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IMMERSIONS OF LORENTZIAN SUBMANIFOLDS INTO R_1^m WITH POINTWISE 2- PLANAR SECTIONS AND ON THE CIRCLES AND PSEUDO SPHERES IN LORENTZIAN GEOMETRY

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ABSTRACT

We planned this paper into two main sections. In the first section, we give an analog for the Lorentzian case of some characterizations given in [2]. There is no difference between the characterizations in both cases of immersions with (pointwise) 2-planar normal sections of Riemannian and Lorentzian manifolds into $\mathbf{R}^m$ and $\mathbf{R}_1^m$, respectively, but the proofs.

In the second part of paper, we deal with the Theorem. 3.2 given in [1] and show that there must be some extra hypothesis to get the characterizations given as Theorems 2.1 and 2.2 in the present paper.

INTRODUCTION

We have taken the references [1], [2] and [4] as a base even notations, used here.

Let $\mathbf{R}_j^m$ be standart semi-Riemannian manifold that j denotes the index of $\mathbf{R}_j^m$. $\tilde{\nabla}$ and ∇ stand for the connections on $\mathbf{R}_j^m$ and M_i^n, respectively, where $M_i^n \subset \mathbf{R}_j^m$ and M_i^n is a submanifold of $\mathbf{R}_j^m$ and has index i. The second fundamental form and shape operator A of M_i^n satisfy following equations;

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{0.1}$$

$$\tilde{\nabla}_X \zeta = - A_\zeta X + D_X \zeta \tag{0.2}$$

for every vector fields X, Y tanget to M_i^n and normal ζ to M_i^n, that is, $X, Y \in \chi(M_i^n)$ ζ $\chi \in (M)_i^{n\perp}$, where D denotes the normal connection on M_i^n. If g is the semi-Riemannian metric on M_i^n induced from the metric on $\mathbf{R}_j^m$ then we have

$$g(A_\zeta X, Y) = g(h(X, Y), \zeta). \tag{0.3}$$

We denote mean curvature vector of M^n by H.

If the equation

$$h(X, Y) = g(X, Y)\, H \tag{0.4}$$

satisfied for every X, Y $\in \chi(M_i^n)$ then M_i^n is called totally umbilic submanifold. Van der Waerden Bortolotti connection on M_i^n will be denoted by $\bar{\nabla}$.

Let t be a unite tangent vector to M at the point p. we define E (p, t) as the affine subspace of $\mathbf{R}_j^m$ passes through p and associated with the vector subspace spanned by t and $(T_pM_i^n)^{\perp}$ and denoted by

$$E(p, t) = p + S_p\, \{t,\ (T_pM_i^n)^{\perp}\}.$$

The section curve $M_i^n \cap E(p, t)$ will denoted by ns (M, p, t) and called normal section curve determined by t. For ns (M, p, t) there are two important possibilities those are;

1) ns (M, p, t) will be 2–planar curve of $\mathbf{R}_j^m$.

2– ns (M, p, t) has 2–planar arc of $\mathbf{R}_j^m$ near p.

If the case 1) holds for every point p and for all tangent vectors t then we say M_i^n has 2–planar normal sections and if the case 2) holds for every point p and for all tangent vectors t then we say M_i^n has pointwise 2–planar normal sections.

Let γ be the arc–length parametrization of the curve ns (M, p,t). If M has (even pointwise) 2–planar normal sections then we have

$$\gamma'(O)\, \Lambda\, \gamma''(O)\, \Lambda \gamma'''(O) = O;\ (\gamma(O) = p)$$

[2], where Λ denotes the exterior product.

Given a curve α. By $k_j(s)$ we denote the j–th curvature of $\alpha(s)$ as in [1]. If $k_j(s) = 0$ for $j > 2$ and if principal vector field Y and binormal vector field Z are space like and if α is time like then we have the following Frenet formulae along α:

$$\alpha'(s) = T_{\alpha(s)}$$

$$\nabla_T T = k_2 Y$$

$$\nabla_T Y = k_2 T + k_2 Z$$

$$\nabla_T Z = - k_2 Y$$

where ∇ denotes the covariant differentiation in M_i (see [1]). If α is space–like and Y is time–like then

$$\alpha'(s) = T_{\chi(s)}$$
$$\nabla_T T = k_2 Y$$
$$\nabla_T Y = k_2 T + k_2 Z$$
$$\nabla_T Z = k_2 Y.$$

Finally, by a Carton frame $\{T, Y, Z\}$ of a null curve α we mean a family of vector fields T, Y, Z along α satisfying the following conditions

$$\alpha'(s) = T,\ g(T, T) = g(Y, Y) = 0$$
$$g(T, Y) = -1,\ g(T, Z) = g(Y, Z) = 0,\ g(Z, Z) = 1$$
$$\nabla_T T = k_1 Z,\ \nabla_T Y = k_2 Z,\ \nabla_T Z = k_2 T + k_1 Y$$

Especially if k_1 and k_2 are positive constants along α then we call the curve α a Cartan framed null curve with constant curvatures [1].

