_j is the $n \times n_i$ matrix containing the n_i eigenvectors corresponding to the n_i eigenvalues represented by the i-th submatrix in Λ (10), and Q'_i is an $n_i \times n$ matrix containing n_i rows of the P^{-1} matrix. It follows from $P\,P^{-1} = I$ that

$$P_1Q'_1 + P_2Q'_2 + P_3Q'_3 + P_4Q'_4 = I \tag{12}$$

and from $P^{-1}\,P = I$ that $Q_i'P_i = I$ and $Q_i'P_j = 0$ for $i \neq j$. From (10),

$$\Lambda^k = \text{Diag.}\,(I, U^k, D^k, O), \qquad k = 1, 2, \ldots \tag{13}$$

and from (3), (4), (10), (11) and (13),

$$A = P_1Q'_1 + P_2UQ'_2 + P_3DQ'_3 \tag{14}$$

$$A^k = P_1Q'_1 + P_2U^kQ'_2 + P_3D^kQ'_3, \qquad k = 1, 2, \ldots \tag{15}$$

The matrix D^k in (15) changes continuously with k and its effect on A^k diminishes only in the limit. The sequence $\{D^k\} \to 0$ by (7) and, therefore, $\{P_3D^kQ'_3\} \to 0$ by Theorem 1. Therefore, when $n_3 \neq 0$, the sequence $\{A^k\}$ has no specific finite behavior and we can only establish its limiting behavior.

The matrix U^k in (15) oscillates with k with a period ι_g, that is, $U^k = U$ for $k = 1, 1 + \iota_g, 1 + 2\iota_g, \ldots$. If ι_g^- also exists, then U^k also changes sign with a period ι_g^-, that is, $U^k = -U$ for $k = 1 + \iota_g^-, 1 + 3\iota_g^-, \ldots$. If ι_g^- exists, then the sequence $\{U^k\}$ contains only ι_g^- distinct elements (disregarding sign) $U, U^2, \ldots, U^{\iota_g^-}$ which change sign with a period ι_g^-. If ι_g^- does not exist, then $\{U^k\}$ contains ι_g distinct elements $U, U^2, \ldots, U^{\iota_g}$ which repeat themselces with a period ι_g without changing sign. Note from (15) that when $n_3 = 0$, the matrix A^k has the same period ι_g as the matrix U^k. In this case, if U^k also changes sign with a period ι_g^-, then A^k can change sign, with the same period, only if $n_1 = 0$, that is, only if the fixed matrix $P_1Q'_1$ does not appear in (15). As an example, let $n_3 = 0$, $\iota_g = 4$ and $\iota_g^- = 2$. Then the sequence $\{U^k\}$ expands as $U, U^2, -U, -U^2, U, U^2, \ldots$, which change sign with period 2 and repeat with period 4. The sequence $\{A^k\}$ can then be expanded as $A, A^2, A^3, A^4, A, A^2, A^3, A^4, \ldots$, where $A = P_1Q'_1 + P_2UQ'_2$, $A^2 = P_1Q'_1 + P_2U^2Q'_2$, $A^3 = P_1Q'_1 - P_2UQ'_2$ and $A^4 = P_1Q'_1 - P_2U^2Q'_2$, which repeat themselves with period 4 without changing sign. If $P_1Q'_1$ does not exist, then the sequence $\{A^k\}$ can be expanded as $A, A^2, -A, -A^2, A, A^2, \ldots$, where $A = P_2UQ'_2$ and $A^2 = P_2U^2Q'_2$, which change sign with period 2 and repeat with period 4.

We will now consider various cases in detail for the behavior of A^k through (15). In each case considered, we set one or more of the n_i values equal to zero (and assume that the other n_i values are in general nonzero) to generate various special cases.

