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ON THE CURVATURE MOTION

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ABSTRACT

Real Spherical curvature motion has been defined by A. Karger and J. Novák [1]. In this paper, this motion has been considered on the dual sphere and the results carried to the line space IR^3 by means of Study map. During the dual spherical curvature motion K/K' the Study map of the orbits drawn on the K' by $\vec{V}_1$, $\vec{V}_2$ and $\vec{V}_3$ are obtained as

$$y_2^2 + y_3^2 - (y_1 - \sqrt{2v})^2 = 2v^2,$$

$$y_3 = c \operatorname{arctg} \frac{y_2}{y_1}, \quad (c=\text{const.})$$

$$y_1^2 + y_2^2 = \frac{(w^* - y_3)^2}{y_3} .$$

Introduction

Oriented lines in an Euclidean 3– dimension Space IR^3 may be represented by unit vectors with three components over the ring of dual numbers. A differentiable curve on the dual unit sphere depending on a real parameter t corresponds to a ruled surface in IR^3. This correspondence is one-to-one and allows the geometry of ruled surfaces to be represented by the geometry of dual spherical curves on a unit dual sphere. Dual spherical geometry, expressed with the help of dual unit vectors, is analogous to real spherical geometry, expressed with the help of real unit vectors. Hence the properties of elementary real spherical geometry can also be carried over by analogy into the geometry of lines in IR^3.

Real spherical curvature motion has been defined by A. Karger and J. Novak [1].

In this paper, we shall study this motion on the dual sphere and the results carry to the line space IR^3 by means of Study map.

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Basic Concepts

Definition 2.1.

If a and a* are real numbers and $\varepsilon^2 = 0$, the expression $A = a + \varepsilon a^*$ is called a dual number.

W.K. Clifford defined the dual numbers and showed that they form a ring, not a field [2].

Two dual numbers $A=a+\varepsilon a^*$ and $B=b+\varepsilon b^*$ are added compa-nentwise

$$A+B = (a+b) + \varepsilon(a^* + b^*),$$

and they are multiplied in the form.

$$AB = ab + \varepsilon(ab^* + a^*b)$$

Division of A by B is denoted by

$$\frac{A}{B} = \frac{a + \varepsilon a^*}{b + \varepsilon b^*} = \frac{a}{b} + \varepsilon \frac{(ba^* - ab^*)}{b^2}$$

where B is not pure dual and $b \neq 0$.

Let ID^3 be dual space of triples of dual numbers with coordinates

$$X_1 = x_1+\varepsilon x_1^* , X_2 = x_2+\varepsilon x_2^*, X_2 = x_2+\varepsilon x_2^*.$$

The set of oriented lines in Euclidean 3–dimensional space IR^3 is one-to-one correspondence with the points of a unit dual sphere in the dual space ID^3.

Theorem 2.1. (E.Study)

The oriented lines in IR^3 are in one-to-one correspondence with the points of the dual unit sphere

$$\|\vec{X}\| = 1 \text{ in } ID^3 \ [3] .$$

If $\vec{X}=\vec{x}+\varepsilon \vec{x}^*$ and $\vec{Y}=\vec{y}+\varepsilon \vec{y}^*$ are two dual vectors, then

$$< \vec{X},\vec{Y} > = \cos \Psi = \cos \psi - \varepsilon \psi^* \sin \psi ,$$

where $\Psi = \psi + \varepsilon\, \psi^*$ is the dual angle between $\vec{X}$ and $\vec{Y}$, ψ is the angle between lines $\vec{x}$ and $\vec{y}$ and ψ^* is the minimal distance between the lines $\vec{x}$ and $\vec{y}$.

The important properties of real vector analysis are valid for the dual vectors.

The six components a_i, a^*_i $(i=1,2,3)$ of a and a^* $(A=a+\varepsilon\, a^*)$ are pluckerian homogeneous line coordinates. Hence the two vectors a and a^* determine the oriented line in IR^3.

Definition 2.2.

A differentiable curve $\vec{X}(t)$ on the dual unit sphere, depending on a real parameter t, represents a differentiable family of straight lines in IR^3 which we call a ruled surface.