Finally we recall two fundamental theorems as follows:

Theorem. A: Let $f: M_r^n \longrightarrow \mathbf{R}^n_s$ be an isometric immersion of a connected pseudo Riemannian manifold, $n \geq 2$. If for every non–null geodesic c of M, foc is a plane curve in $\mathbf{R}_s^n$, then L is constant for all unit vectors $X \in TM$ and we have the following cases:

$L > 0$: Each foc is a part of an $S^1 \subset \mathbf{R}_1^2$ or an $S_1^1 \subset \mathbf{R}_1^2$, each of radius $(1 / \sqrt{L})$.

$L < 0$: Each foc is a part of an $\mathbf{H}^2 \subset \mathbf{R}_1^2$ or an $\mathbf{H}^1_2 \subset \mathbf{R}_2^2$, each of radius $(-1 / \sqrt{L})$.

$L = 0$: Each foc is either a line segment or a curve in a degenerate plane $\mathbf{R}^2_{0,1}$ or $\mathbf{R}^2_{1,1}$

where $L = \langle h(X, X), h(X, X)\rangle$ [3].

Theorem. B: If the curve $s \longrightarrow \gamma(s)$ time–like circle then γ satisfies the following third order differential equation,

$$\nabla_X \nabla_X X - g(\nabla_X X, \nabla_X X)\, X = 0 \tag{0.5}$$

where $X(\gamma(s)) = \gamma'(s)$ is the velocity vector of γ, [1].

1. IMMERSIONS WITH (POINTWISE) PLANAR NORMAL SECTIONS OF LORENTZIAN SURFACES

The facts of being planar in Lorentzian space $\mathbf{R}_1^m$ alike in the case of Euclidean spaces, that is, a curve, time–like or space–like, is planar R_1^m, m–dimensional standart Lorentzian space, iff $k_2 = 0$ where k_2 is the second curvature function of the curve. In addition we conclude that if β is a Cartan framed null curve in Lorentzian surfaces M_1 in E^3 and a planar curve then β is a geodesic in M_1. Conversly, if "$k_1 = 0$ and $k_2 = 0$" or "$k_1 = 0$ and for a fixed point $\beta(0)$, the vectors $\beta(s)-\beta(0)$ and $\beta'(0)$ are linearly dependent" then β is planar.

Let M^n, $(n \geq 2)$, be an n–dimensional Lorentzian submanifold of the Lorentzian space R^{n+m}, $(m \geq 1)$. If the normal section curve $\gamma = ns(M, p, t)$ is space–like or time–like then $\nabla_t T = 0$ and if γ is null then $\nabla_t T = \lambda t$; $(\lambda \neq 0)$ where; $\alpha(0) = p$, $\gamma'(s) = T$, $\gamma'(0) = t \in T_pM$. If γ is Cartan framed null curve then $\nabla_t T = 0$.

Following characterization of Lorentzian submanifolds with normal sections is an easy analog of the Riemannian case.

Theorem 1.1. Let M be a Lorentzian submanifold of the Lorentzian space R_1^{n+p}, $(p \geq 1, n \geq 2)$. Then, M has pointwise planar normall sections iff

$$(\bar{\nabla}_t h)(t, t) \wedge h(t, t) = 0. \tag{1.1}$$

Theorem 1.2. Let M^n be an n–dimensional Lorentzian submanifold of R_1^{n+1} $(n \geq 2)$. If all null curves in M^n Cartan framed then for all $p \in M^n$ we have that.

"$\bar{\nabla}_p h \equiv 0 \Leftrightarrow M^n$ has pointwise 2–planar normal sections at p and the point p is a vertex point for all normal section curves pass through p".

Of course, we have to point out one thing about the proof which takes place for sufficiency of the theorem.

Since M^n_1 has pointwise 2–planar normal sections, then

$$(\bar{\nabla}_t h)(t, t) \wedge h(t, t) = 0$$

that gives us $(\bar{\nabla}_t h)(T, T) = \zeta h(t, t)$. On the other hand if the point p is a vertex point so

$$\frac{d^2k}{ds}(o) = 0$$

where k is the first curvature function of the normal section curve. Thus;

$$\frac{d^2k}{ds}(o) = g\left((\bar{\nabla}_t h)(T, T),\ h(t, t)\right) = 0$$

or

$$g(\zeta\, h(t, t),\ h(t, t)) = 0$$

or

$$\zeta\, g(h(t, t),\ h(t, t)) = 0 \tag{1.2}$$

If $\zeta = 0$ then $(\bar{\nabla}_t h)(t, t) = 0$. If $\zeta \neq 0$ then define

$$U = \{t \in T_pM \mid h(t, t) = 0\}$$

but int $(U) \neq \varnothing$ so

$$(\bar{\nabla}_t h)(T, T) = D_t h(T, T) - 2h(\nabla_t T, T)$$

that is

$$(\bar{\nabla}_t h)(t, t) = 0.$$

Following theorem has the same proof of Theorem. 2 in [2] and we just express it here.

