Commun. Fac. Sci. Univ. Ank. Series A_1
V. 43. pp. 31-40 (1994)

DISCRETE HYPERGEOMETRIC FUNCTIONS AND THEIR PROPERTIES

By

MUMTAZ AHMAD KHAN

(Received July 13, 1994; Accepted June 17, 1994)

Dept. of Appl. Maths., Z. H. College of Engg. A.M.U., Aligarh, INDIA

ABSTRACT

In 1979, Harman [8] in connection of the study of q-analytic functions [7]., introduced a discrete analogue $z^{(n)}$ of the classical power function z^n. This paper deals with a study of a class of functions called discrete hypergeometric functions defined in (2.2) by using the discrete power function $z^{(n)}$.

1. INTRODUCTION

Harman [7], in 1978, introduced the concept of q-analyticity of a function by replacing derivatives by q-difference operators $D_{q,x}$ and $D_{q,y}$ which are defined as follows:

$$D_{q,x}[f(z)] = \frac{f(z) - f(qx, y)}{(1-q)x} \tag{1.1}$$

$$D_{q,y}[f(z)] = \frac{f(z) - f(x, qy)}{(1-q)iy} \tag{1.2}$$

where f is a discrete function.

The two operators involve a 'basic triad' of points denoted by

$$T(z) = \{(x, y), (qx, y), (x, qy)\} \tag{1.3}$$

Let D be a discrete domain. Then a discrete function f is said to be 'q-analytic' at $z \in D$ if

$$D_{q,x}[f(z)] = D_{q,y}[f(z)] \tag{1.4}$$

If in addition (1.4) holds for every $z \in D$ such that $T(z) \subseteq D$ then f is said to be 'q-analytic' in D. (1.5)

For simplicity if (1.4) or (1.5) holds, the common operator D_q is used where

$$D_q \equiv D_{q,x} \equiv D_{q,y} \tag{1.6}$$

The function z^n is of basic importance in complex analysis since its use in infinite series leads to the Weierstrassian concept of an analytic function. Harman [8] defined, for a nonnegative integer n, a q–analytic function $z^{(n)}$ to denote the discrete analogue of $z^{(n)}$, if it satisfies the following conditions:

$$\left.\begin{aligned} D_q\,[z^{(n)}] &= \frac{(1-q^n)}{(1-q)}\, z^{(n-1)} \\ z^{(o)} &= 1 \\ 0^{(n)} &= 0,\ n > 0 \end{aligned}\right\} \tag{1.7}$$

The operator C_y, given by

$$C_y \equiv \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j}\,(iy)^j\, D^j_{q,x} \tag{1.8}$$

when applied to the real function x^n, yields $z^{(n)}$.

In fact, Harman [8] defined $z^{(n)}$ by

$z^{(n)} \equiv C_y\,(x^n)$; n a non–negative integer

$$= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j}\,(iy)^j\, D^j_{q,x}\,(x^n), \tag{1.9}$$

which on simplification, yields

$$z^{(n)} = \sum_{j=0}^{n} \binom{n}{j}_q x^{n-j}\,(iy)^j \tag{1.10}$$

or alternatively,

$$z^{(n)} = \sum_{j=0}^{n} \binom{n}{j}_q x^j\,(iy)^{n-j} \tag{1.11}$$

To justify that $z^{(n)}$ is a proper analogue of z^n, Harman [8] proved that $z^{(n)}$ is a q–analytic function and satisfies the three requirements of (1.7).

We shall also use the following notations due to Hahn [3]: Let

$$f(x) = \sum_{r=0}^{\infty} a_r x^r \tag{1.12}$$

be a power series in x. Then

$$f([x-y]) = \sum_{r=0}^{\infty} a_r (x-y)_r \tag{1.13}$$

$$f\left(\frac{t}{[x-y]}\right) = \sum_{r=0}^{\infty} a_r \frac{t^r}{(x-y)_r} \tag{1.14}$$

where $(x-y)_\alpha = x^\alpha \prod_{n=0}^{\infty} \left[\frac{1-(y/x)\,q^n}{1-(y/x)\,q^{\alpha+n}}\right]$. (1.15)

For various other definitions, notations and results used in this paper one is referred to remarkable books on q–Hypergeometric series by Exton [1], Gasper and Rahman [2] and Slater [13].