CASE 1 $\quad n_2 = n_3 = 0 \qquad (\lambda_i = 0, 1)$

In this case, it follows from (15) that

$$A^k = P_1Q'_1, \qquad k = 1, 2, \ldots$$

Since the eigenvalues here consist of only 0 and 1, if A is a nonsymmetric matrix with distinct eigenvalues, then it should be a 2×2 matrix. If A is a nonsymmetric matrix with repeated eigenvalues for which a a set of linearly independent eigenvectors exists, then A can be of any size. This is the case of an idempotent matrix. A matrix A is called an idempotent matrix if $A = A^2$. The eigenvalues being 0 and 1 is a necessary and sufficient condition for idempotency of a symmetric matrix, but it is only a necessary condition for a nonsymmetric matrix [2]. The result given above for A^k proves sufficiency for a nonsymmetric matrix also but only under the stated conditions.

This is the only case where A^k is a fixed matrix which does not depend on k.

CASE 2 $\quad n_3 = 0 \qquad (|\lambda_i| = 0, 1)$

In this case, it follows from (15) and the periodic behavior of the U^k matrix that

$$A^k = P_1Q'_1 + P_2U^kQ'_2, \qquad k = 1, 2, \ldots$$
$$A^k = P_1Q'_1 + P_2UQ'_2, \qquad k = 1, 1 + \iota_g, 1 + 2\iota_g, \ldots$$
$$A^k = P_1Q'_1 - P_2UQ'_2, \qquad k = 1 + \iota_g^-, 1 + 3\iota_g^-, \ldots$$

the last equation being applicable only if ι_g^- exists. Here A^k has a period ι_g but it does not change sign. The sequence $\{A^k\}$ contains ι_g distinct elements $A, A^2, \ldots, A^{\iota_g}$ which repeat themselves with a period ι_g.

If $\iota_g^- = 1$, which happens iff $U = -I$, then $\iota_g = 2$, $\lambda_i = 0, 1, -1$ and

$$A^k = P_1Q'_1 - P_2Q'_2, \qquad k = 1, 3, \ldots$$
$$A^k = P_1Q'_1 + P_2Q'_2, \qquad k = 2, 4, \ldots$$

Here A^k oscillates between two matrices, A and A^2, depending on whether k is odd or even. Note again that if A is a nonsymmetric matrix with distinct eigenvalues, then it is a 3×3 matrix here. This is the case of a tripotent matrix. A matrix A is called a tripotent matrix if $A = A^3$.

The eigenvalues being 0,1 and -1 is a necessary and sufficient condition for tripotency of a symmetric matrix, but it is only a necessary condition for a nonsymmetric matrix [2]. The results given above for A^k proves sufficiency for a nonsymmetric matrix also but only under the stated conditions. If $n_4 = 0$ also ($\lambda_i = 1, -1$), then it follows from (12) and A^k above for even k that $A^k = I$, $k = 2, 4, \dots$.

CASE 3 $\quad n_1 = n_3 = 0 \qquad (|\lambda_i| = 0,1$ but $\lambda_i \neq 1)$

In this case,

$$A^k = P_2 U^k Q'_2, \qquad k = 1, 2, \dots$$

$$A^k = P_2 U Q'_2, \qquad k = 1, 1 + \iota_g, 1 + 2\iota_g, \dots$$

$$A^k = -P_2 U Q'_2, \qquad k = 1 + \iota_g^-, 1 + 3\iota_g^-, \dots$$

Here A^k has a period ι_g, and it changes sign with a periods ι_g^-, if ι_g^- exist. If ι_g^- exists, then the sequence A^k contains ι_g^- distinct elements $A, A^2, \dots, A^{\iota_g^-}$ which change sign with a period ι_g^-. If ι_g^- does not exist, then $\{A^k\}$ contains ι_g distinct elements $A, A^2, \dots, A^{\iota_g}$ which repeat themselves with a period ι_g.

If $\iota_g^- = 1$, that is $U = -I$, then $\iota_g = 2$, $\lambda_i = 0, -1$ and

$$A^k = -P_2 Q'_2, \qquad k = 1, 3, \dots$$

$$A^k = P_2 Q'_2, \qquad k = 2, 4, \dots$$

Here A^k oscillates between A and -A depending on whether k is odd or even.

Cases 2 and 3 contain all the special cases where the matrix A^k has a period. In Case 2, A^k cannot change sign but in Case 3 it can.