Theorem 1.3. Let $M^n{}_1$ be an n–dimensional Lorentzian submanifold of Lorentzian space $\mathbf{R}_1{}^{n+m}$ $(n \geq 2,\ m \geq 1)$. Then

"$(\bar{\nabla}_t h)(t, t) = 0$ for every t in T_pM iff $\bar{\nabla} h \equiv 0$"

Corollary: Let $M^n{}_1$ $(n \geq 2)$ be an n–dimensional Lorentzian submanifold of the Lorentzian space $\mathbf{R}_1{}^{n+1}$ and assume that all null curves in $M_1{}^n$ are Cartan framed. Then the following are equivalent

i) $(\bar{\nabla}_t h)(t, t) = 0$ for all $t \in T_pM$

ii) $\bar{\nabla}_p h \equiv 0$

iii) M has pointwise 2–planar normal sections and p is a vertex point for all normal section curves that pass through p.

Now we give the following definition;

Definition 1.1. Let $M^n{}_1$ $(n \geq 2)$ be an n–dimensional Lorentzian submanifold of the Lorentzian space $\mathbf{R}_1{}^{n+1}$. If $\gamma(s) = ns\,(M, p, t)$ is a null curve and assume that $\gamma(s)$ is not Cartan framed then the number λ $(\lambda \neq 0)$ which is defined by

$$\nabla_t T = \lambda t, \quad (\gamma'(0) = t)$$

will be called planar normal section curvature of for the sake of simplicity P.N.S curvature of γ. Furthermore, the critical points of the function

$$A : I \longrightarrow R, \ A(s) = g(\gamma''(s), \gamma''(s))$$

will be called vertex points of γ.

It is clear that, if γ is not a null curve then the above definition coincides with the well known definition of vertex points and the curvature.

Theorem 1.4. Let M^n_1 be an n–dimensional Lorentzian submanifold of the Lorentzian space $\mathbf{R}_1^{n+m}$ $(n \geq 2, \ m \geq 1)$. Let γ be the null normal section curve ns (M, p, t) such that γ is not Cartan framed and has P.N.S. curvature λ. Assume that;

$$\nabla_t \nabla_T T = \lambda_1 t, \ \lambda_1 \neq 0, \ \gamma'(s) = T$$

and h (T, T) is constant along γ then $\gamma(0) = p$ is a vertex point, furthermore

$$(\bar{\nabla}_t h)(t, t) \wedge h(t, t) = 0.$$

Proof:

Since

$$A(s) = g(\gamma''(s), \gamma''(s))$$
$$= g(\nabla_T T, \ \nabla_T T) + g(h(T, T), \ h(T, T))$$

then

$$A'(s) = \frac{dA}{ds}(s) = 2\lambda_1 \lambda \ g(t, t) = 0$$

so the point $p = \gamma(0)$ is a vertex point of γ. On the other hand

$$(\bar{\nabla}_t h)(T, T) = D_t h(T, T) - 2h(t, \nabla_t T) = -2\lambda h(t, t)$$

so

$$(\bar{\nabla}_t h)((T, T) \wedge h(t, t) = -2\lambda h(t, t) \wedge h(t, t) = 0$$

2. A CHARACTERIZATION FOR SEMI–SPHERES IN $\mathbf{R}_1^m$

We belive that Theorem. 3.2 in [1] is false because of the method used for the proof which is based on "changing Y into –Y″, where Y is the second Frenet vector of the curve. Since if one changes Y into –Y then the curve that has –Y as the second Frenet vector is not the same curve any more which has Y as the second Frenet vector. Indeed, for the curve α which is as before we have

$$(\nabla_{X(s)}X_{(s)})_{s=0} = (1/r)\, Y$$

and

$$\left.\begin{aligned}\nabla_{X(s)}X_{(s)} &= k\, Y_s \\ \nabla_{X(s)}Y_{(s)} &= k\, X_s\end{aligned}\right\} \qquad (2.1)$$

where k is a positive constant and X_s is space–like and first Frenet vector of the curve. But $\{X, -Y\}$ does not satisfy the equations in (2.1). In fact, if

$$\left.\begin{aligned}\nabla_{X(s)}X_{(s)} &= k\, Y_s \\ \nabla_{X(s)}Y_{(s)} &= k\, X_s\end{aligned}\right. \quad ;\ (k > 0 \text{ and } k \text{ is constant})$$

then

$$k\,(-Y_s) = -\nabla_{X(s)}X_s$$

or

$$\nabla_{X(s)}Y_{(s)} = k\, Y_s$$

thus

$$\nabla_{X(s)}(-Y_{(s)}) = -\nabla_{X(s)}Y_{(s)} = -k\, X_s$$

that is

$$\nabla_{X(s)}(-Y_{(s)}) \neq k\, X_s$$

which means that the equations in (2.1) does not hold for $\{X, -Y\}$.