2. DISCRETE HYPERGEOMETRIC FUNCTIONS

Using Harman's discrete analogue $z^{(n)}$ for the classical function z^n, we now introduce a discrete analogue ${}_rM_s\,[(a_r);\,(b_s);\,q,\,z]$ of the q–hypergeometric function ${}_r\Phi_s^{(q)}\,[(a_r);\,(b_s);\,z]$.

It is well known that

$$D_q\left\{{}_r\Phi_s^{(q)}[(a_r);\,(b_s);\,x]\right\} = \frac{(1-q^{a_1})\,\ldots\ldots(1-q^{a_r})}{(1-q)\,(1-q^{b_1})\ldots(1-q^{b_s})}$$

$${}_r\Phi_s^{(q)}\,[1+(a_r);\,1+(b_s);\,x]$$

and so it seems reasonable to assume that for n, a non–negative integer a q–analytic function ${}_rM_s\,[(a_r);\,(b_s);\,q,\,z]$ will denote the discrete analogue of ${}_r\Phi_s^{(q)}\,[(a_r);\,(b_s);\,z]$ if it satisfies the following conditions:

(i) $D_q \{{}_rM_s [(a_r); (b_s); q, z]\} = \dfrac{(1-q^{a_1}) \ldots\ldots\ldots (1-q^{a_r})}{(1-q)(1-q^{b_1}) \ldots\ldots (1-q^{b_s})}$

$${}_rM_s [1 + (a_r); 1 + (b_s); q, z], \tag{2.1}$$

(ii) The first term of the series is 1.

(iii) ${}_rM_s [(a_r); (b_s); q, 0] = 1.$

Such a function is obtained by applying the operator C_y defined in (1.8) to the q–hypergeometric function ${}_r\Phi_s [(a_r); (b_s); x]$, with real argument x.

In fact, ${}_rM_s [(a_r); (b_s); q, z]$ is defined by

$${}_rM_s [(a_r); (b_s); q, z] \equiv C_y \left\{ {}_r\Phi_s^{(q)} [(a_r); (b_s); x] \right\}$$

$$= \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n z^{(n)}}{(q)_n (q^{(b_s)})_n} \tag{2.2}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{(a_r)})_{n+k} x^n (iy)^k}{(q)_n (q)_k (q^{(b_s)})_{n+k}} \tag{2.3}$$

The following theorem shows that ${}_rM_s [(a_r); (b_s); q, z]$ satisfies (2.1) and hence can be taken as a discrete analogue of ${}_r\Phi_s^{(q)} [(a_r); (b_s); z]$.

Theorem 1. ${}_rM_s [(a_r); (b_s); q, z]$ is q–analytic and satisfies the requirements of (2.1).

Proof:

$${}_rM_s [(a_r); (b_s); q, z] = \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n z^{(n)}}{(q)_n (q^{(b_s)})_n}$$

$$= \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n}{(q)_n (q^{(b_s)})_n} \sum_{j=0}^{n} \binom{n}{j}_q x^{n-j} (iy)^j ,$$

and hence

$D_{q,x} \{ {}_rM_s [(a_r); (b_s); q, z] \}$

$$= \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n}{(q)_n (q^{(b_s)})_n} \sum_{j=0}^{n-1} \binom{n}{j}_q \frac{(1-q^{n-j})}{(1-q)} x^{n-j-1} (iy)^j$$