In Cases 1 to 3, $n_3 = 0$ and, therefore, the sequence $\{A^k\}$ has a specific finite behavior. When $n_3 \neq 0$, $\{A^k\}$ has no such behavior and we can only establish its limiting behavior. The following cases will cover these situations.

CASE 4 $\quad n_1 = n_2 = 0 \qquad (|\lambda_i| < 1)$

In this case,

$$A^k = P_3 D^k Q'_3, \qquad k = 1, 2, \dots$$

$$\lim_{k \to \infty} A^k = 0$$

Note that Theorem 2 gives a more general and more stronger form of this result.

CASE 5 $n_2 = 0$ ($|\lambda_i| < 1$, and $\lambda_i = 1$ for at least one λ_i)

In this case,

$$A^k = P_1Q'_1 + P_3D^kQ'_3, \qquad k = 1, 2, \ldots$$

$$\lim_{k\to\infty} A^k = P_1Q'_1$$

In Cases 4 and 5, the matrix A^k in the limit is a fixed matrix, that is, more formally, the sequence $\{A^k\}$ converges.

CASE 6 $n_1 = 0$ ($|\lambda_i| \leq 1$ but $\lambda_i \neq 1$)

In this case,

$$A^k = P_2U^kQ'_2 + P_3D^kQ'_3, \qquad k = 1, 2, \ldots$$

$$A^k = P_2UQ'_2 + P_3D^kQ'_3, \qquad k = 1,\ 1 + \iota_g,\ 1 + 2\iota_g, \ldots$$

$$A^k = -P_2UQ'_2 + P_3D^kQ'_3, \qquad k = 1 + \iota_g^-,\ 1 + 3\iota_g^-, \ldots$$

Since $\{P_3D^kQ'_3\} \to 0$, the matrix A^k in the limit has a period ι_g, and it changes sign with a period ι_g^-, if ι_g^- exists. If ι_g^- exists, then the sequence $\{A^k\}$ in the limit contains ι_g^- distinct elements, $P_2U^iQ'_2$, $i = 1, 2, \ldots, \iota_g^-$, which change sign with a period ι_g^-. If ι_g^- does not exist, then $\{A^k\}$ in the limit contains ι_g distinct elements, $P_2U^iQ'_2$, $i = 1, 2, \ldots, \iota_g$, which repeat themselves with a period ι_g.

If $\iota_g^- = 1$, that is $U = -I$, then $\iota_g = 2$, $|\lambda_i| < 1$, and $\lambda_i = -1$ for at least one λ_i. In this case,

$$A^k = -P_2Q'_2 + P_3D^kQ'_3, \qquad k = 1, 3, \ldots$$

$$A^k = P_2Q'_2 + P_3D^kQ'_3, \qquad k = 2, 4, \ldots$$

Here the matrix A^k in the limit oscillates between the matrices $-P_2Q_2'$ and $P_2Q'_2$ for odd and even k.

CASE 7 n_1, n_2, n_3 all nonzero ($|\lambda_i| \leq 1$, and $\lambda_i = 1$ for at least one λ_i)

In this case,

$$A^k = P_1Q'_1 + P_2U^kQ'_2 + P_3D^kQ'_3, \qquad k = 1, 2, \ldots$$

$$A^k = P_1Q'_1 + P_2UQ'_2 + P_3D^kQ'_3, \qquad k = 1,\ 1 + \iota_g,\ 1 + 2\iota_g, \ldots$$

$$A^k = P_1Q'_1 - P_2UQ'_2 + P_3D^kQ'_3, \qquad k = 1 + \iota_g^-,\ 1 + 3\iota_g^-, \ldots$$

Here the matrix A^k in the limit has a period ι_g but it does not change sign. The sequence $\{A^k\}$ in the limit contains ι_g distinct elements, $P_1Q'_1 + P_2U^iQ'_2$, $i = 1, 2, \ldots, \iota_g$, which repeat themselves with a period ι_g.

