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THE RELATIVE ENTROPY OF TOPOLOGICAL DYNAMICAL SYSTEM WITH CONTINUOUS TIME

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ABSTRACT

In this study, first the partition entropy is reminded then the basic properties of the partition entropy are given without into details.

Then the properties of the topological dynamical system with continuous time are investigated and some important properties of it are proved.

INTRODUCTION

In 1958 Kolmogorov [4] Introduced the concept of entropy in ergodic theory and investigated the fundamental properties of entropy. In 1959 the definition of entropy of a dynamical system was given by Sinai [6].

The properties of the partition entropy of dynamical system are investigated by Rochlin [5] and Billingsley [3]. Then the definitions of flow entropy and skew product dynamical system are given by Abromov and some properties of these are proved [1] and [2].

The entropy of a dynamical system is defined in three steps. The entropy of a finite measurable partition $P = \{A_1, \ldots, A_n\}$ is defined by

$$H_m(P) = -\sum_{i=1}^{n} m(A_i) \log m(A_i).$$

The entropy a finite measurable partition P relative to f is

$$h_m(P, f) = \lim_{n\to\infty} \sup \frac{1}{n} H_m \left(\bigvee_{i=0}^{n-1} f^{-1} P \right)$$

It turns out that the limit superior here is really an ordinary limit. Finally the entropy of a dynamical system is

$$h_m(f) = \sup h_m(P, f)$$

where the supremum extends over all finite measurable partition.

In this article by using the methods of Abromov's and Rochlin's the relative entropy of topological dynamical system with continuous time is defined and some basic properties of it are proved.

1. ENTROPY

Let (X, A, m) be a measure space and Let $f: X \to X$ be a continuous map $M(X, f)$ denotes the space of all f-invariant measures which are defined on the measurable space (X, A). $M(X, f)$ is convex and compact in the weak-topology.

1.1. Definition. If $P = \{A_1, A_2, \ldots, A_n\}$ is a finite measurable partition of X and for every $m \varepsilon M(X, f)$ then the function

$H_m(P) = \sum_{i=1}^{n} z(m(A_i))$ is called entropy of partition P.

Where the function

$$z(t) = \begin{cases} -t\log t & \text{if } t > 0 \\ 0 & \text{if } t = 0, \quad t = 1 \end{cases}$$

is non-negative, continuous and strictly concave function. All logarithms in this paper will be taken to the natural base.

1.2. Proposition. Let P, Q be two finite partitions of X and $m \varepsilon M(X, f)$. Then

i) a) $H_m(P) \geq 0$

b) $H_m(P) = 0$ iff $P = \{X, \emptyset\}$

ii) $H_m(P) \geq H_m(Q)$ if $P \geq Q$

iii) $H_m(PvQ) \geq H_m(P) + H_m(Q)$

iv) Let $(P_n)_{n \geq 1}$ be a sequence of measurable partitions of X

If $P_n \to P$ as $n \to \infty$ then $H_n(P_n) \to H_m(P)$ [5].

1.3. Definition. Let P and Q be two finite measurable partitions of X and for every $m \varepsilon M(X, f)$ then the function

$$H_m(P \mid Q) = -\sum_{i,j} m(A_i \cap C_j) \log \frac{m(A_i \cap C_j)}{m(C_j)}$$

is called conditional entropy of the partition P given Q.