$$= \frac{1}{(1-q)} \sum_{n=1}^{\infty} \frac{(q^{(a_r)})_n z^{(n-1)}}{(q)_{n-1} (q^{(b_s)})_n}$$

$$= \frac{(1-q^{a_1}) \ldots\ldots.(1-q^{a_r})}{(1-q)(1-q^{b_1}) \ldots(1-q^{b_s})} {}_rM_s [1 + (a_r); 1 + (b_s); q, z].$$

similarly,

$D_{q,y} \{ {}_rM_s [(a_r); (b_s); q, z] \}$

$$= \frac{(1-q^{a_1}) \ldots\ldots.(1-q^{a_r})}{(1-q)(1-q^{b_1}) \ldots(1-q^{b_s})} {}_rM_s [1 + (a_r); 1 + (b_s); q, z].$$

Hence ${}_rM_s [(a_r); (b_s); q, z]$ is q–analytic and satisfies condition (i) of (2.1).

Since $z^{(0)} = 1$ and $0^{(n)} = 0$, $n > 0$, by definition and so ${}_rM_s [(a_r); (b_s); q, z]$ satisfies (ii) and (iii) also of (2.1). This proves the theorem.

It is of interest to note the similarity of ${}_rM_s [(a_r); (b_s); q, z]$ to the function ${}_r\Phi_s^{(q)} [(a_r); (b_s); [x + y]]$ defined by Jackson [10] as follows:

$$ {}_r\Phi_s^{(q)}\left([a_r);\ (b_s);\ [x+y]\ \right] $$

$$ = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^{(a_r)})_{m+n}\, x^m\, (iy)^n\, q^{\frac{1}{n}\, n\,(n-1)}}{(q)_m\,(q)_n\,(q^{(b_s)})_{m+n}} $$

$$ = \sum_{N=0}^{\infty} \frac{(q^{(a_r)})_N}{(q)_N\,(q^{(b_s)})_N}\,(x+iy)\,(x+iqy)\ \ldots\ (x+iq^{N-1}\,y). \qquad (2.4) $$

The discrete hypergeometric function defined in (2.2) can be written in either of the following two forms:

$$ {}_rM_s\,[(a_r);\ (b_s);\ q,\ z\,] = \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n\,(iy)^n}{(q)_n\,(q^{(b_s)})_n}\ {}_r\Phi_s^{(q)}\,[(a_r)+n;\ (b_s)+n;\ x\,] \qquad (2.5) $$

or alternatively as,

$$ {}_rM_s\,[(a_r);\ (b_s);\ q,\ z\,] = \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n\,x^n}{(q)_n\,(q^{(b_s)})_n}\ {}_r\Phi_s^{(q)}\,[(a_r)+n;\ (b_s)+n; iy\,]. \qquad (2.6) $$

From (2.5) and (2.6), we observe that a discrete hypergeometric function can be regarded as a 'generating function' for the q–hypergeomotric functions of the form

$$ {}_r\Phi_s^{(q)}\ [(a_r)+n;\ (b_s)+n;\ x\,] \text{ or } {}_r\Phi_s^{(q)}\ [(a_r)+n;\ (b_s)+\ n;\ iy\,]. $$

We further observe that for $x = 0$, ${}_rM_s\ [(a_r);\ (b_s);\ q,\ z\,]$ reduced to ${}_r\Phi_s^{(q)}\ [(a_r); (b_s); iy\,]$ while for $y = 0$ it becomes ${}_r\Phi_s^{(q)}\ [(a_r);\ (b_s);\ x\,]$.