If $\iota_g^- = 1$, that is $U = -I$, then $\iota_g = 2$, $|\lambda_i| < 1$, and $\lambda_i = 1$ and $\lambda_i = -1$ at least once. In this case,

$$A^k = P_1Q'_1 - P_2Q'_2 + P_3D^kQ'_3, \qquad k = 1, 3, \ldots$$

$$A^k = P_1Q'_1 + P_2Q'_2 + P_3D^kQ'_3, \qquad k = 2, 4, \ldots$$

Here the matrix A^k in the limit oscillates between the matrices $P_1Q_1' - P_2Q'_2$ and $P_1Q'_1 + P_2Q'_2$ for odd and even k.

In Cases 6 and 7, the matrix A^k in the limit has a period. In Case 6, A^k in the limit can change sign whereas in Case 7 it cannot.

Cases 1 to 7 include all special cases for $|\lambda_i| \leq 1$ where a simultaneous period ι_g exists for the powers of eigenvalues with unit absolute values. In Cases 1, 4 and 5, the sequence $\{A^k\}$ converges. In the others, $\{A^k\}$ diverges by oscillation. The remaining two cases which include divergence of the sequence $\{A^k\}$ to infinity and its divergence by drifting are stated below for the sake of completeness.

CASE 8 $|\lambda_i| > 1$ for at least one λ_i.

$$\lim_{k \to \infty} A^k = \infty$$

CASE 9 $|\lambda_i| \leq 1$ and there exists no simultaneous period ι_g for the powers of eigenvalues with unit absolute values.

In this case, the sequence A^k diveregs by drifting. As mentioned previously, this mode of divergence requires the existence of at least one complex eigenvalue with unit absolute value having no period for its powers. Divergence by drifting cannot accur for symmetric matrices due to nonexistence of complex eigenvalues.

It should be noted from (14) and (15) that the zero eigenvalues have no direct effect of A and A^k. However, the presence or absence of zero eigenvalues, or of their eigenvectors P_4, affects the matrices P_1, P_2, P_3, Q_1, Q_2 and Q_3 through (12), and therefore affects A and A^k as well. In most of the Cases 1 to 9, setting $n_4 = 0$ does not affect the nature of the results.

Symmetric Matrices

The power of a symmetric matrix with distinct or repeated eigenvalues can be analyzed as a special case with considerable simplifications due to the nonexistence of complex eigenvalues and the existence of real orthonormal eigenvectors.

NUMERICAL EXAMPLES

In this section, numerical examples are provided to verify the analytical results previously obtained. All the results can be illustrated simply by using 2×2 or 3×3 matrices, with no loss of generality. It is well known that for a triangular matrix, the diagonal elements are the eigenvalues. This can be used to construct real nonsymmetric matrices with any desired real eigenvalues. We utilize this in three of the examples.

In Table 1 below, for each example, we indicate the case number (Cases 1 to 9 of the previous section) to which the example belongs, we give the A matrix, its eigenvalues λ_i, and the A^k matrix, either for finite k or in the limit as $k\rightarrow\infty$. For Cases 5, 6, and 7, we also provide the P (2) and P^{-1} matrices so that the limiting results can be verified using the eigenvectors p_i, which are the columns of P, and the vectors q'_i, which are the rows of P^{-1}. Although the P matrix is not unique, the limiting results are invariant with respect to the choice of P. For Cases 5, 6, and 7, the limiting results are all realized for $10 \leq k \leq 22$. The reader can verift the results given for A^k either numerically by actually obtaining them through successive multiplication of the A matrix or analytically by using the equations given under Cases 1 to 9 of the previous section.

The A matrix in the example for Case 5 is acceptable as the transition probability matrix of a time-homogeneous two-state Markov chain. The k-step transition probabilities are given by A^k, and the steady-state probabilities, 0.25 and 0.75, of the two states are given by the elements of the identical rows of the limiting A^k matrix.

The last two examples in the table involve matrices with repeated eigenvalues. For the A matrix with eigenvalues $\lambda_i = 1, 1,5$, there exists a set of linearly independent eigenvectors as given by the nonsingular P matrix in the table. Therefore, our results are applicable for this case, and since $\lambda_3 = 5 > 1$, this belongs to Case 8 and $\{A^k\}$ diverges to infinity. For the A matrix with eigenvalues $\lambda_i = 1, 1$, the eigenvectors in their general form are given by the P matrix in the table. Since this P matrix is always singular, there cannot exist linearly independent eigenvectors in this case, and therefore our results are not applicable. It can be verified numerically that the sequence ($\{A^k\}$ diverges to inifinity here. If there were linearly independent eigenvectors, this would belong to Case 1 and A^k would be a fixed matrix for all k.