In general, a dual unit vector, a function of one dual variable $T=t+\varepsilon t^*$.

$$\vec{X}(T) = \vec{X}(t) + \varepsilon t^* \dot{\vec{X}}(t) \ , \ \|\vec{X}\| = 1$$

is a differentiable line-system [4].

Definition 2.3.

Consider a non-singular linear transformation between two dual orthonormal coordinate systems linked to the unit dual spheres K (moving) and K′ (fixed), respectively. These dual unit spheres have a common center 0 and K is the moving sphere with respect to the fixed sphere K′.

In this paper, this motion is considered as one-dual parameter spherical motion and denoted by K/K′. If the unit dual spheres K and K′ correspond to the line spaces H and H′, respectively, then K/K′ corresponds to the spatial motion which will be denoted by H/H′. Then H is the moving space with serpect to the fixed space H′.

Dual Spherical Curvature Motion

Consider a fixed dual orthonormal frame $\{\vec{U}_1, \vec{U}_2, \vec{U}_3\}$. This frame will be represented by unit dual sphere K'. Let a circle S^1_1 with radius $r_1 = \frac{\sqrt{2}}{2}$ be given on the sphere K' and also let a great circle S^1_2 be given on the sphere K'. Let us consider two points as $A \in S^1_1$ and $B \in S^1_2$ where dual arc length $\widehat{AB}$ has length $\frac{\pi}{2}$. Let $\vec{V}_1$ and $\vec{V}_2$ be position vectors of the endpoints of the dual arc length $\widehat{AB}$. Since $\widehat{AB}$ is with length $\frac{\pi}{2}$ then a vector $\vec{V}_3$ is defined by the relation $\vec{V}_3 = \vec{V}_1 \wedge \vec{V}_2$. Thus, a moving orthonormal frame $\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ is chosen. This moving frame will be represented by unit dual sphere K (moving). Let 0_1 be the center of circle S_1^1. The dual arc length $\widehat{AB}$ moves so that all time $A \in S_1^1$ and $B \in S_2^1$.

This motion is analogous to the real spherical curvature motion [1]. In this paper this motion will be called as dual spherical curvature motion. As the parameter of the motion we choose the dual angle $W = w + \varepsilon w^*$ of the vectors $\vec{U}_1$ and $\vec{V}_2$, see figure 3.1.

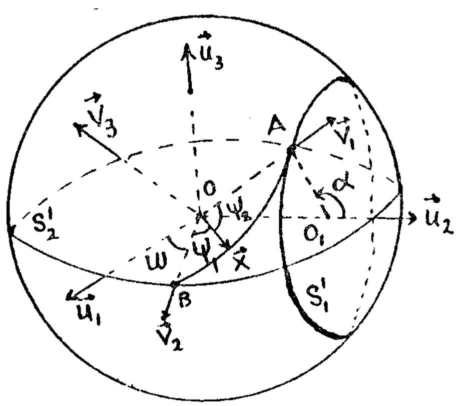

Figure 3.1

In this case, dual curvature motion is represented by K/K'. Let us denote by $A=\alpha+\varepsilon\alpha^*$ the angle of the vectors $\vec{A0}_1$ and $\vec{U}_2$. Then the following expressions may be written as

$$\vec{V}_1 = \frac{\sqrt{2}}{2}\vec{U}_1 + \frac{\sqrt{2}}{2}\cos A\vec{U}_2 + \frac{\sqrt{2}}{2}\sin A\vec{U}_3$$

$$\vec{V}_2 = \cos W\vec{U}_1 + \sin W\vec{U}_2 ,$$

$$<\vec{V}_1,\vec{V}_2> = 0 \text{ i.e. } \frac{\sqrt{2}}{2}\cos W + \frac{\sqrt{2}}{2}\cos A\sin W = 0. \tag{3.1}$$