3 PARTICULAR CASES

As particular cases of (2.5) and (2.6), we have the following interesting results:

$${}_0M_0\ [—;\ —;\ q,\ z] = e_q(x)\ e_q(iy), \tag{3.1}$$

$${}_2M_1\ [a, b; c; q, z],$$

$$= \frac{1}{(1-x)_{a+b-c}} \sum_{n=0}^{\infty} \frac{(q^a)_n (q^b)_n (iy)^n}{(q)_n (q^c)_n (xq^{a+b-c})_n}$$

$${}_2\Phi_1 \left[\begin{matrix} q^{c-a},\ q^{c-b};\ xq^{a+b-c+n} \\ q^{c+n}; \end{matrix} \right], \tag{3.2}$$

$${}_0M_1\ [—;\ a;\ q,\ z] = (q)_{a-1} \left(-\frac{1}{\sqrt{x}}\right)^{a-1} \sum_{n=0}^{\infty} \frac{(y/\sqrt{x})^n}{(q)_n} q^{j}\, a+n-1\ (2i\sqrt{x}), \tag{3.3}$$

Further, summing up the ${}_r\Phi_s$ –function by means of known summation theorems, we have

$${}_1M_0\ [a;—;\ q,\ z] = \frac{1}{(1-x)_a}\ {}_1\Phi_0\ [q^a;—;\ \frac{iy}{[1-xq^a]}], \tag{3.4}$$

$${}_1M_0\ [a;\ —;\ q,\ z] = \frac{1}{(1-x)_a}\ {}_1\Phi_1\ [q^a;\ xq^a;\ iy], \tag{3.5}$$

$${}_1M_0\ [a;—;\ q,\ z] = \frac{1}{(1-iy)_a}\ {}_1\Phi_0\ [q^a;—;\ \frac{x}{[1-iyq^a]}], \tag{3.6}$$

$${}_1M_0\ [a;—;\ q,\ z] = \frac{1}{(1-iy)_a}\ {}_1\Phi_1\ [q^a;\ iyq^a;\ x] \tag{3.7}$$

$${}_2M_1\ [a,\text{-}n;\ b;\ q,\ (q,\ y)] = \frac{(q^{b-a})_n\, q^{an}}{(q^b)_n}\ {}_2\Phi_1 \left[\begin{matrix} q^a,\ q^{-n};\ -iyq^{1-b} \\ q^{1+a-b-n};\ q^{-1} \end{matrix} \right] \tag{3.8}$$

§ 4. INTEGRAL REPRESENTATIONS

We also note the following simple integral representations for ${}_2M_1$ [a, b; c; q, z] and ${}_3M_2$ [a, b, c; d, e; g, z] ${}_2M_1$ [a, b; c; q, z]

$$= \frac{\Gamma_q(c)}{\Gamma_q(b)\ \Gamma_q(c-b)} \int_0^1 t^{b-1} (1-qt)_{c-b-1}\ {}_1M_0\ [a; -; q, zt]\ d\ (t; q), \tag{4.1}$$

provided Rl (b) > 0, $|x| < 1$, $|y| < 1$

$${}_3M_2\ [a, b, c; d, e; q, z] = \frac{\Gamma_q(d)\ \Gamma_q(e)}{\Gamma_q(b)\ \Gamma_q(c)\ \Gamma_q(d-b)\ \Gamma_q(e-c)}\ x$$

$$\int_0^1 \int_0^1 t^{b-1} (1-qt)_{d-b-1}\ v^{c-1} (1-qv)_{e-c-1}\ {}_1M_0\ [a;-; ztv]\ d\ (t; q)\ d\ (v; q) \tag{4.2}$$

provided Rl (b) > 0, Rl (c) > 0, $|x| < 1$, $|y| < 1$.

One can similarly, write down the integral representation for ${}_rM_s$ –function.

§ 5. CONTINUOUS DISCRETE HYPERGEOMETRIC FUNCTIONS

Any two discrete hypergeometric functions,

${}_rM_s$ [(a_r); (b_s); q, z] and ${}_rM_s$ [(a'_r); (b'_s); q, z]

are said to be continuous, when all their parameters are equal except one pair, and this pair of parameter differs only by unity.

