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A CONVEXITY STUDY IN SPHERE

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ABSTRACT

Convex subsets, convex bodies and foot points in the unit n-sphere S^n are defined. Some geometric properties of convex bodies and foot points in S^n -as a manifold with focal points- are derived in comparison with the corresponding properties of convex bodies in both Euclidean space E^n and a Riemanninan manifold W without focal points.

1- INTRODUCTION

In [8], P.J. Kelly and M.L. Weiss proved some interesting results when they studied the basic geometric properties of convex bodies in the Euclidean space E^n. Recently, we studied the same properties when the ambient space is a complete simply connected Riemannian manifold W without focal points [3, 4]. It was found in [3, 4] that almost all the results in E^n -concerning convexity, foot points and sectional curvatures- are still valid in W. As far as we know, properties of convex bodies which have been considered in [3, 4, 8] have not yet been considered for the same sort of bodies in the unit n-sphere S^n. Consequently, this work is mainly devoted to study some geometric properties of convex bodies in sphere. Illustrative examples in E^n and S^n are given as a comparison study to show how results are affected by the existence of conjugate points in the ambient space S^n.

In the following we give some of the results proved in [3, 4, 8].

Proposition (1-1)

Closed geodesic balls in W are convex bodies. Geodesic spheres in W are closed convex hypersurfaces. Horodiscs in W are convex subsets.

Proposition (1–2)

If p is an interior point in a convex body $B \subset W$ (or E^n) with smooth boundary ∂B, then each geodesic ray from p intersects the hypersurface ∂B exactly at one point and the intersection is transversal.

Proposition (1–3)

For a convex body $B \subset W$ (or E^n) with smooth boundary ∂B

(i) each tangent geodesic γ to ∂B has the property $\gamma \cap \text{Int}(B) = \varnothing$,

(ii) B lies on one side of the tangent geodesic hypersurface of ∂B at each point $p \in \partial B$

(iii) no two outer geodesic rays perpendicular to ∂B meet.

Proposition (1-4)

For a compact smooth hypersurface M in W there exists a point $p \in M$ such that the sectional curvature $K_p(M)$ of M at p satisfies $K_p(M) \geq K_p(W)$.

The remarkable L. Amaral's theorem [1] can be concluded directly and easily from the last proposition (1–4) if we replace W by the hyperbolic space H^n (See [3].

For other interesting results, specially concerning foot points, we refer the reader to [2, 3, 4, 8]. From now on the unit sphere S^n is always taken as an imbedded hypersurface

$$S^n = \{x : x \in E^n, \langle x, x \rangle = 1\}$$

in E^n. Geometric properties of S^n may be found in any text book in differential geometry. All curves are parametrized by arc length. All manifolds are sufficiently smooth for dicussions to make sense.

2– Definitions and Background

Let us begin with introducing the convexity concept in a general Riemannian manifold M.

Definition (2–1)

A set A in M is called convex if for each pair of points $p, q \in A$ there is a unique minimal geodesic segment from p to q and this segment is in A.

Definition (2-2)

A set B in M is a convex body if it is a compact convex subset of M with a non-empty interior. A strictly convex body is a convex body

B such that the boundary ∂B of B does not contain any geodesic segment of M.

Definition (2-3)

A set B° is a closed convex (resp. strictly convex) hypersurface of M if it could be made as a boundary of a convex (resp. strictly convex) body B in M, i.e $B^\circ = \partial B$.

In the light of the above definitions we can easily prove the following remarks.

Remarks

(a) The whole of S^n is not a convex set in contrary to the convexity of E^n (or W).

(b) Any convex body B in S^n is contained in an open hemisphere S_υ^n of S^n with some point $\upsilon \in S^n$ as its center.

(c) No two points of a convex body $B \subset S^n$ form a conjugate pair. Same thing is valid in a general Riemannian manifold M.

(d) Any convex body $B \subset S^n$ can be mapped geodesically onto a convex body in E^n by using the Beltrami (or central projection) map [2, 6].

(e) Any closed geodesic ball $\overline{B(p, r)}$ centered at an arbitrary point $p \in S^n$ with radius $r < \pi / 2$ (small geodesic ball) in S^n is a convex body. This fact is true, in general, for any geodesic ball in either E^n or W.