From (3.1)

$$\cos W = -\cos A\sin W$$

or

$$\cos A = -\operatorname{cotg} W$$

is obtained. In this case, $\sin A = (1-\operatorname{cotg}^2W)^{1/2}$. Thus, moving dual orthonormal frame $\{\vec{V}_1,\vec{V}_2,\vec{V}_3\}$ is obtained as follows

$$\vec{V}_1 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\operatorname{cotg} W, (1-\operatorname{cotg}^2W)^{1/2}\right) \tag{3.2}$$

$$\vec{V}_2 = (\cos W, \sin W, 0), \tag{3.3}$$

$$\vec{V}_3 = \vec{V}_1\wedge\vec{V}_2=\left(-\frac{\sqrt{2}}{2}(-\cos 2W)^{1/2}, \frac{\sqrt{2}}{2}\operatorname{cotg} W(-\cos 2W)^{1/2}, \frac{\sqrt{2}}{2\sin W}\right) \tag{3.4}$$

Consider a fixed dual point $\vec{X}$ of K on the arc $\widehat{AB}$ which lies on the K. During the motion K/K', fixed dual point $\vec{X}$ draws an orbit on the sphere K'.

The Study Map Of The Obrit Which Are Drawn By $\vec{V}_1$, $\vec{V}_2$ And $\vec{V}_3$

Denote the dual angles of $\widehat{BX}$ and $\widehat{XA}$ by $\Psi_1 = \psi_1 + \varepsilon\, \psi_1^*$, $\Psi_2 = \psi_2 + \varepsilon\psi_2^*$ respectively. Then the vector $\vec{X}$ may be written as fallows [5],

$$\vec{X} = \frac{\vec{V}_1 \sin \Psi_1 + \vec{V}_2 \sin \Psi_2}{\sin (\Psi_1 + \Psi_2)} = \vec{V}_1 \sin \Psi_1 + \vec{V}_2 \sin \Psi_2 \tag{4.1}$$

where $\sin (\Psi_1 + \Psi_2) = 1$.

The vectors $\vec{V}_1$, $\vec{V}_2$ and the angle $\sin \Psi_1$, $\sin \Psi_2$ may be written as follows

$$\vec{V}_1 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \frac{\cos w - \varepsilon w^* \sin w}{\sin w + \varepsilon w^* \cos w}, \frac{\sqrt{2}}{2} \left(1 - \frac{(\cos w - \varepsilon w^* \sin w)^2}{(\sin w + \varepsilon w^* \cos w)^2}\right)^{1/2}\right) \tag{4.2}$$

$$\vec{V}_2 = (\cos w - \varepsilon w^* \sin w, \sin w + \varepsilon w^* \cos w, 0) \tag{4.3}$$

$$\sin\Psi_1 = \sin\psi_1 + \varepsilon\psi^*_1 \cos\psi_1, \sin\Psi_2 = \sin\psi_1 + \varepsilon\psi_2^* \cos\psi_2. \tag{4.4}$$

Using (4.2), (4.3) and (4.4) the real and dual parts of $\vec{X}$ are respectively obtained as follows

$$\vec{x} = \left(\frac{\sqrt{2}}{2} \sin\psi_1 + \cos w \sin\psi_2, -\frac{\sqrt{2}}{2} \sin\psi_1 \operatorname{cotg} w + \sin\psi_2 \sin w, \frac{\sqrt{2}}{2} \sin\psi_1 (1 - \operatorname{cotg}^2 w)^{1/2}\right) \tag{4.5}$$

and

$$\vec{x}^* = \left(\frac{\sqrt{2}}{2} \psi_1^* \cos\psi_1 + \psi_2^* \cos w \cos\psi_2 - w^* \sin w \sin\psi_2, \frac{\sqrt{2}}{2} \psi_1^* \operatorname{Cos}\psi_1\right.$$

$$\cot g w + \frac{\sqrt{2}}{2} w^* \frac{\sin\psi_1}{\sin^2 w} + \psi_2{}^* \sin w \cos\psi_2 + w^* \cos w \sin\psi_2, \qquad \frac{\sqrt{2}}{2}\psi_1{}^* \cos\psi_1$$

$$(1-\mathrm{Cotg}^2 w)^{1/2} - \frac{\sqrt{2}}{2} w^* \frac{\cot g w (1-\cot g^2 w)^{1/2}}{\mathrm{Cos} 2w} \mathrm{Sin}\psi_1) \ . \tag{4.6}$$