If we use the notations $(\alpha_r, + i)_n$ and $(\alpha_r, -i)_n$ to denote

$(\alpha_1)_n (\alpha_2)_n \ldots. (\alpha_{i-1})_n (\alpha_i + 1)_n (\alpha_{i+1})_n \ldots. (\alpha_r)_n$

and

$(\alpha_1)_n (\alpha_2)_n \ldots. (\alpha_{i-1})_n (\alpha_{i+1})_n (\alpha_{i+1})_n \ldots. (\alpha_r)_n$

respectively, where $1 \leq i \leq r$, with similar notations for (β_s), we have

$$
{}_rM_s\,[\alpha_r, + i); (\beta_s); q, z] = \frac{1}{(1-q^{\alpha_i})}\{ {}_rM_s\,[\alpha_r); (\beta_s); q, z] - q^{\alpha_i}\, {}_rM_s\,[(\alpha_r); (\beta_s); q, qz]\}, \tag{5.1}
$$

$$
{}_rM_s\,[(\alpha_r, -i); (\beta_s); q, z] = (1-q^{\alpha_i-1}) \sum_{n=0}^{\infty} q^{n(\alpha_i-1)}\, {}_rM_s\,[(\alpha_r); (\beta_s); q, q^n z], \tag{5.2}
$$

$$
{}_rM_s\,[(\alpha_r); (\beta_s, + j); q, z] = (1-q^{\beta_j}) \sum_{n=0}^{\infty} q^{n\beta_j}\, {}_rM_s\,[(\alpha_r); (\beta_s); q, q^n z], \tag{5.3}
$$

and

$$
{}_rM_s\,[(\alpha_r); (\beta_s - j); q, z] = \frac{1}{(1-q^{\beta_j-1})}\, {}_rM_s\,[(\alpha_r); (\beta_s); q, z] - q^{\beta_j-1}\, {}_rM_s\,[(\alpha_r); (\beta_s); q, qz]. \tag{5.4}
$$

REFERENCES

[1] EXTON, H., q–Hypergeometric Functions and Applications. John Wiley and Sons (Halsted Press), New York; Ellis Horwood, Chichester (1983).

[2] GASPER, G. and RAHMAN, M., Basic Hypergeometric Series. Encyclopedia of Mathematics and its Applications, Volume 35. Cambridge University Press (1990).

[3] HAHN, W., Beitrage zür theorie der Heineschen Reihen, die 24 integrale der hypergeometrischen q–differenzengleichung, Das q–Analogen der Laplace Transformation, *Math. Nachr.*, 2 (1949), pp. 340–379.

[4] HARMAN, C.J., A note on a discrete analytic function. *Bull. Austral. Math. Soc.*, 10 (1974), pp. 123–134.

[5] HARMAN, C.J., A new definition of discrete analytic functions. *Bull Austral. Math. Soc.* 10 (1974), pp. 281–291.

[6] HARMAN, C.J., Polynomial in discrete geometric function theory. *Ganita*, 29 (1978), pp. 93–101.

[7] HARMAN, C.J., Discrete geometric function theory I. *Applicable Analysis*, 7 (1978), pp. 315–336.

[8] HARMAN, C.J., Discrete geometric function theory II. *Applicable Analysis*, 9 (1979), pp. 191–203.

[9] HARMAN, C.J., Discrete Analysis on a Radical Lattice, J. Math. Anal. and Appl. 79 (1981), pp. 323–330.

[10] JACKSON, F.H., On basic double hypergeometric functions. *Quart. J. Math. (Oxford)*, 15 (1944), pp. 49–61.

[11] KHAN, M.A., Contribution to the theory of Basic Hypergeometric Series. Ph.D. Thesis, University of Lucknow (1977).

[12] KHAN, M.A., Transformations of Discrete Hypergeometric Functions. *Mathematica Balkanica*, New Series Vol. 6 (1992), Fase. 3, pp. 199–206.

[13] SLATER, L.J., Generalized hypergeometric functions, Cambridge University Press (1966).