(f) Any closed geodesic ball of radius $r \geq \pi / 2$ is not a convex body in S^n and consequently the closed half-space of S^n is not a convex body in contrary to the same property in either E^n or W [3] (By half-space in W we mean horodiscs). A geodesic ball of radius $r = \pi / 2$ will be called great geodesic ball.

(g) Any geodesic sphere S (p, r) of center p and radius $r < \pi / 2$ (small geodesic sphere) in S^n is a convex hypersurface.

(h) Totally geodesic hypersurfaces (great spheres) in S^n are non-convex hypersurfaces. Also, geodesics of S^n are non-convex subsets of S^n.

(i) A geodesic sphere S (p, r) of radius r in S^n has constant sectional curvature $K = 1 / \sin^2 r$ [2]. In this way, a totally geodesic hypersphere in S^n has sectional curvature $K = 1$ while small geodesic spheres in S^n are of constant sectional curvatures greater than 1.

(j) The closure of an open convex subset in S^n is not necessarily convex. Open hemispheres in S^n are good examples of this case. In E^n (or W), the closure of any open convex subset is always convex (See [2, 3]). The proof of this fact depends basically on the truncated geodesic cone concept defined below.

(k) Any subset A of S^n with diameter $d(A) > \pi$ does not have a convex hull H(A) where H(A) always exists for any subset $A \subset E^n$. If $A \subset S^n$ has $d(A) < \pi$, then H(A) exists. We can show that Beltrami maps from S^n to E^n [2, 6] preserves convex subsets as well as convex hulls.

Definition (2-4)

Let γ be a geodesic ray in S^n from p. A truncated geodesic cone ${}_pC\gamma$ in S^n with vertex p and axis γ is the family of all geodesic segments emanating from p with the same initial angle with γ and each segment is of length less than π.

The length π is excluded in the above definition so as to avoid conjugate points of p on the surface of the cone ${}_pC\gamma$.

We can give another definition of the cone ${}_pC\gamma$ as follows.

Consider the exponential map $\exp_p$: $T_pS^n \to S^n$ restricted to a ball $B(0, r) \subset T_pS^n$ of radius $r < \pi$. This map is a diffeomorphism on B (0, r). The cone ${}_pC\gamma$ will be taken as the image of a cone ${}_OC_L$ with vertex 0 and axis the straight line segment L in T_pS^n such that ${}_OC_L \subset B(0, r)$.

One of the main results we shall use later on is the following which relates the height function of a submanifold N in a Riemannian manifold M with its second fundamental form [7].

Proposition (2–1)

Let N be an immersed hypersurface of a Riemannian n-manifold M. Let p be a point of N. Then the second fundamental form of N at p is the Hessian of the height function of N with respect to its tangent space T_pN as a hyperplane of T_pM.

3- Main Results on Convexity

Lemma (3–1)

Let B be a convex body in S^n with smooth boundary ∂B. If p is an interior point of B, i.e $p \in \text{Int}(B)$, then each geodesic ray from p intersects of hypersurface ∂B for the first time transversally.

Proof

Let γ: $[0, \infty) \rightarrow S^n$ be an arbitrary geodesic ray such that $p = \gamma(0)$. This ray γ can not be contained wholly inside B otherwise B will not be contained in an open hemisphere of S^n contradicting remark (b). Assume in contrary to the lemma that γ has a tangential first intersection, say $\gamma(a)$, with ∂B. Clearly, the geodesic segment $\gamma[0, a]$ is free from conjugate points of $\gamma(a)$. Draw a thin geodesic cone ${}_{\gamma(a)}C_{\gamma}$ with vertex $\gamma(a)$, axis γ and base D in B (See Fig. (1)). Then there exists a minimal geodesic segment $\bar{\gamma}$ from $\gamma(a)$ to $x \in D$ such that $\bar{\gamma} \not\subset B$ contradicting the convexity of B and the proof is complete.