Table 1. Numerical Examples for Powers of Nonsymmetric Matrices

Case No.	A	λ_i	A^k
1	$\begin{pmatrix} 0.7 & 0.21 \\ 1.0 & 0.30 \end{pmatrix}$	$\lambda_1=1$ $\lambda_2=0$	$A^k=A, \quad k=1,2,\ldots$
2	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\lambda_1=1$ $\lambda_2=i$ $\lambda_3=-i$	$A^2=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A^3=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A^4=I$ $A^k=A, \quad k=1,5,9,\ldots$ $A^k=A^2, \quad k=2,6,10,\ldots$ $A^k=A^3, \quad k=3,7,11,\ldots$ $A^k=A^4, \quad k=4,8,12,\ldots$
2	$\begin{pmatrix} 0 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$	$\lambda_1=1$ $\lambda_2=-1$ $\lambda_3=0$	$A^2=\begin{pmatrix} 0 & 2 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $A^k=A, \quad k=1,3,\ldots$ $A^k=A^2, \quad k=2,4,\ldots$
2	$\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$	$\lambda_1=1$ $\lambda_2=-1$	$A^k=A, \quad k=1,3,\ldots$ $A^k=I, \quad k=2,4,\ldots$
3	$\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$	$\lambda_1=i$ $\lambda_2=-i$	$A^2=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $A^k=A, \quad k=1,5,9,\ldots$ $A^k=A^2, \quad k=2,6,10,\ldots$ $A^k=-A, \quad k=3,7,11,\ldots$ $A^k=-A^2, \quad k=4,8,12,\ldots$
3	$\begin{pmatrix} 2 & -1 \\ 6 & -3 \end{pmatrix}$	$\lambda_1=-1$ $\lambda_2=0$	$A^k=A, \quad k=1,3,\ldots$ $A^k=-A, \quad k=2,4,\ldots$
4	$\begin{pmatrix} 0.1 & 0.05 \\ -2 & 0.3 \end{pmatrix}$	$\lambda_1=0.2+0.3i$ $\lambda_2=0.2-0.3i$	$\lim_{k\to\infty} A^k=0$
5	$\begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix}$	$\lambda_1=1$ $\lambda_2=0.2$	$P=\begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}, P^{-1}=\frac{1}{4}\begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$ $\lim_{k\to\infty} A^k = p_1q'_1 = \frac{1}{4}\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$
6	$\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0.5 \end{pmatrix}$	$\lambda_1=0$ $\lambda_2=-1$ $\lambda_3=0.5$	$P=\begin{pmatrix} 1 & 1 & 16 \\ 1 & 0 & 18 \\ 0 & 0 & 3 \end{pmatrix}, P^{-1}=\frac{1}{3}\begin{pmatrix} 0 & 3 & -18 \\ 3 & -3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ $\lim_{k(\text{even})\to\infty} A^k = p_2q'_2 = \frac{1}{3}\begin{pmatrix} 3 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lim_{k(\text{odd})\to\infty} A^k = -p_2q'_2$