Since

$$<\vec{X},\vec{V}_1> = \sin\Psi_1 = \sin\psi_1 + \varepsilon\ \psi_1{}^* \cos\psi_1 = \text{Const.} \tag{4.7}$$

which means that $\psi_1 = \text{Const.}$, $\psi_1{}^* = \text{Const.}$, the equations (4.5), (4.6) and (4.7) permit us to write the following relations :

$$\left.\begin{array}{l} <\vec{x}, \vec{x}> = 1 \\ <\vec{x}, \vec{x}^*> = 0 \\ <\vec{x}, \vec{v}_1> - \sin\psi_1 = 0 \\ <\vec{x}, \vec{v}_1{}^*> + <\vec{x}^*, \vec{v}_1> - \psi_1{}^* \sin\psi_1 = 0 \end{array}\right\} \tag{4.8}$$

The equations (4.8) have only two parameters w and w* so (4.8) represents a line congruence in IR^3.

Now, we may calculate the equations of this congruence in plucker coordinates. Let $\vec{Y}$ be a point of this congruence, then $\vec{Y}$ may be written as follows [6]

$$\vec{Y} = \vec{x}(w,w^*) \wedge \vec{x}^*(w,w^*) + \nu \vec{x}(w,w^*) \tag{4.9}$$

Calculating the $\vec{x} \wedge \vec{x}^*$ in (4.9), the following equations are obtained:

$$c_1 = \frac{\sqrt{2}}{2} \sin w (1-\cot g^2 w)^{1/2} [\psi_1{}^* \cos\psi_1 \sin\psi_2 - \psi_2{}^* \sin\psi_1 \cos\psi_2] + \frac{w^*}{2}$$

$$(1-\cot g^2 w)^{1/2} \left[\frac{\sin^2\psi_1}{\cos^2 w} - \frac{2\sqrt{2}\ \sin\psi_1 \sin\psi_2\ \cos^3 w}{\cos 2w} \right],$$

$$c_2 = \frac{\sqrt{2}}{2} \cos w (1-\cot g^2 w)^{1/2}\ [\psi_2{}^* \sin\psi_1 \cos\psi_2 - \psi_1{}^* \sin\psi_2 \cos\psi_1] + \frac{w^*}{2\sin w}$$

$$(1-\cot g^2 w)^{1/2} \quad . \quad [\ \frac{\sin^2\psi_1 \cos w}{\cos 2w} + \sqrt{2}\ \sin\psi_1\ \sin\psi_2]\ ,$$

$$c_3 = w^* \ [\ \frac{\sin^2\psi_1}{2\sin^2 w} + \frac{\sqrt{2}}{2}\ \frac{\cos w \sin\psi_1 \sin\psi_2}{\sin^2 w} + \sin^2\psi_2] - \frac{\sqrt{2}}{2}\psi_1^*$$

$$\frac{\cos\psi_1 \sin\psi_2}{\sin w} + \frac{\sqrt{2}}{2}\ \psi_2^*\ \frac{\sin\psi_1 \cos\psi_2}{\sin w}$$

wheie c_1, c_2 and c_2 are components of $\vec{x} \wedge x^*$.

If the coordinates of $\vec{Y}$ are denoted by (y_1, y_2, y_2), then from the equation (4.9) we have

$$y_1 = \frac{\sqrt{2}}{2}\ \sin w\ (1-\cot g^2 w)^{1/2} [\psi_1\ ^*\cos\psi_1 \sin\psi_2 - \psi_2^*\ \sin\psi_1 \cos\psi_2] +$$

$$+ \frac{w^*}{2}\ (1-\cot g^2 w)^{1/2}\ [\ \frac{\sin^2\psi_1}{\cos 2w} - 2\sqrt{2}\ \frac{\sin\psi_1 \sin\psi_2 \cos^3 w}{\cos 2w}\ ] +$$

$$+ \frac{\sqrt{2}}{2}\ v\sin\psi_1 + v\cos w \sin\psi_2, \qquad (4.10)$$

$$y_2 = \frac{\sqrt{2}}{2}\ \cos w\ (1-\cot g^2 w)^{1/2} [\psi_2\ ^*\sin\psi_1 \cos\psi_2 - \psi_1\ ^*\sin\psi_2 \cos\psi_1] +$$