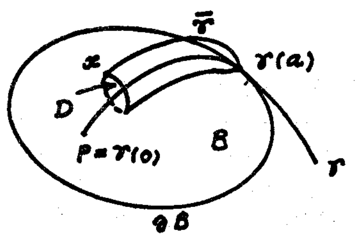

Fig. (1)

Corollary (3–1)

Let p be an interior point of a convex body $B \subset S^n$ with smooth boundary ∂B. Let γ: $[0,2\pi] \rightarrow S^n$ be a closed geodesic through p such that $p = \gamma(0) = \gamma(2\pi)$. Then the first and the last intersections of γ with ∂B are transversal.

Lemma (3-2)

Let p be an interior point of a convex body $B \subset S^n$ and γ: $[0,2\pi] \rightarrow S^n$ be a closed geodesic through p such that $p = \gamma(0)$. Let $\gamma(a)$ and $\gamma(b)$, $b > a$, be the first and the last intersections of γ with ∂B. Then $\gamma(a, b) \cap B = \varnothing$.

Proof

Let γ be a closed geodesic through p as given in the lemma. Assume in contrary that $\gamma(a, b) \cap B \neq \varnothing$. Without loss of generality, we consider the case when $\gamma(a, b) \cap B$ is a single point, say $q = \gamma(c)$, for $a < c < b$. (See Fig. (2)).

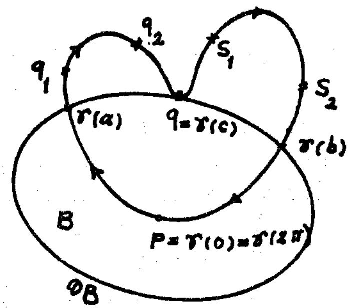

Fig. (2)

The geodesic segment $\gamma[a, c]$ should not be minimal otherwise B will be no -convex. Consequently $\gamma[a, c]$ contains a pair (q_1, q_2) of of conjugate points. Also, the geodesic segment $\gamma[c, b]$ contains a conjugate pair (s_1, s_2) of points contradicting the geometry of geodesics in S^n. The other possibilities can be discussed similarly.

From the above two lemmas we arrive at the following.

Proposition (3–1)

Any closed geodesic γ through an interior point p of a convex body $B \subset S^n$ with smooth boundary ∂B intersects ∂B exactly twice. The intersections are all transversal.

Corollary (3–2)

The smooth boundary ∂B of a convex body $B \subset S_\rho$ is diffeomorphic to S^{n-1}.

This corollary can be justified as follows.

Let B becontained in the open hemisphere $S_p{}^n \subset S^n$ of center p. Draw a small geodesic sphere $S(p, \delta)$ of center $p \in \text{Int}(B)$ and sufficiently small radius δ such that $S(p, \delta) \subset \text{Int}(B)$. By the convexity of B and $B(p, \delta)$, any geodesic ray from p intersects -for the first time- both $S(p,\delta)$ and ∂B transversally. In this way, we can build up a central projection diffeomorphism $\beta : \partial B \to S(p, \delta)$. If we compose β with $\exp_p{}^{-1}$ restricted to $S_p{}^n$, we obtain a diffeomorphism $\exp^{-1}{}_p o\beta : \partial B \to S(0, \delta) \subset T_pS^n \cong E^n$.

Proposition (3–2)

Let $B \subset S^n$ be a convex body with smooth boundary ∂B and $p \in \partial B$ an arbitrary point. Then B lies on one side of the great hypersphere tangent to ∂B at p.

Proof

Assume in contrary to the proposition that B does not lie on one side of the great hypersphere tangent to ∂B at p. Consequently, there exists a closed geodesic γ, with orientation indicated in Fig. (3), tangent to ∂B at p which intersects ∂B at least twice, say at q, $r \in \partial B$, transversally (Lemma (3–1)). The geodesic segment γ_{pq} -as it lies outside B- is not minimal and consequently there exists a point $s \in \gamma_{pq}$ which is conjugate to p. The geodesic segment γ_{qp} is not contained in B which contradicts the convexity of B and the proof is complete.

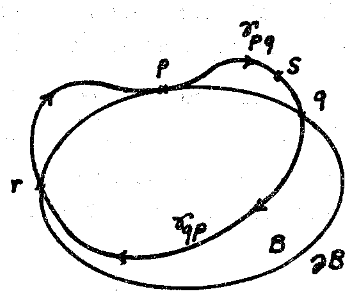

Fig. (3)

Another proof of the above proposition (2–1) may be given if we notice from Fig. (3) that p has two different conjugate points, the first is on γ_{pq} and the other is on γ_{rp}, contradicting the sphere geometry.