Table 1. Cont'd

Case No.	A	λ_i	A^k
7	$\begin{pmatrix} -1 & 2 & 3 \\ 0 & 0.5 & 4 \\ 0 & 0 & 1 \end{pmatrix}$	$\lambda_1=1$ $\lambda_2=-1$ $\lambda_3=0.5$	$P=\begin{pmatrix} 19 & 1 & 4 \\ 16 & 0 & 3 \\ 2 & 0 & 0 \end{pmatrix}$, $P^{-1}=\frac{1}{6}\begin{pmatrix} 0 & 0 & 3 \\ 6 & -8 & 7 \\ 0 & 2 & -16 \end{pmatrix}$ $\lim_{k(odd)\to\infty} A^k = p_1q'_1-p_2q'_2 = \frac{1}{6}\begin{pmatrix} -6 & 8 & 50 \\ 0 & 0 & 48 \\ 0 & 0 & 6 \end{pmatrix}$ $\lim_{k(even)\to\infty} A^k = p_1q'_1+p_2q'_2 = \frac{1}{6}\begin{pmatrix} 6 & -8 & 64 \\ 0 & 0 & 48 \\ 0 & 0 & 6 \end{pmatrix}$
8	$\begin{pmatrix} 5 & -10 \\ -2 & -3 \end{pmatrix}$	$\lambda_1=1+2i$ $\lambda_2=1-2i$	$\lim_{k\to\infty} A^k=\infty$
9	$\begin{pmatrix} 1 & -0.4 \\ 1 & 0.6 \end{pmatrix}$	$\lambda_1=0.8+0.6i$ $\lambda_2=0.8-0.6i$	$\{A^k\}$ diverges by drifting
8	$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}$	$\lambda_1=1$ $\lambda_2=1$ $\lambda_3=5$	$P=\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$, $\lim_{k\to\infty} A^k=\infty$
-	$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$	$\lambda_1=1$ $\lambda_2=1$	$P=\begin{pmatrix} 0 & 0 \\ C_1 & C_2 \end{pmatrix}$, $C_1\neq 0$, $C_2\neq 0$ $\lim_{k\to\infty} A^k=\infty$

SUMMARY AND CONCLUSIONS

The present study investigates the finite and the limiting behavior of the power of a real nonsymmetric matrix with distinct eigenvalues through its spectral decomposition, which proves to be very convenient for such an analysis. The bahavior of the power of such a matrix can be determined from the eigenvalues and eigenvectors of the matrix. Analytical results for all special cases of practical interest are obtained, and numerical examples are provided. The results are valid also for nonsymmetric matrices with repeated eigenvalues provided that a set of linearly independent eigenvectors exists. Powers of symmetric matrices, with or withour distinct eigenvalues, can be analyzed as a special case. Since a detailed summary of reşults is given as Case 1 to 9 in the paper, we will content with a general summary here concerning the power A^k, $k = 1, 2, ...,$ of a nonsymmetric matrix A satisfying the conditions stated above:

i) The presence of an eigenvalue of A with absolute value greater than one causes the sequence $\{A^k\}$ to diverge to infinity.

ii) When the eigenvalues of A are less than or equal to one in absolute value and there exists no simultaneous period ι_g for the powers of eigenvalues with unit absolute values, the sequence $\{A^k\}$ diverges by drifting. That is, the sequence changes continuously without oscillation or without diverging to infinity. This mode of divergence requires the existence of at least one complex eigenvalue with unit absolute value having no period for its powers. Divergence by drifting cannot occur for symmetric matrices due to nonexistence of complex eigenvalues.

iii) When the absolute values of the eigenvalues of A belong to the set (0,1) and ι_g exists, A^k is either a fixed matrix which does not depend on k or it oscillates with period ι_g.

iv) When there are nonzero eigenvalues with absolute values less than one (and no eigenvalue with absolute value greater than one) and ι_g exists, the matrix A^k is neither fixed nor it oscillates, but it exhibits such behavior only in the limit. That is, the sequence $\{A^k\}$ either converges or diverges by oscillation.

v) The sequence $\{A^k\}$ converges to the null matrix if and only if all the eigenvalues of A are less than one in absolute value. This result is actually true for all matrices.

vi) The matrix A^k oscillates, either for finite k or in the limit, only if there exist eigenvalues with unit absolute values other than the unit eigenvalue and ι_g exists.

SİMETRİK OLMAYAN MATRİSLERİN KUVVETLERİ

ÖZET

Simetrik olmayan ve karakteristik değerleri farklı olan gerçek bir matrisin kuvvetinin sonlu ve limit davranışları matrisin spektral ayrıştırımı yoluyla analiz edilmektedir. Pratik değeri olan bütün özel durumlar için analitik neticeler elde edilmekte ve sayısal örnekler verilmektedir. Neticeler, doğrusal olarak bağımsız karakteristik vektörlerin mevcut olması şartıyla, bazı karakteristik değerleri aynı olan simetrik olmayan matrisler için de geçerlidir.

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