$$\frac{w^*}{2\sin w}\ (1-\cot g^2 w)^{1/2}\ .\ \ [\ \frac{\sin^2\psi_1 \cos w}{\cos 2w} + \sqrt{2}\ \sin\psi_1 \sin\psi_2]$$

$$- \frac{\sqrt{2}}{2}\ v\sin\psi_1 \cot g w + v\sin\psi_2 \sin w, \qquad (4.11)$$

$$y_3 = w^*\ [\ \frac{\sin^2\psi_1}{2\sin^2 w} + \frac{\sqrt{2}}{2}\ \frac{\cos w \sin\psi_1 \sin\psi_2}{\sin^2 w} + \sin^2\psi_2] - \frac{\sqrt{2}}{2}$$

$$\psi_1^*\ \frac{\cos\psi_1 \sin\psi_2}{\sin w} + \frac{\sqrt{2}}{2}\ \psi_2^*\ \frac{\sin\psi_1 \cos\psi_2}{\sin w} + \frac{\sqrt{2}}{2}\ v\sin\psi_1$$

$$(1-\cot g^2 w)^{1/2}\ . \qquad (4.12)$$

If we put $\Psi_1 = 0$ i.e. $\psi_1, \psi_1^* = 0$, in the equation (4.1), then we have $\vec{X} = \vec{V}_2$. Thus, from (4.10), (4.11) and (4.12) for the obrit drawn by $\vec{X}$ or $\vec{V}_2$ the equation

$$\left.\begin{aligned} y_1 &= v\cos w \\ y_2 &= v\sin w \\ y_2 &= w^* \end{aligned}\right\} \tag{4.13}$$

are obtained. The equation (4.13) represents a line congruence in IR^3. That is, during the dual spherical curvature motion the Study map of the orbit which is drawn on the K′ by $\vec{X}$ is a congruence whose axe is y_3. This congruence has the form

$$\left.\begin{aligned} y_1^2 + y_2^2 &= v^2 \\ y_3 &= w^* \end{aligned}\right\} \tag{4.14}$$

If in particular $w^* = cw$ ($c = \text{const.}$), then (4.14) reduces to

$$y_3 = c\,\text{arctg}\,\frac{y_2}{y_1} \tag{4.15}$$

which represents a right helicoid. Thus, we may give the following theorem.

Tehorem 4.1.

During the dual spherical curvature motion the Study map of the orbit drawn on the K′ by $\vec{X}$ (or $\vec{V}_2$) is the congruence (4.14). In the case that $w^* = cw$ the right helicoid $y_3 = c\,\text{arctg}\,\frac{y_2}{y_1}$ is obtained.

If we put $\Psi_2 = 0$ i.e. $\psi_2 = 0$, $\psi_2^* = 0$ in the (4.1), then we have $\vec{X} = \vec{V}_1$. Thus from (4.10), (4.11) and (4.12) for the orbit drawn by $\vec{X}$ or $\vec{V}_1$ the equation

$$2y_1 = w^* \frac{(-\cos 2w)^{1/2}}{\sin w \cos 2w} + \sqrt{2}\, v$$

$$2y_2 = w^* \frac{(-\cos 2w)^{1/2} \cdot \cos w}{\sin^2 w \cos 2w} - \sqrt{2}\, v \frac{\cos w}{\sin w} \tag{4.16}$$

$$2y_3 = \frac{w^*}{\sin^2 w} + \sqrt{2}\, v \frac{(-\cos 2w)^{1/2}}{\sin w}$$

are obtained.

From the equation (4.16) we have

$$y_2^2 + y_3^2 - (y_1 - \sqrt{2}\, v)^2 = 2v^2 \tag{4.17}$$

which represents one parameter family of one-Sheeted hyperboloids. Intersection of each hyperboloid and the corresponding planes $y_1 = \sqrt{2}$ v is the family of circles

$$y_2^2 + y_3^2 = 2v^2.$$

Thus, we have the following theorem

Theorem 4.2

During the dual spherical curvature motion the Study map of the orbit drawn by $\vec{X}$ (or $\vec{V}_1$) is one parameter family of one-sheeted hyperboloids.