The converse of the above proposition is not generally true as any closed hemisphere S_v^n centered at $v \in S^n$ in S^n lies on one side of each tangent great hypersphere to ∂S_v^n while S_v^n is not a convex body in S^n. Actually, S_v^n has only one tangent great hypersphere to ∂S_v^n which is itself ∂S_v^n. The above proposition (3–2) together with its converse are true in E^n as well as in W [3, 4].

In the light of the above proposition (3–2) and taking into account the property mentioned in proposition (2–1), we have the following consequences.

Corollary (3–3)

For a convex body $B \subset S^n$, each boundary point p is a global minimum point of the height function -with respect to the inner direction and p as a base point- of either B or ∂B.

From this corollary, we can easily show that at each point of ∂B the Hessian of the height function as well as the second fundamental form are definite (or semi-definite) [3] and so we have the following results.

Corollary (3–4)

For a convex body $B \subset S^n$, all the boundary points of B have sectional curvature K satisfying $K \geq 1$.

Corollary (3–5)

For a convex body $B \subset S^n$, there exists at least one point $p \in \partial B$ with sectional curvature K strictly greater than 1.

The reason is that B is contained inside a closed ball $\overline{B(p, r)}$ for some $p \in S^n$ and $r < \pi / 2$ where $\partial B \cap \overline{\partial B(p, r)} \neq \varnothing$. For an arbitrary point $q \in \partial B \cap \overline{B(p, r)}$, we have easily -using remark (i) and the height function concept and its relation with sectional curvature -that K_q of ∂B at q satisfies

$$K_q \geq 1 / \sin^2 r > 1.$$

Corollary (3–5) can be proved for any closed (not necessarily convex) hypersurface M of S^n such that M is contained in an open hemisphere of S^n. In this case, we shall obtain a result similar to that of L. Amaral [1] but in the spherical ambient space.

The following proposition (3–3)gives a necessary and sufficient condition for boundary points of a convex body in S^n to be global maximum points of the height functions.

Proposition (3–3)

For a convex body $B \subset S^n$, each height function on B -with respect to inner direction and boundary base point -has a global maximum point on the boundary ∂B if and only if the diameter d(B) of B satisfies $d(B) \leq \pi / 2$. (The proof is direct)

The following example shows how the condition $d(B) \leq \pi / 2$ is substantial in the above proposition.

Example (3-1)

Let us consider the closed geodesic ball $\overline{B(p, r)} \subset S^n$ centered at $p \in S^n$ with radius $\pi/4 < r < \pi/2$ as a convex body in S^n. Without loss of generality, let us take $\overline{B(p, r)}$ to be contained in the upper hemisphere of S^n and let $q \in \overline{B(p, r)} \cap S(\upsilon, \pi/2)$ where υ is the north pole of S^n. It is clear that υ which is an interior point of $\overline{B(p, r)}$ is the global maximum point of the height function based at q (The height of $x \in B$ is $d(x, S(\upsilon, \pi/2))$ (See § 4)). Notice that in E^n, global maximum points of each height function are always boundary points (See Fig. 4).

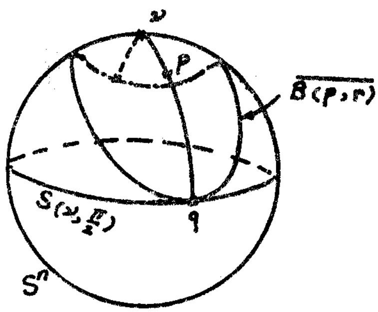

Fig. (4-a)

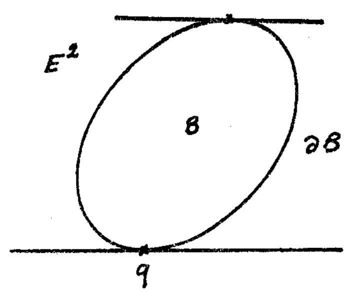

Fig. (4-b)

In Euclidean space E^n, it has been proved that for a convex body $B \subset E^n$ all the points of the segment L joining an arbitrary point $p \in$ Int (B) and a boundary point $q \in \partial B$ are interior points (except q.) The corresponding result in W is also valid [4] but in S^n -as a manifold with conjugate (or focal) points -we should add a restrictive condition as follows.