Because of the above reason, we give the Theorem. 3.2 in [1] is false and we assert the following two theorems instead.

On the other hand, by using the bilinearity of g together with the equations (0.1), (0.2) and (1.3) we obtain;

$$g\,(\tilde{\nabla}_X X, \tilde{\nabla}_X X) = g\,(\nabla_X X, \nabla_X X) + g\,(H, H) \qquad (2.7)$$

Thus, (2.3), (2.7) and (2.8) imply that

$$\tilde{\nabla}_X\tilde{\nabla}_X X - g\,(\tilde{\nabla}_X X, \tilde{\nabla}_X X)\, X = 0$$

that is γ is a time–like circle in $\mathbf{R}_1^{n+p}$. Since M^n_1 has parallel mean curvature vector field, we have

$$D_X h(X, X) = D_X H = 0$$

thus

$$(\bar{\nabla}_t h)(X, X) = D_t h(X, X) - 2h(t, \nabla_t X) = 0$$

and as a consequence of that we get

$$(\bar{\nabla}_t h)(t, t) \wedge h(t, t) = 0$$

where $\bar{\nabla}$ denotes the Van der Waerden–Bortolotti connection (see [2]). So $M^n{}_2$ has pointwise 2–planar normal sections because of Theorem A.

Proof of Theorem 2.2.

Hypothesis ii) together with Theorem. B implies that $M_1{}^n$ has 2–planar normal sections of the same curvature. Thus, Theorem. C holds for $M^n{}_1$. Because of that, every geodesic γ in $M^n{}_1$ with initial value $\gamma'(0) = x$ is an arc of an $S^1 \subset \mathbf{R}_1{}^2$ and each of radius is constant and has the value of

$$\frac{1}{\| h(X, X) \|}$$

where R^2 and $\mathbf{R}_1{}^2$ stands for the planes that passes through $\gamma(0)$ and lies in $T_{\gamma(0)} M^n{}_1$. This arc is the solution curve of the following equations

$$\left.\begin{aligned} \tilde{\nabla}_X X &= \| h(X, X) \| \, Y \\ \tilde{\nabla}_X Y &= \| h(X, X) \| \, X \end{aligned}\right\} \qquad (2.1)$$

with the initial values that

$$X_p = x$$

$$Y_p = y$$

where x, y are orthonormal tangent vectors. Furthermore,

$$Y = - \frac{h(X, X)}{\| h(X, X) \|}$$

and that $Y_{\gamma(o)}$ is uniquely determined and independent of the chosen x since im (h) = 1.

Thus the curvature center

$$C = \gamma(0) + \frac{1}{\| h(x, x) \|} \, Y_{\gamma(o)}$$

of γ is independent of the chosen x, that is c is constant. What we get is that the geodesics pass through the point $\gamma(0) = p$ ile on the pseudo -sphere whose center is c and radius is $-\frac{1}{\| h(x, x) \|}$. Thus the point $\gamma(0) = p$ has a neighborhood in M^n_1 that lies on a pseudosphere so M^n_1 is totally umbilic and has parallel mean curvature vector field.

Remark:

Theorem. 4.2 and Theorem. 5.3 in [1] was proved by using the method just described at the beginning of this section when analyzing the proof of Theorem. 3.3 in [1] So we think that those theorems still open problems to be solved In addition, Theorem 4.1. given in [6] is false. The reason follows.

If we set $\{X, Y\}$ and $\{\tilde{X}, \tilde{Y}\}$ as the Frenet frames of circles c_1 and c_2, respectively on the condition that

$$c_1(o) = p$$
$$c'_1(o) = u$$
$$(\nabla_{c'_1} c'_1)(o) = kv; \; (k > 0 \text{ and constant})$$

and

$$c_2(o) = p$$
$$c'_2(o) = u$$
$$(\nabla_{c'_2} c_2)(o) = -kv; \; (k > 0 \text{ and constant})$$

as in [6]. For both cases we have

$$(\bar{\nabla}_u B)(X, X) + 3k\, B(u, v) = 0 \tag{1}$$

$$(\bar{\nabla}_u B)(X, X) - 3k\, B(u, v) = 0. \tag{2}$$

But we can't have

$$(\bar{\nabla}_u B)(X, X) = 0$$

and

$$B(u, v) = 0$$

from (1) and (2), since $B(\tilde{X}, \tilde{X}) \neq B(X, X)$.

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