1.4. Proposition. Let P, P_1, P_2, Q, Q_1, Q_2 be finite measurable partitions of X and for every $m \varepsilon M(X, f)$. Then

1) $H_m(P_1 \vee P_2 \mid Q) = H_m(P_2 \mid P_1 \vee Q) + H_m(P_1 \mid Q)$
2) $H_m(P_1 \vee P_2) = H_m(P_2 \mid P_1) + H_m(P_1)$
3) $H_m(P_1 \vee P_2 \mid Q) \leq H_m(P_1 \mid Q) + H_m(P_2 \mid Q)$
4) $H_m(P \mid Q \vee P_1) \leq H_m(P \mid Q) \leq H_m(P)$
5) $H_m(P \mid Q_2) \leq H_m(P \mid Q_1)$ if $Q_1 \leq Q_2$
6) $H_m(P_1 \mid Q) \leq H_m(P_2 \mid Q)$ if $P_1 \leq P_2$
7) $H_m(P_1 \mid Q) \leq H_m(P_1 \vee P_2 \mid Q)$
8) If f is a measure-preserving map then
$H_m(fP \mid fQ) = H_m(P \mid Q)$ [7].

2. DYNAMICAL SYSTEMS WITH CONTINUOUS TIME AND BASIC PROPERTIES

2.1. Definition. Let (X, A, m) and (Y, B, m_0) be two measure spaces a map $f:X \to Y$ is said to be measure preserving if $m(f^{-1}(B)) = m_0(B)$ for all $B \varepsilon B$. f is an invertible measure-preserving map if it is 1—1 onto measure-preserving map and f^{-1} is also measure preserving map, f is an automorphism of measure space (X ,A, m).

If f is 1—1 map of the space X onto itself such that for all $A \varepsilon A$ we have $f(A)$, $f^{-1}(A) \varepsilon A$ and $m(A) = m(f(A)) = m(f^{-1}(A))$.

The measure m is said to be a f-invariant measure for the automorphism f (X, A, m, f) is called a dynamical system. If X is a compact metric space then the (X, A, m, f) is known as a topological dynamical system.

2.2. Definition. Suppose $\{f_t\}_{t \varepsilon R}$ is a one-parameter group of automorphism of the measure space (X, A, m) for all $t_1, t_2 \varepsilon R$, $x \varepsilon X$

i) $f_{t1} \circ f_{t2}(x) = f_{t1+t2}(x)$

ii) If for any measurable function $\varphi(x)$ on X the function $\varphi(f_t x)$ is measurable on the cartesian product X x R then

$\{f_t\}_{t \varepsilon R}$ is said to be flow

$F = \{f_t\}_{t \varepsilon R}$ is a measure-preserving flow if $m(FA) = m(A)$ for all $m \varepsilon M(X, F) = \bigcap_{t \varepsilon R} M(X, f_t)$ if for all $A \varepsilon A$, $\lim_{t \to \infty} m(f_t A \,\Delta\, A) =$

$= 0$ then $F = \{f_t\}$ is said to be continuous flow, $(X, A, m, \{f_t\}_{t \varepsilon R})$ is called a dynamical system with continuous time and will be expressed as $(X, \{f_t\})$.

Suppose $(X, A. m \{f_t\}_{t \varepsilon R})$ is a dynamical system $\{f_t\}_{t \varepsilon R}$ is said to be measurable flow, if for $A \varepsilon A$ all $t \varepsilon R$ and $x \varepsilon A$ $f_t x$ is an element of A. Let $F = \{f_t\}_{t \varepsilon R}$ and $G = \{g_t\}_{t \varepsilon R}$ be flows defined on X and Y respectively. Let $\pi: X \rightarrow Y$ be a surjective continuous and measure-preserving map. If for every $t \varepsilon R$ and $x \varepsilon X$.

$\pi o F(x) = G o \pi(x)$ then

Dynamical system $(Y, \{g_t\})$ is said to be a factor of dynamical system $(X, \{f_t\})$.

$F = \{f_t\}_{t \varepsilon R}$ flow is called ergodic flow if for every $t \varepsilon R$ and $A \varepsilon A$ $F(A) = A$ $m(A) = 0$ or $m(A) = 1$.