In addition, during the same motion we can study the orbit drawn on the spheıe K′ by $\vec{V}_3$.

Since

$$\vec{V}_3 = \left(- \frac{\sqrt{2}}{2} (-\cos 2w)^{1/2}, \frac{\sqrt{2}}{2} \operatorname{cotg} w\, (-\cos 2w)^{1/2}, \frac{\sqrt{2}}{2\sin w} \right), \tag{4.18}$$

making necessary calculations, from (4.18)

$$\vec{V}_3 = \left(- \frac{\sqrt{2}}{2} (-\cos 2w)^{1/2}, \frac{\sqrt{2}}{2} \operatorname{cotg} w\, (-\cos 2w)^{1/2}, \frac{\sqrt{2}}{2\sin w} \right) \tag{4.19}$$

and

$$\vec{v}_3^{\,*} = \Big(\frac{\sqrt{2}}{2} w^* \frac{\sin 2w\,(-\cos 2w)^{1/2}}{\cos 2w}, \; -\sqrt{2}\, w^* \frac{(-\cos 2w)^{1/2}\cos^2 w}{\cos 2w}$$

$$-\frac{\sqrt{2}}{2} w^* \frac{(-\cos 2w)^{1/2}}{\cos 2w}, \; -\frac{\sqrt{2}}{2} w^* \frac{\cos w}{\sin^2 w} \Big) \qquad (4.20)$$

are obtained where $\vec{v}_3$ and $\vec{v}_3^{\,*}$ the real and dual parts of the $\vec{V}_3$, respectively. Similar to previous discussion

$$\vec{G} = (y_1, y_2, y_3) = \vec{v}_3 \wedge \vec{v}_3^{\,*} + p\vec{v}_3 \qquad (4.21)$$

is written. From (4.21) we have

$$2y_1 = w^* \frac{(-\cos 2w)^{1/2}}{\sin w} + 2w^* \frac{\cos^2 w(-\cos 2w)^{1/2}}{\sin w \cos 2w}$$

$$-\sqrt{2}\, p(-\cos 2w)^{1/2}, \qquad (4.27)$$

$$2y_2 = -w^* \frac{\cos w(-\cos 2w)^{1/2}}{\sin^2 w} + 2w^* \frac{\cos w(-\cos 2w)^{1/2}}{\cos 2w} +$$

$$+\sqrt{2}\, p \frac{\cos w(-\cos 2w)^{1/2}}{\sin w}, \qquad (4.23)$$

$$2y_3 = -w^* \frac{\cos^2 w}{\sin^2 w} + w^* + \frac{\sqrt{2}\, p}{\sin w}. \qquad (4.24)$$

If $p = 0$ in (4.21), (4.22) and (4.23), then

$$y_1^2 + y_2^2 = \frac{(w^* - y_3)^2}{y_3} \qquad (4.25)$$

is obtained. Thus, we have the following theorem

Tehorem 4.3.

During the dual spherical curvature motion the Study map of the orbit drawn on the sphere K′ by $\vec{V}_3$ is a family of surface in the form

$$y_1^2 + y_2^2 = \frac{(w^* - y_3)^2}{y_3}.$$

ÖZET

Reel küresel eğrilik hareketi A. Karger ve J. Novak tarafından [1] de verildi. Bu çalışmada, bu hareket dual küreye taşınarak dual küresel eğrilik hareketi tanımlandı. Burada elde edilen sonuçlar Study dönüşümü vasıtasıyla IR^3 doğrular uzayına taşındı. K / K′ dual küresel eğrilik hareketi esnasında $\vec{V}_1$, $\vec{V}_2$ ve $\vec{V}_3$ tarafından K′ de çizilen yörüngelerin Study dönüşümleri sırayla

$$y_2^2 + y_3^2 - (y_1 - \sqrt{2}\, v)^2 = 2v^2,$$

$$y_3 = c \operatorname{arctg} \frac{y_2}{y_1}, \quad (c = \text{sab.})$$

$$y_1^2 + y_2^2 = \frac{(w^* - y_3)^2}{y_3}$$

şeklinde elde edildi.

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