Proposition (3-4).

Let B a convex body in S^n, $p \in$ Int (B) and $q \in \partial B$. Then all the points of the unique minimal geodesic segment γ joining p and q are interior points except q.

The proof can be carried out by using the truncated geodesic cones concept as it has been done in [3, 4].

The converse of the above proposition (3-4) is not generally true. One might consider any closed hemisphere in S^n as an example. The converse is true if we replace S^n by E^n (or W). The reason is that if all the points of the geodesic segment γ joining $p \in$ Int (B) and $q \in \partial B$ are interior points (except q), then B will be a starshaped subset of E^n and p will lie in the kernel (Ker (B)) subset of B. In addition, if this is true for all the interior points of B, then Ker (B) = B and consequently B is convex.

4– Foot Points

The distance d (p, S) from a point p to a non-empty subset S of a Riemannian manifold M is defined as [8].

$$d\,(p.\ S) = \text{glb}\ \{d\,(p, x) : x \in S\}$$

Definiton (4–1)

A point p has a foot q in a subset $S \subset M$ if

(i) $q \in S$,

(ii) $q\,(p, q) = d\,(p, S)$

One may understand that the foot point of p in S is the nearest point of S to p. Moreover, if $\overline{S}$ is the closure of $\overline{S}$ and $p \notin \overline{S}$, then d (p, S) = d (p, ∂S), i.e the foot point of a point p, in the closure $\overline{S}$ of S, is always a boundary point provided that $p \notin \overline{S}$. If $p \in S$, then p is the unique foot point of itself. Clearly, the foot point of a point $p \notin S$ in a closed subset S with smooth boundary ∂S is a critical point of the distance function $d_p : \partial S \to R$ defined as $d_p(x) = d\,(p, x)$. Also the critical geodesic segment from p to the foot point strikes ∂S orthogonally at the foot point (See [5] p. 216).

Proposition (4–1)

Let B be a convex body in S^n and $p \in S^n$ such that $d(p, B) < \pi / 2$. Then p has a unique foot point in B.

Proof

If d (p, B) = 0, then $p \in B$ and p is the unique foot point of itself as indicated above.

If $0 < d\,(p, B) < \pi / 2$, then $p \in B$. Assume in contrary that p has two different foot points, say q_1 and q_2. Let $d\,(p, q_1) = d\,(p, q_2) = \iota$, where $\iota < \pi / 2$. Draw the closed geodesic ball $\overline{B\,(p, \iota)}$. We have that $B \cap \overline{B\,(p, \iota)} =) \{q_1, q_2\}$. As $\overline{B\,(p, \iota)}$ has diameter less than π, then $\overline{B\,(p, \iota)}$ is a convex body in S^n. In this way, the unique minimal geodesic segment γ joining q_1 and q_2 is contained in $\overline{B(p, \iota)}$. Hence, $\gamma \not\subset B$ which contradicts the convexity of B. A similar discussion can be carried out if p has more than two foot points.

Remark (4–1)

If $d(p, B) \geq \pi/2$, then the above proposition (4–1) is not neccessarily true in the light of the following example.

Example (4–1)

Consider B to be a convex body in S^2 which has a part γ_1 of its smooth boundary $\partial B = \gamma_1 \cup \gamma_2$ as a sufficiently small geodesic segment The south pole p of S^2 which satisfies $d(p, B) = \pi / 2$ has all the points of γ_1 as foot points. (See Fig. (5)).

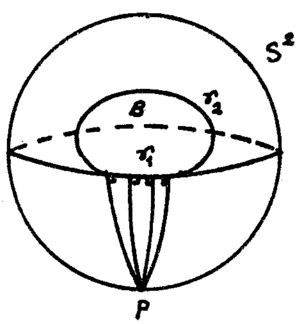

Fig. (5)

We can show that the above proposition (4–1) is still true for $d(p, B) \leq \pi / 2$ on the condition that B is strictly convex.

The concept of the foot points in subsets as well as convex subsets in sphere (or more generally in a Riemannian manifold) deserves to devote a separate work. We hope to acheive such a study in the near future.

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