2.3. Proposition. If $\{f_t\}_{t \varepsilon R}$ is an ergodic flow then its factor is also ergodic.

Proof: Let $B \varepsilon B$ be G-invariant set. Then by property of a factor $F(\pi^{-1}(B)) = \pi^{-1} G(b)$. Since B is an G-invariant set from this equality we obtain $F(\pi^{-1}(B)) = \pi^{-1}(B)$. This implies that B is an F-invariant set. Since B is an ergodic set, $m(\pi^{-1}(B)) = 0$ or $m(^{-1}(B)) = 1$. Since π is a measure-preserving map, $m(\pi^{-1}(B)) = m_0(B)$ follows. From the last equality $m_0(B) = 0$ or $m_0(B) = 1$. This implies that G is an ergodic flow.

2.4. Proposition. If $\{a_n\}_{n \geq 1}$ satisfies $a_n \geq 0$, $a_{n+m} \geq a_n + a_m$ every m, n then $\lim\limits_{n \to \infty} \frac{a_n}{n}$ exists and equals to $\inf\limits_n \frac{a_n}{n}$ [7].

3. RELATIVE ENTROPY FOR A FLOW

3.1. Proposition. Suppose P is a finite measurable partition of X and ε_y denotes the partitions of Y into points. Then

$$\lim_{n \to \infty} \frac{1}{n} H_m \left(\bigvee_{i=0}^{n-1} F_i P \mid \pi^{-1}(\varepsilon_y) \right) \text{ exists.}$$

Proof: By lemma 2.4 if we take $a_n = H_m \left(\bigvee_{i=0}^{n-1} F_i P \mid \pi^{-1}(\varepsilon_y) \right)$, the result is obtained.

3.2. Definition. Let P be a finite partition and $H_m(P) < \infty$ We define the limit which is obtained from prop. 3.1 as follow

$$h_m(f_t \mid g_t, P) = \lim_{n\to\infty} \frac{1}{n} H_m \left(\bigvee_{i=0}^{n-1} F_i P \mid \pi^{-1}(\varepsilon_y) \right)$$

the following function,

$$h_m(f_t \mid g_t) = \sup \{h_m(f_t \mid g_t, P)\}.$$

is called the relative entropy of dynamical system with continuous time where the supremum is taken over all finite partitions of X.

3.3. Proposition. Suppose P, Q are two finite measurable partitions of X for all $t \varepsilon R$

i) $h_m(f_t \mid g_t, P \vee Q) \leq h_m(f_t \mid g_t, P) + h_m(f_t \mid g_t, Q)$

the equality takes place if the partitions P and Q are independent.

ii) $h_m(f_t \mid g_t) \leq h_m(f_t \mid g_t, Q)$ if $P < Q$

iii) $h_m(f_t \mid g_t, P) \leq h_m(f_t \mid g_t, Q) + H_m(P \mid Q \vee \pi^{-1}(\varepsilon_y))$

Proof. By (iii) of proposition 1.4

$$H_m \left(\bigvee_{i=0}^{n-1} F_i P \vee \bigvee_{i=0}^{n-1} F_i Q \mid \pi^{-1}(\varepsilon_y) \right) \leq H_m \left(\bigvee_{i=0}^{n-1} F_i P \mid \pi^{-1}(\varepsilon_y) \right)$$

$$+ H_m \left(\bigvee_{i=0}^{n-1} F_i Q \mid \pi^{-1}(\varepsilon_y) \right)$$

dividing the above by $n > 0$ and taking the limit for $n \to \infty$ by Proposition 3.1, for every $t \varepsilon R$

$h_m(f_t \mid g_t, P \vee Q) \leq h_m(f_t \mid g_t, P) + h_m(f_t \mid g_t, Q)$ the result is obtained.

ii) If $P \leq Q$ then $\bigvee_{i=0}^{n-1} F_i P < \bigvee_{i=0}^{n-1} F_i Q$. Therefore

$H_m \left(\bigvee_{i=0}^{n-1} F_i P \mid \pi^{-1}(\varepsilon_y) \right) \leq H_m \left(\bigvee_{i=0}^{n-1} F_i Q \mid \pi^{-1}(\varepsilon_y) \right)$ using Proposition 3.1

Hence $h_m(f_t \mid g_t, P) < h_m(f_t \mid g_t, Q)$

iii) By (1), (4) and (8) of Proposition 1.4

$$H_m \left(\bigvee_{i=0}^{n-1} F_i P \mid \pi^{-1}(\varepsilon_y) \right) \leq H_m \left(\bigvee_{i=0}^{n-1} F_i P \vee \bigvee_{i=0}^{n-1} F_i Q \mid \pi^{-1}(\varepsilon_y) \right)$$

$$H_m\left(\bigvee_{i=0}^{n-1} F_iQ \mid \pi^{-1}(\varepsilon_y)\right) + H_m\left(\bigvee_{i=0}^{n-1} F_iP \mid \bigvee_{i=0}^{n-1} F_iQ \vee \pi^{-1}(\varepsilon_y)\right)$$

dividing by $n > 0$ both sides,

$$\frac{1}{n} H_m\left(\bigvee_{i=0}^{n-1} F_iQ \mid \pi^{-1}(\varepsilon_y)\right) + \frac{1}{n}\sum_{i=0}^{n-1} H_m(F_i P \mid Q \vee \pi^{-1}(\varepsilon_y))$$

$$= \frac{1}{n} H_m\left(\bigvee_{i=0}^{n-1} F_i P \mid \pi^{-1}(\varepsilon_y)\right) + H_m(P \mid Q \vee \pi^{-1}(\varepsilon_y))$$

taking limit for $n \to \infty$

$$h_m(f_t \mid g_t, P) \leq h_m(f_t \mid g_t, Q) + H_m(P \mid Q \vee \pi^{-1}(\varepsilon_y))$$

3.4. Theorem. Suppose $(Y, \{g_t\})$ dynamical system is a factor of $(X, \{f_t\})$ dynamical system. Then for all $t \varepsilon R$.

$$h_m(f_t \mid g_t) = |t| \, h_m(f \mid g)$$

Proof: Assuming $t > 0$ we shall first prove tha $0 < u < t$ implies

$$h_m(f_t \mid g_t) = \frac{t}{u} t_m(f_u \mid g_u)$$

Suppose k is a positive integer $\delta = \frac{1}{k}$ and P is definite partition of space X and ε_y is a partition of Y into points.

Put $Q = P v f_{\delta u} P v f_{2\delta u} P v \ldots v f_{(k-1)\delta u} P$

further fix a positive integer n and denote by $l = l(n)$ some natural number such that $nt < lu < (n+1)t$ for $p = 1, 2, \ldots, n$ denote by $r(p)$ the natural number satisfying $r(p)\delta u \leq pt \leq [r(p)+1]\delta u$ $h_m(f_t \mid g_t) = \sup_p h_m(f_t \mid g_t, P)$ by definition 3.2

$$h_m(f_t \mid g_t, P) = \lim_{n\to\infty} \frac{1}{n} H_m\left(\bigvee_{i=0}^{n-1} f_{it} P \mid \pi^{-1}(\varepsilon_y)\right)$$ by proposition 3.1 using the properties of the entropy of a partition we can write

$$H_m\left(\bigvee_{i=1}^{n} f_{it} P \mid \pi^{-1}(\varepsilon_y)\right) = H_m(f_t P \vee f_{2t}P \vee \ldots f_{nt} P \mid \pi^{-1}(\varepsilon_y))$$

$$\leq H_m(Q \vee f_u Q \vee \ldots \vee f_{lu} Q \mid \pi^{-1}(\varepsilon_y)) + H_m(f_t P \vee \ldots \vee f_{tn} P \mid Q \vee f_u Q \vee \ldots \vee d_{lu} Q \vee \pi^{-1}(\varepsilon_y)).$$ Therefore

$$H_m\left(\bigvee_{i=1}^{n} f_{it} P \mid \pi^{-1}(\varepsilon_y)\right) = H_m(Q \vee f_u Q\vee \dots \vee f_{lu} Q \mid \pi^{-1}(\varepsilon_y)) +$$

$H_m [f_t P \vee \dots \vee f_{nt} P \mid P \vee f_u P\vee \dots \vee f_{(k-1)u} P \vee \dots f_{lu} P \vee \dots \vee$

$f_{lu+(k-1)u}(P) \vee \pi^{-1}(\varepsilon_y)]$ is obtained.

In fact $lu + (k-1)\delta u = (k(l+1)-1)\delta u$

$$H_m\left(\bigvee_{i=1}^{n} f_{it} P \mid \pi^{-1}(\varepsilon_y)\right) \leq H_m(Q\vee f_u Q\vee \dots \vee f_{lu} Q \mid \pi^{-1}(\varepsilon_y)) +$$

$$H_m(f_t P \vee \dots \vee f_{tn} P \mid P\vee f_{\delta u} P \vee \dots \vee f_{(k-1)\delta u} P \vee \dots \vee f_{lu} P\vee \dots \vee f_{(k(l+1)-1)\delta u} P \vee \pi^{-1}(\varepsilon_y)) < H_m[Q \vee f_u Q \vee \dots \vee f_{lu} Q \mid \pi^{-1}(\varepsilon_y)$$

$$+ \sum_{p=1}^{n} H_m(f_{pt} P \mid f_{r(p)\delta u} P\vee\pi^{-1}(\varepsilon_y)]$$

is obtained. But

$H_m(f_{pt} P \mid f_{r(p)\delta u} P\vee \pi^{-1}(\varepsilon_y)) = H_m(f_s P \mid P \vee \pi^{-1}(\varepsilon_y))$

where $s = pt - r(p)\delta u < \delta u$

choose an arbitrary $\varepsilon > 0$ Since the flow $\{f_t\}$ is continuous $\lim_{\delta\to 0} m(f_\delta A\Delta A) = 0$. Therefore for any sufficiently small $\delta > 0$ we have the inequality $H_m(f_s P \mid P\vee \pi^{-1}(\varepsilon_y)) < \varepsilon$

therefore we get

$H_m(f_t P\vee \dots \vee f_{nt} P \mid \pi^{-1}(\varepsilon_y)) = H_m(Q\vee \dots \vee f_{l(n)u} Q \mid \pi^{-1}(\varepsilon_y)$

$$+ n\varepsilon \lim_{n\to\infty} \frac{k(n)}{n} = \frac{t}{u},$$

the last inequality implies

$$\lim_{n\to\infty} \frac{1}{n} H_n(f_t P \vee \dots \vee f_{nt} P \mid \pi^{-1}(\varepsilon_y)) \leq \frac{t}{u} \lim_{n\to\infty} \frac{1}{l(n)} H_n(Q \vee \dots \vee f_{l(n)u} Q \mid \pi^{-1}(\varepsilon_y)) + \varepsilon$$

Since ε was arbitrary we get

$$h_m(f_t \mid g_t) = \frac{t}{u} h_m(f_u \mid g_u)$$

Now suppose the positive integer r satisfies $\frac{t}{r} < u$ therefore

$$h_m(f_u \mid g_u) = \frac{u}{t/r} h_m(f_{t/r} \mid g_{t/r}) = \frac{r.u}{t} h_m(f_{t/r} \mid g_{t/r})$$

Since $h_m(f_{t/r} \mid g_{t/r}) = \frac{1}{r} h_m(f_t \mid g_t)$ we get

$$h_m(f_u \mid g_u) < \frac{u}{t} h_m(f_t \mid g_t) \text{ i.e.}$$

$$h_m(f_t \mid g_t) = \frac{t}{u} h_m(f_t \mid g_t)$$

hence $h_m(f_t \mid g_t) = \frac{t}{u} (f_t \mid g_t)$.

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